**SCIENTIA**
Series A: *Mathematical Sciences*, Vol. 33 (2023), 73–88
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446


# A novel approach to evaluating improper integrals

Russell A. Gordon* and Seán M. Stewart†

Abstract. We explain and apply a recently developed method for evaluating improper integrals of the form $\int_0^\infty f(t)\,dt$ using Laplace transforms. A number of examples are provided to illustrate the method, along with some results that streamline the computations. We show how the method can be used to readily determine values for entire classes of certain integrals which, using other more familiar methods, are difficult to find. We also indicate how the method can determine the values of integrals for which other methods fail.

## 1. Introduction

The purpose of this paper is to discuss and illustrate a recently developed method for evaluating certain integrals of the form $\int_0^\infty f(t)\,dt$. It is possible to evaluate some of the integrals that appear here in other ways, but in such cases, we find that the new method makes the computations much easier. In addition, we show how to apply the method to evaluate some integrals that do not fall within the realm of more traditional methods. Hence, this new method increases the number of options that can be considered for evaluating integrals.

This recently developed method for evaluating integrals was initially introduced heuristically by Kempf, Jackson, and Morales (see [**10**], [**11**]) in the context of quantum field theory; they referred to it as "integration by differentiation." It was later made rigorous by Jia, Tang, and Kempf [**9**] for certain classes of functions and integrals. Our goal here is to explain and illustrate this new method without overwhelming the reader with some of the technical details. Careful proofs can be found in [**9**] and further expanded upon in [**6**], so we focus on helping other mathematicians appreciate the advantages of this newer method.

2000 *Mathematics Subject Classification.* Primary 44A10, 33B10, Secondary 40A10.
*Key words and phrases.* Laplace transform, improper integrals, Dirichlet integral.

## 2. Main Result

Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is analytic on $\mathbb{R}$. We express $f$ as $f(x) = \sum_{k=0}^{\infty} a_k x^k$, noting that $f$ is then defined for all complex numbers $z$. We define a differential operator $f(\beta\partial_x)$, where $\beta$ is a complex number and the symbol $\partial_x^k \circ \phi(x)$ means to take the $k$th derivative of $\phi$ with respect to $x$, as follows: given an analytic function $\phi$, we have

$$f(\beta\partial_x) \circ \phi(x) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k(\beta\partial_x)^k \circ \phi(x) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k \beta^k \phi^{(k)}(x) = \sum_{k=0}^{\infty} a_k \beta^k \phi^{(k)}(x)$$

for all real numbers $x$ for which the corresponding series converges. To preview what is to come, suppose further that $f$ has a Laplace transform defined for all $s > 0$. Now consider the following informal set of equations (see [**11**]) to compute the Laplace transform of $f$, where the substitution $u = st$ is made in the first step:

$$\begin{aligned}
\mathscr{L}\{f(t)\}(s) &= \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-u} f(u/s)\,\frac{du}{s} \\
&= \int_0^\infty \sum_{k=0}^{\infty} \frac{a_k}{s^{k+1}} u^k e^{-u}\,du = \sum_{k=0}^{\infty} \frac{a_k}{s^{k+1}} \int_0^\infty u^k e^{-u}\,du \\
&= \sum_{k=0}^{\infty} a_k \cdot \frac{k!}{s^{k+1}} = \sum_{k=0}^{\infty} a_k(-\partial_s)^k \circ \frac{1}{s} = f(-\partial_s) \circ \frac{1}{s}
\end{aligned}$$

for all $s > 0$. Given this representation for the Laplace transform, it then seems reasonable to conclude that

$$\int_0^\infty f(t)\,dt = \lim_{s\to 0^+} f(-\partial_s) \circ \frac{1}{s}.$$

However, there are several issues to address here. How do we justify the interchange of the integral and the infinite sum? How can we apply our differential operator to a function that is not analytic? As we shall see, it turns out that this integration formula is valid for a certain class of functions.

We first give a specific example to illustrate this differential operator. For any complex number $\beta$ and analytic function $\phi$, we find that

$$e^{\beta\partial_x} \circ \phi(x) = \sum_{k=0}^{\infty} \frac{\beta^k}{k!} \phi^{(k)}(x) = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(x)}{k!} (x + \beta - x)^k = \phi(x + \beta).$$

Hence, the differential operator $e^{\beta\partial_x}$ merely gives a translation of the function $\phi$. It then follows easily that the differential operators $\sin(\beta\partial_x)$ and $\cos(\beta\partial_x)$ behave in much the same way when the sine and cosine functions are written as their complex counterparts. Next, we use the differential operator to find the Laplace transform of the function $t^2 \cos(3t)$:

$$\begin{aligned}\mathscr{L}\{t^2\cos(3t)\}(s) &= (-\partial_s)^2\cos(-3\partial_s)) \circ \frac{1}{s} = \frac{1}{2}\big(e^{3i\partial_s} + e^{-3i\partial_s}\big)\partial_s^2 \circ \frac{1}{s} \\ &= \frac{1}{2}\big(e^{3i\partial_s} + e^{-3i\partial_s}\big) \circ \frac{2}{s^3} = \frac{1}{(s+3i)^3} + \frac{1}{(s-3i)^3} \\ &= \frac{(s+3i)^3 + (s-3i)^3}{(s^2+9)^3} = \frac{2s^3 - 54s}{(s^2+9)^3}.\end{aligned}$$

We have used the translation property of the operators $e^{3i\partial_s}$ and $e^{-3i\partial_s}$ on the function $2/s^3$, even though this function is not analytic. We explain below why this computation is guaranteed to be accurate.

Let $S$ be the collection of all functions that are analytic and bounded on $\mathbb{R}$ and that can be expressed as finite linear combinations of terms of the form $t^k e^{\beta t}$, where $k$ is an integer and $\beta$ is a complex number with $\operatorname{Re}\beta \leqslant 0$, using complex numbers for the scalars. For a few examples of functions that belong to the set $S$, we note that

$$\begin{aligned}t^4 e^{-t}\sin(3t) &= \frac{1}{2i}\,t^4 e^{(-1+3i)t} - \frac{1}{2i}\,t^4 e^{(-1-3i)t}; \\ \frac{\sin t - t}{t^3} &= \frac{1}{2i}\,t^{-3}e^{it} - \frac{1}{2i}\,t^{-3}e^{-it} - t^{-2}; \\ \frac{\Big(\cos t - 1 + \frac{1}{2}t^2\Big)^2}{t^7} &= \frac{\frac{1}{2}}{t^7} + \frac{\frac{1}{2}\cos(2t)}{t^7} - \frac{(2-t^2)\cos t}{t^7} + \frac{1 - t^2 + \frac{1}{4}t^4}{t^7} \\ &= \frac{1}{2}\,t^{-7}\cos(2t) - 2t^{-7}\cos t + t^{-5}\cos t + \frac{3}{2}\,t^{-7} - t^{-5} + \frac{1}{4}\,t^{-3} \\ &= \frac{1}{4}\,t^{-7}e^{2it} + \frac{1}{4}\,t^{-7}e^{-2it} - t^{-7}e^{it} - t^{-7}e^{-it} + \frac{1}{2}\,t^{-5}e^{it} + \frac{1}{2}\,t^{-5}e^{-it} \\ &\quad + \frac{3}{2}\,t^{-7} - t^{-5} + \frac{1}{4}\,t^{-3}.\end{aligned}$$

As indicated in these examples, we may let $\beta = 0$ so that terms of the form $t^k$ can be used in the linear combinations for elements of $S$.

The fact that negative exponents appear when some of these analytic functions are expanded raises a question about the differential operator. Referring to the second example above, we have

$$\frac{\sin(\partial_s) - \partial_s}{\partial_s^3} = \frac{1}{2i}\,\partial_s^{-3}e^{i\partial_s} - \frac{1}{2i}\,\partial_s^{-3}e^{-i\partial_s} - \partial_s^{-2}.$$

How should we interpret the operator $\partial_s^n$ when $n$ is a negative integer? The answer, which should be intuitively clear, is that $\partial_s^n \circ \phi(s)$ represents the $n$th antiderivative of $\phi$ when $n < 0$. The next question that arises concerns the constant of integration. We assert that any such constants that might appear can be omitted. To verify this, suppose that $f(t)/t^n$ is an analytic function. Then it must be true that $f(t) = \sum\limits_{k=n}^{\infty} a_k t^k$. Suppose that $\phi$ is some function and let $\phi_n$ denote any $n$th order antiderivative of $\phi$.

We then have (under the assumption that the appropriate series converge)

$$\frac{f(-\partial_s)}{\partial_s^n} \circ \phi(s) = f(-\partial_s) \circ \phi_n(s) = \sum_{k=n}^{\infty} (-1)^k a_k \partial_s^k \circ \phi_n(s) = \sum_{k=n}^{\infty} (-1)^k a_k (\phi_n(s))^{(k)}.$$

Since we are taking at least $n$ derivatives of the function $\phi_n$, any constants that would have appeared in the antiderivatives (turning into polynomials of degree less than or equal to $n-1$) will disappear. Hence, we omit all constants of integration in what follows. Furthermore, since translation does not impact derivatives or antiderivatives (in the sense of order of operations), we find that

$$e^{\beta\partial_s} \partial_s^n \circ \phi(s) = e^{\beta\partial_s} \circ \Big(\partial_s^n \circ \phi(s)\Big) = \partial_s^n \circ \Big(e^{\beta\partial_s} \circ \phi(s)\Big) = \partial_s^n e^{\beta\partial_s} \circ \phi(s)$$

for any complex number $\beta$ and integer $n$.

We now record our key result for evaluating certain improper integrals.

**Theorem 1:** If $f$ belongs to the collection $S$, then

$$\mathscr{L}\{f(t)\}(s) = f(-\partial_s) \circ \frac{1}{s} \qquad \text{and} \qquad \int_0^{\infty} f(t)\, dt = \lim_{s\to 0^+} f(-\partial_s) \circ \frac{1}{s}.$$

**Proof:** A careful and detailed proof of this result is given in [**6**]. ■

For later reference, we list the following three simple facts:

$$e^{\alpha\partial_s} \circ \frac{1}{s} = \frac{1}{s+\alpha}, \qquad \partial_s^n \circ \frac{1}{s} = \frac{(-1)^n n!}{s^{n+1}}, \qquad \partial_s^{-(n+1)} \circ \frac{1}{s} = \frac{s^n}{n!}\big(\log s - h_n\big),$$

where $h_n$ is the $n$th harmonic number with $h_0 = 0$; the third equation follows easily using integration by parts and induction. To illustrate Theorem 1, we begin with four relatively simple integrals.

**Example 1:** Verify that $\displaystyle\int_0^{\infty} t^4 e^{-2t}\, dt = \frac{3}{4}$. Using Theorem 1, we find that

$$\int_0^{\infty} t^4 e^{-2t}\, dt = \lim_{s\to 0^+} e^{2\partial_s} \partial_s^4 \circ \frac{1}{s} = \lim_{s\to 0^+} e^{2\partial_s} \circ \frac{4!}{s^5} = \lim_{s\to 0^+} \frac{24}{(s+2)^5} = \frac{3}{4}.$$

Of course, this is a well-known integral, but a solution using Theorem 1 indicates how easy it is to evaluate this integral with no prior knowledge.

**Example 2:** Verify that $\displaystyle\int_0^\infty t^4 e^{-t}\sin t\,dx = -3$. Using Theorem 1, we find that

$$\begin{aligned}
\int_0^\infty t^4 e^{-t}\sin t\,dx &= \lim_{s\to 0^+}\Big(-e^{\partial_s}\sin(\partial_s)\partial_s^4\circ\frac{1}{s}\Big)\\
&= \lim_{s\to 0^+}\Big(-\frac{1}{2i}\Big(e^{(1+i)\partial_s}-e^{(1-i)\partial_s}\Big)\circ\frac{24}{s^5}\Big)\\
&= 12i\Big(\frac{1}{(1+i)^5}-\frac{1}{(1-i)^5}\Big)\\
&= 12i\Big(\frac{-1+i}{8}-\frac{-1-i}{8}\Big)\\
&= 12i\cdot\frac{i}{4}\\
&= -3.
\end{aligned}$$

The omitted computations involving complex numbers are routine.

These first two examples indicate why this technique has been called integration by differentiation (see [**9**]). In each case, an integral has been evaluated using differentiation. As indicated in our next two examples (and as we shall later see), the more interesting examples involve antidifferentiation.

**Example 3:** Verify that $\displaystyle\int_0^\infty \frac{\sin^3 t}{t^3}\,dt = \frac{3\pi}{8}$, an example of a Dirichlet type integral. We first note that

$$\frac{\sin^3 t}{t^3} = \frac{1}{t^3}\Big(\frac{e^{it}-e^{-it}}{2i}\Big)^3 = \frac{i}{8}\Big(\frac{e^{3it}-3e^{it}+3e^{-it}-e^{-3it}}{t^3}\Big).$$

Theorem 1 then yields (the integrand is an even function so $f(-\partial_s) = f(\partial_s)$)

$$\begin{aligned}
\int_0^\infty \frac{\sin^3 t}{t^3}\,dt &= \frac{i}{8}\lim_{s\to 0^+}\big(e^{3i\partial_s}-3e^{i\partial_s}+3e^{-i\partial_s}-e^{-3i\partial_s}\big)\,\partial_s^{-3}\circ\frac{1}{s}\\
&= \frac{i}{8}\lim_{s\to 0^+}\big(e^{3i\partial_s}-3e^{i\partial_s}+3e^{-i\partial_s}-e^{-3i\partial_s}\big)\circ\frac{s^2}{2}\Big(\log s-\frac{3}{2}\Big)\\
&= \frac{i}{16}\lim_{s\to 0^+}\Big((s+3i)^2\Big(\log(s+3i)-\frac{3}{2}\Big)-3(s+i)^2\Big(\log(s+i)-\frac{3}{2}\Big)\\
&\qquad +3(s-i)^2\Big(\log(s-i)-\frac{3}{2}\Big)-(s-3i)^2\Big(\log(s-3i)-\frac{3}{2}\Big)\Big)\\
&= \frac{i}{16}\Big(-9\Big(\log(3i)-\frac{3}{2}\Big)+3\Big(\log(i)-\frac{3}{2}\Big)-3\Big(\log(-i)-\frac{3}{2}\Big)+9\Big(\log(-3i)-\frac{3}{2}\Big)\Big)\\
&= \frac{i}{16}\Big(-9\log(3i)+3\log(i)-3\log(-i)+9\log(-3i)\Big)\\
&= \frac{3i}{16}\Big(-3\log(3)-3\log(i)+\log(i)-\log(-i)+3\log(3)+3\log(-i)\Big)\\
&= \frac{3i}{8}\big(\log(-i)-\log(i)\big) = \frac{3i}{8}\Big(-\frac{\pi i}{2}-\frac{\pi i}{2}\Big) = \frac{3\pi}{8}.
\end{aligned}$$

Note the use of the principal value of the complex logarithm.

**Example 4:** Evaluate $\displaystyle\int_0^\infty \frac{e^{-at}\sin(bt)}{t}\,dt$, where $a$ and $b$ are positive constants.

**Solution:** By Theorem 1, the value of the integral is

$$\begin{aligned}\int_0^\infty \frac{e^{-at}\sin(bt)}{t}\,dt &= \lim_{s\to 0^+} \frac{e^{a\partial_s}\sin(b\partial_s)}{\partial_s} \circ \frac{1}{s}\\ &= \frac{1}{2i}\lim_{s\to 0^+}\big(e^{(a+bi)\partial_s} - e^{(a-bi)\partial_s}\big)\circ \log s\\ &= \frac{1}{2i}\lim_{s\to 0^+}\big(\log(s+a+bi) - \log(s+a-bi)\big)\\ &= \frac{1}{2i}\log\Big(\frac{a+bi}{a-bi}\Big) = \arctan(b/a).\end{aligned}$$

Note that the value of this integral has been determined with minimal computations.

## 3. Streamlining the Computations

As is clear from the last two examples presented in the previous section requiring antidifferentiation, limits involving complex powers of the exponential function appear regularly in the evaluation of the type of integrals under consideration. To streamline the computations, we prove the following theorem and its relevant corollaries.

**Theorem 2:** Suppose that $a$ and $b$ are positive real numbers and that $n$ is a nonnegative integer. Expressing $(a+bi)^n$ as $c+di$, we then have

$$\begin{aligned}\lim_{s\to 0^+}\Big(\frac{e^{a\partial_s}\sin(b\partial_s)}{\partial_s^{n+1}}\circ\frac{1}{s}\Big) &= \frac{1}{n!}\Big(d\log\sqrt{a^2+b^2} + c\arctan(b/a) - dh_n\Big);\\ \lim_{s\to 0^+}\Big(\frac{e^{a\partial_s}\cos(b\partial_s)}{\partial_s^{n+1}}\circ\frac{1}{s}\Big) &= \frac{1}{n!}\Big(c\log\sqrt{a^2+b^2} - d\arctan(b/a) - c\,h_n\Big).\end{aligned}$$

**Proof:** Using the properties of the operator $\partial_s$, we have (recall that $\overline{z^n} = \overline{z}^n$ for complex conjugates)

$$\begin{aligned}\lim_{s\to 0^+}\Big(e^{a\partial_s}\sin(b\partial_s)\partial_s^{-n-1}\circ\frac{1}{s}\Big) &= \frac{1}{(2i)n!}\lim_{s\to 0^+}\Big(\big(e^{(a+bi)\partial_s} - e^{(a-bi)\partial_s}\big)\circ s^n\big(\log s - h_n\big)\Big)\\ &= \frac{1}{(2i)n!}\lim_{s\to 0^+}\Big((s+a+bi)^n\big(\log(s+a+bi) - h_n\big)\\ &\qquad\qquad - (s+a-bi)^n\big(\log(s+a-bi) - h_n\big)\Big)\\ &= \frac{1}{(2i)n!}\Big((c+di)\big(\log(a+bi) - h_n\big) - (c-di)\big(\log(a-bi) - h_n\big)\Big)\\ &= \frac{1}{(2i)n!}\Big(c\log\Big(\frac{a+bi}{a-bi}\Big) + di\log(a^2+b^2) - 2dih_n\Big)\\ &= \frac{1}{n!}\Big(c\arctan(b/a) + d\log\sqrt{a^2+b^2} - dh_n\Big)\end{aligned}$$

and

$$\begin{aligned}
\lim_{s\to 0^+}\Big(e^{a\partial_s}\cos(b\partial_s)\partial_s^{-n-1}\circ\frac{1}{s}\Big) &= \frac{1}{2n!}\lim_{s\to 0^+}\Big(\big(e^{(a+bi)\partial_s}+e^{(a-bi)\partial_s}\big)\circ s^n\big(\log s-h_n\big)\Big)\\
&= \frac{1}{2n!}\Big((c+di)\big(\log(a+bi)-h_n\big)+(c-di)\big(\log(a-bi)-h_n\big)\Big)\\
&= \frac{1}{2n!}\Big(c\log(a^2+b^2)+di\log\Big(\frac{a+bi}{a-bi}\Big)-2c\,h_n\Big)\\
&= \frac{1}{n!}\Big(c\log\sqrt{a^2+b^2}-d\arctan(b/a)-c\,h_n\Big).
\end{aligned}$$

In each case, we are using the principal branch of the complex logarithm. ∎

**Corollary 3:** If $b$ is a positive real number and $p$ is a positive integer, then

$$\begin{aligned}
\lim_{s\to 0^+}\frac{\sin(b\partial_s)}{\partial_s^{2p-1}}\circ\frac{1}{s} &= (-1)^{p-1}\cdot\frac{b^{2p-2}}{(2p-2)!}\cdot\frac{\pi}{2};\\
\lim_{s\to 0^+}\frac{\cos(b\partial_s)}{\partial_s^{2p}}\circ\frac{1}{s} &= (-1)^{p}\cdot\frac{b^{2p-1}}{(2p-1)!}\cdot\frac{\pi}{2};\\
\lim_{s\to 0^+}\frac{\sin(b\partial_s)}{\partial_s^{2p}}\circ\frac{1}{s} &= (-1)^{p-1}\cdot\frac{b^{2p-1}}{(2p-1)!}\cdot(\log b-h_{2p-1});\\
\lim_{s\to 0^+}\frac{\cos(b\partial_s)}{\partial_s^{2p-1}}\circ\frac{1}{s} &= (-1)^{p-1}\cdot\frac{b^{2p-2}}{(2p-2)!}\cdot(\log b-h_{2p-2}).
\end{aligned}$$

**Proof:** These four limits can be obtained from Theorem 2 by setting $a=0$ and noting that the expression $\arctan(b/a)$ becomes $\pi/2$. The values of $c$ and $d$ vary depending on whether $n$ is even or odd; the elementary details are left to the reader. ∎

**Corollary 4:** If $a$ is a positive real number and $n$ is a nonnegative integer, then

$$\lim_{s\to 0^+}\Big(\frac{e^{a\partial_s}}{\partial_s^{n+1}}\circ\frac{1}{s}\Big)=\frac{a^n}{n!}\big(\log a-h_n\big).$$

**Proof:** Letting $b=0$ in the cosine portion of Theorem 2 (which implies $c=a^n$ and $d=0$), we find that

$$\lim_{s\to 0^+}\Big(\frac{e^{a\partial_s}}{\partial_s^{n+1}}\circ\frac{1}{s}\Big)=\frac{1}{n!}\Big(c\log a-c\,h_n\Big)=\frac{a^n}{n!}\Big(\log a-h_n\Big).$$

This completes the proof. ∎

**Corollary 5:** If $p$ is a positive integer, then $\lim\limits_{s\to 0^+}\dfrac{1}{\partial_s^{p+1}}\circ\dfrac{1}{s}=0.$

**Proof:** Using simple properties of the logarithm function, we find that

$$\lim_{s\to 0^+}\frac{1}{\partial_s^{p+1}}\circ\frac{1}{s}=\lim_{s\to 0^+}\frac{s^p}{p!}\big(\log s-h_p\big)=0.$$

This establishes the desired limit. ∎

We refer to the evaluation of integrals using Theorem 1 along with Theorem 2 and its corollaries as the *Laplace transform operator method*, the LTO method for short.

## 4. Some Further Examples

We now give some examples to illustrate the LTO method, starting by redoing Example 3 to show how the limits presented in Theorem 2 and its corollaries simplify the computations.

**Example 5:** Verify that $\displaystyle\int_0^\infty \frac{\sin^3 x}{x^3}\,dx = \frac{3\pi}{8}$. Using the LTO method, along with a trigonometric identity for the function $\sin^3 x$ and the appropriate limit given in Corollary 3 (the first of the listed limits using $p = 2$ and $b = 1$, $b = 3$, respectively), we find that

$$\begin{aligned}
\int_0^\infty \frac{\sin^3 x}{x^3}\,dx &= \int_0^\infty \frac{\frac{3}{4}\sin x - \frac{1}{4}\sin(3x)}{x^3}\,dx \\
&= \lim_{s\to 0^+}\Big(\frac{3}{4}\cdot\frac{\sin(\partial_s)}{\partial_s^3} - \frac{1}{4}\cdot\frac{\sin(3\partial_s)}{\partial_s^3}\Big)\circ\frac{1}{s} \\
&= \frac{3}{4}(-1)^1\cdot\frac{1^2}{2!}\cdot\frac{\pi}{2} - \frac{1}{4}(-1)^1\cdot\frac{3^2}{2!}\cdot\frac{\pi}{2} \\
&= \frac{\pi}{2}\Big(-\frac{3}{8}+\frac{9}{8}\Big) \\
&= \frac{3\pi}{8}.
\end{aligned}$$

Comparing the computations here with those in Example 3 illustrates the advantages of Theorem 2 and its corollaries; these advantages increase with more complicated integrals.

**Example 6:** Verify that $\displaystyle\int_0^\infty \frac{(\sin x - x)^4}{x^7}\,dx = \frac{152}{45}\log 2 - \frac{81}{40}\log 3 - \frac{1}{90}$.

**Solution:** Referring to the trigonometric identities

$$\begin{aligned}
\sin^2 x &= \frac{1}{2}\big(1-\cos(2x)\big); \\
\sin^3 x &= \frac{1}{4}\big(3\sin x - \sin(3x)\big); \\
\sin^4 x &= \frac{1}{4}\big(1 - 2\cos(2x) + \cos^2(2x)\big) = \frac{1}{8}\big(3 - 4\cos(2x) + \cos(4x)\big);
\end{aligned}$$

we find that the integrand can be written as

$$\frac{\frac{3}{8}}{x^7} - \frac{\frac{1}{2}\cos(2x)}{x^7} + \frac{\frac{1}{8}\cos(4x)}{x^7} - \frac{3\sin x}{x^6} + \frac{\sin(3x)}{x^6} + \frac{3}{x^5} - \frac{3\cos(2x)}{x^5} - \frac{4\sin x}{x^4} + \frac{1}{x^3}.$$

Using the LTO method to evaluate the integral, we see that three of the corresponding limits will be zero by Corollary 5. Noting that all of the functions are odd (so that

$f(-\partial_s) = -f(\partial_s)$), Corollary 3 can be used to determine the other six limits:

$$\lim_{s\to0^+} \frac{\cos(2\partial_s)}{2\partial_s^7} \circ \frac{1}{s} = \frac{1}{2}(-1)^3 \frac{2^6}{6!}\big(\log 2 - h_6\big) = -\frac{2^5}{6!}\big(\log 2 - h_6\big) = -\frac{2}{45}\log 2 + \frac{2}{45}h_6;$$
$$\lim_{s\to0^+} \frac{\cos(4\partial_s)}{-8\partial_s^7} \circ \frac{1}{s} = -\frac{1}{8}(-1)^3 \frac{4^6}{6!}\big(\log 4 - h_6\big) = \frac{2^9}{6!}\big(\log 4 - h_6\big) = \frac{64}{45}\log 2 - \frac{32}{45}h_6;$$
$$\lim_{s\to0^+} \frac{3\sin(\partial_s)}{\partial_s^6} \circ \frac{1}{s} = 3(-1)^2 \frac{1^5}{5!}\big(\log 1 - h_5\big) = -\frac{3}{5!}h_5 = -\frac{1}{40}h_5;$$
$$\lim_{s\to0^+} \frac{-\sin(3\partial_s)}{\partial_s^6} \circ \frac{1}{s} = (-1)^3 \frac{3^5}{5!}\big(\log 3 - h_5\big) = -\frac{3^5}{5!}\big(\log 3 - h_5\big) = -\frac{81}{40}\log 3 + \frac{81}{40}h_5;$$
$$\lim_{s\to0^+} \frac{3\cos(2\partial_s)}{\partial_s^5} \circ \frac{1}{s} = 3(-1)^2 \frac{2^4}{4!}\big(\log 2 - h_4\big) = \frac{3\cdot 2^4}{4!}\big(\log 2 - h_4\big) = 2\log 2 - 2h_4;$$
$$\lim_{s\to0^+} \frac{4\sin(\partial_s)}{\partial_s^4} \circ \frac{1}{s} = 4(-1)^1 \frac{1^3}{3!}\big(\log 1 - h_3\big) = \frac{4}{3!}h_3 = \frac{2}{3}h_3.$$

The value of the integral is thus

$$\begin{aligned}\int_0^\infty \frac{(\sin x - x)^4}{x^7}\,dx &= \Big(-\frac{2}{45} + \frac{64}{45} + 2\Big)\log 2 - \frac{81}{40}\log 3 + \Big(-\frac{2}{3}h_6 + 2h_5 - 2h_4 + \frac{2}{3}h_3\Big)\\ &= \frac{152}{45}\log 2 - \frac{81}{40}\log 3 + \frac{2}{5} - \frac{2}{3}\Big(\frac{1}{4} + \frac{1}{5} + \frac{1}{6}\Big)\\ &= \frac{152}{45}\log 2 - \frac{81}{40}\log 3 - \frac{1}{90}.\end{aligned}$$

Some computer algebra systems are unable to evaluate this integral exactly. Requesting an antiderivative for the integrand generates a function with 29 terms involving the sine, cosine, and cosine integral functions. Hence, the LTO method is quite helpful and much simpler for integrals of this type.

**Example 7:** Find the value of $\displaystyle\int_0^\infty \frac{e^{-ax}\sin^3(bx)}{x^3}\,dx$, where $a$ and $b$ are positive constants.

**Solution:** Since (see Example 6 for the appropriate trigonometric identity)

$$\frac{e^{-ax}\sin^3(bx)}{x^3} = \frac{e^{-ax}\big(3\sin(bx) - \sin(3bx)\big)}{4x^3},$$

the LTO method for evaluating the integral yields two limits. For the first limit, noting that $(a+bi)^2 = (a^2-b^2) + 2abi$, we find that

$$\begin{aligned}\frac{3}{4}\lim_{s\to0^+} \frac{e^{a\partial_s}\sin(b\partial_s)}{\partial_s^3} \circ \frac{1}{s} &= \frac{3}{4}\cdot\frac{1}{2!}\Big(2ab\cdot\log\sqrt{a^2+b^2} + (a^2-b^2)\arctan(b/a) - 2abh_2\Big)\\ &= \frac{3ab}{8}\log(a^2+b^2) + \frac{3}{8}(a^2-b^2)\arctan(b/a) - \frac{9ab}{8}.\end{aligned}$$

For the second limit, noting that $(a+3bi)^2 = (a^2-9b^2)+6abi$, we have

$$\begin{aligned}
-\frac{1}{4}\lim_{s\to 0^+} &\frac{e^{a\partial_s}\sin(3b\partial_s)}{\partial_s^3}\circ\frac{1}{s} \\
&= -\frac{1}{4}\cdot\frac{1}{2!}\Big(6ab\cdot\log\sqrt{a^2+9b^2}+(a^2-9b^2)\arctan(3b/a)-6abh_2\Big) \\
&= -\frac{3ab}{8}\log(a^2+9b^2)-\frac{1}{8}(a^2-9b^2)\arctan(3b/a)+\frac{9ab}{8}.
\end{aligned}$$

The value of the integral is thus

$$\frac{3ab}{8}\log\Big(\frac{a^2+b^2}{a^2+9b^2}\Big)+\frac{3}{8}(a^2-b^2)\arctan(b/a)-\frac{1}{8}(a^2-9b^2)\arctan(3b/a).$$

Some computer algebra systems have a great deal of difficulty with this integral, but the LTO method provides a very easy solution.

**Example 8:** In this example, we consider two families of integrals. For each positive integer $n$, let

$$I_n = \int_0^\infty \frac{\sin^{2n+1}x}{x}\,dx \qquad \text{and} \qquad J_n = \int_0^\infty \frac{\sin^{2n+1}x}{x^3}\,dx.$$

Using the trigonometric identity

$$\sin^{2n+1}x = \frac{(-1)^n}{2^{2n}}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}\sin((2n-2k+1)x),$$

along with the value of the Dirichlet integral (namely $I_0 = \pi/2$), it is not difficult to verify directly that

$$I_n = \frac{\pi}{2^{2n+1}}\binom{2n}{n}.$$

Performing integration by parts twice on the second integral (omitting the elementary details), we find that

$$J_n = n(2n+1)I_{n-1}-\frac{1}{2}(2n+1)^2 I_n$$

for each positive integer $n$. A little algebra then yields

$$J_n = \frac{\pi}{2^{2n+2}}\cdot\frac{2n+1}{2n-1}\binom{2n}{n}.$$

We have thus found the values of these two families of integrals without use of the LTO method; the $I_n$ values are Entry **3.821.7** in [**7**] and the $J_n$ values are Entry **3.821.11** in [**7**].

However, using the LTO method for the $J_n$ values, we obtain

$$\begin{aligned}\int_0^\infty \frac{\sin^{2n+1} x}{x^3}\,dx &= \lim_{s\to 0^+} \frac{(-1)^n}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \frac{\sin((2n-2k+1)\partial_s)}{\partial_s^3} \circ \frac{1}{s} \\ &= \frac{(-1)^n}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} (-1)^1 \frac{(2n+1-2k)^2}{2!} \cdot \frac{\pi}{2} \\ &= \frac{\pi}{4} \cdot \frac{(-1)^{n+1}}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} (2n+1-2k)^2\end{aligned}$$

for all $n \geqslant 1$. Given the two forms for the value of $J_n$, it must be true that

$$\sum_{k=0}^n (-1)^k \binom{2n+1}{k} (2n+1-2k)^2 = (-1)^{n+1} \frac{2n+1}{2n-1} \binom{2n}{n}.$$

This identity is indeed valid, but a direct proof using properties of binomial coefficients is nontrivial. Several other intriguing (and more complicated) binomial identities that arise from the use of the LTO method for evaluating integrals can be found in [13].

**Example 9:** Verify that

$$\int_0^\infty \frac{(4n+2)!}{2x^{4n+3}} \Big(\cos x - \sum_{k=0}^n \frac{(-1)^k}{(2k)!} x^{2k}\Big)^2 dx = 2^{4n}\log 2 - 2^{4n} h_{4n+2} + \sum_{k=n+1}^{2n+1} \binom{4n+2}{2k} h_{2k},$$

where $n$ is any nonnegative integer and $h_n$ represents the $n$th harmonic number. Letting $b_k = (-1)^k/(2k)!$ to simplify the notation, we note that

$$\frac{\Big(\cos x - \sum_{k=0}^n b_k x^{2k}\Big)^2}{x^{4n+3}} = \frac{\frac{1}{2}}{x^{4n+3}} + \frac{\frac{1}{2}\cos(2x)}{x^{4n+3}} - \frac{\sum_{k=0}^n (2b_k \cos x) x^{2k}}{x^{4n+3}} + \frac{\Big(\sum_{k=0}^n b_k x^{2k}\Big)^2}{x^{4n+3}}.$$

Using the LTO method, we know that two of these terms will produce limits of zero. For the other two relevant limits, we have (note that the functions are odd in this case)

$$\begin{aligned}\lim_{s\to 0^+} \Big(\frac{\cos(2\partial_s)}{-2\partial_s^{4n+3}} \circ \frac{1}{s}\Big) &= -\frac{1}{2}\,(-1)^{2n+1}\, \frac{2^{4n+2}}{(4n+2)!}(\log 2 - h_{4n+2}) \\ &= \frac{2}{(4n+2)!}\big(2^{4n}\log 2 - 2^{4n} h_{4n+2}\big); \\ \lim_{s\to 0^+} \sum_{k=0}^n \Big(\frac{2b_k \cos(\partial_s)}{\partial_s^{4n-2k+3}} \circ \frac{1}{s}\Big) &= \sum_{k=0}^n \frac{2b_k(-1)^{2n-k+1}}{(4n-2k+2)!}(\log 1 - h_{4n-2k+2}) \\ &= \sum_{k=0}^n \frac{2}{(2k)!(4n-2k+2)!} h_{4n-2k+2} \\ &= \frac{2}{(4n+2)!} \sum_{k=0}^n \binom{4n+2}{2k} h_{4n-2k+2}.\end{aligned}$$

Adding these results gives an equivalent form for the value of the integral. For a more standard approach, one that involves the sine integral and cosine integral functions, to evaluating these integrals, see [**4**]. Once again, seeing another method for evaluating a given integral reveals the ease and efficiency with which the LTO method works.

**Example 10:** Find the value of $\displaystyle\int_0^\infty e^{-2x}\frac{\big(\sin x - x\big)^2}{x^4}\,dx$.

**Solution:** Using the LTO method, we find that

$$\begin{aligned}\int_0^\infty e^{-2x}\frac{\big(\sin x - x\big)^2}{x^4}\,dx &= \int_0^\infty e^{-2x}\Big(\frac{\frac{1}{2}-\frac{1}{2}\cos(2x)-2x\sin x+x^2}{x^4}\Big)\,dx\\ &= \lim_{s\to0^+}\Big(\frac{e^{2\partial_s}}{2\partial_s^4}\circ\frac{1}{s}-\frac{e^{2\partial_s}\cos(2\partial_s)}{2\partial_s^4}\circ\frac{1}{s}-\frac{2e^{2\partial_s}\sin(\partial_s)}{\partial_s^3}\circ\frac{1}{s}+\frac{e^{2\partial_s}}{\partial_s^2}\circ\frac{1}{s}\Big).\end{aligned}$$

We now compute each of the limits separately. For the first limit, we have $n=3$ and $a=2$ in Corollary 4 and this yields

$$\lim_{s\to0^+}\Big(\frac{e^{2\partial_s}}{2\partial_s^4}\circ\frac{1}{s}\Big)=\frac{1}{2}\cdot\frac{2^3}{6}\big(\log 2-h_3\big)=\frac{2}{3}\log 2-\frac{11}{9}.$$

For the cosine result, we note that $n=3$, $a=2$, $b=2$, $c=-16$, and $d=16$ are the appropriate values in Theorem 2. It follows that

$$\lim_{s\to0^+}\Big(-\frac{e^{2\partial_s}\cos(2\partial_s)}{2\partial_s^4}\circ\frac{1}{s}\Big)=-\frac{1}{2}\cdot\frac{1}{6}\Big(-8\log 8-16\cdot\frac{\pi}{4}+16\cdot\frac{11}{6}\Big)=2\log 2+\frac{\pi}{3}-\frac{22}{9}.$$

For the sine result, we note that $n=2$, $a=2$, $b=1$, $c=3$, and $d=4$, thus obtaining

$$\lim_{s\to0^+}\Big(-\frac{2\,e^{2\partial_s}\sin(\partial_s)}{\partial_s^3}\circ\frac{1}{s}\Big)=-2\cdot\frac{1}{2}\Big(2\log 5+3\arctan\tfrac{1}{2}-4\cdot\frac{3}{2}\Big)=-2\log 5-3\arctan\tfrac{1}{2}+6.$$

For the fourth limit, we have $n=1$ and $a=2$ in Corollary 4 giving

$$\lim_{s\to0^+}\frac{e^{2\partial_s}}{\partial_s^2}\circ\frac{1}{s}=2\big(\log 2-h_1\big)=2\log 2-2.$$

Hence, the value of the integral is

$$\frac{2}{3}\log 2-\frac{11}{9}+2\log 2+\frac{\pi}{3}-\frac{22}{9}-2\log 5-3\arctan\tfrac{1}{2}+6+2\log 2-2,$$

which simplifies to

$$\frac{1}{3}+\frac{\pi}{3}+\frac{14}{3}\log 2-2\log 5-3\arctan\tfrac{1}{2}.$$

Some software packages give this value exactly, while others only generate an approximate value.

We next present a list of a variety of integrals for which the LTO method works well; the reader is encouraged to carry out the details for some of these integrals. Since there are other ways to evaluate most of these integrals, it is possible to compare methods and decide which approach provides the clearest and shortest path to the value of the integral.

(1) $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$ (Dirichlet type integral)

(2) $\displaystyle\int_0^\infty \frac{(\sin x - x)^2}{x^5}\,dx = \frac{1}{3}\log 2 - \frac{1}{12}$

(3) $\displaystyle\int_0^\infty \frac{\sin(ax)\sin(bx)\sin(cx)}{(ax)(bx)(cx)}\,dx = \frac{\pi}{2a}$, where $a > b > c > 0$ and $a > b + c$ (Borwein integral, see [**2**])

(4) $\displaystyle\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x}\,dx = \log b - \log a$, where $a, b > 0$ (Frullani integral, see [**1**] and [**8**])

(5) $\displaystyle\int_0^\infty \frac{(\sin x - x)^4}{x^8}\,dx = \frac{47\pi}{7!}$

(6) $\displaystyle\int_0^\infty x^n e^{-x}\sin x\,dx = \frac{n!}{2^{(n+1)/2}}\sin\Big((n+1)\frac{\pi}{4}\Big)$, where $n \geqslant 0$ is an integer (Entry **3.944.5** in [**7**])

(7) $\displaystyle\int_0^\infty \Big(\frac{\sin x}{x}\Big)^n dx = \frac{\pi}{2}\cdot\frac{1}{2^{n-1}(n-1)!}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{k}(n-2k)^{n-1}$ for $n \in \mathbb{Z}^+$ (Entry **3.836.2** in [**7**])

(8) $\displaystyle\int_0^\infty \frac{1}{x^{4n+4}}\Big(\sin x - \sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}x^{2k+1}\Big)^2 dx = \frac{\pi/2}{(4n+3)((2n+1)!)^2}$, where $n$ is a nonnegative integer (see [**12**], [**4**], and [**5**] for three different approaches)

(9) $\displaystyle\int_0^\infty \frac{e^{-ax}\sin^2(bx)}{x^2}\,dx = b\arctan(2b/a) - \frac{a}{4}\log\Big(\frac{a^2+4b^2}{a^2}\Big)$, where $a$ and $b$ are positive constants

For our last example, we tackle an integral whose exact value may never have been computed. The details are tedious but not difficult, revealing the power of the LTO method.

**Example 11:** Find the value of $\displaystyle\int_0^\infty e^{-x}\frac{\big(\sin x - x\big)^4}{x^{11}}\,dx.$

**Solution:** Both Maple and Mathematica generate a very lengthy expression for an antiderivative of this function (involving a few pages of functions), but neither system can find an exact value for this integral. An approximate value is 0.0004093129. Using the LTO method (which, admittedly, is rather tedious in this case, but it does require just nine limits), we find the value of the integral to be

$$-\frac{227}{86400}+\frac{7}{1080}\arctan 1-\frac{44581}{1814400}\arctan 2-\frac{481}{22680}\arctan 3$$
$$+\frac{113221}{3628800}\arctan 4+\frac{37}{945}\ln 2+\frac{109699}{4838400}\ln 5-\frac{14579}{774144}\ln 17.$$

Using trigonometric identities for powers of $\sin x$ (see Example 6), we find that

$$\begin{aligned}\frac{(\sin x-x)^4}{-x^{11}}&=\frac{\sin^4 x-4x\sin^3 x+6x^2\sin^2 x-4x^3\sin x+x^4}{-x^{11}}\\&=-\frac{\frac{3}{8}}{x^{11}}+\frac{\frac{1}{2}\cos(2x)}{x^{11}}-\frac{\frac{1}{8}\cos(4x)}{x^{11}}+\frac{3\sin x}{x^{10}}-\frac{\sin(3x)}{x^{10}}-\frac{3}{x^9}\\&\quad+\frac{3\cos(2x)}{x^9}+\frac{4\sin x}{x^8}-\frac{1}{x^7}.\end{aligned}$$

Using the LTO method, we need to evaluate nine limits, which we do one by one.

For the first limit, we have $a=1$, $n=10$ in Corollary 4:

$$-\frac{3}{8}\lim_{s\to 0^+}\frac{e^{\partial_s}}{\partial_s^{11}}\circ\frac{1}{s}=-\frac{3}{8\cdot 10!}\big(\log 1-h_{10}\big)=\frac{3h_{10}}{8\cdot 10!}.$$

For the second limit, we have $a=1$, $b=2$, $n=10$, $c=237$, $d=-3116$ in Theorem 2:

$$\frac{1}{2}\lim_{s\to 0^+}\frac{e^{\partial_s}\cos(2\partial_s)}{\partial_s^{11}}\circ\frac{1}{s}=\frac{1}{2\cdot 10!}\big(237\log\sqrt{5}+3116\arctan 2-237h_{10}\big).$$

For the third limit, use $a=1$, $b=4$, $n=10$, $c=1093425$, $d=905768$ in Theorem 2:

$$-\frac{1}{8}\lim_{s\to 0^+}\frac{e^{\partial_s}\cos(4\partial_s)}{\partial_s^{11}}\circ\frac{1}{s}=-\frac{1}{8\cdot 10!}\big(1093425\log\sqrt{17}-905768\arctan 4-1093425h_{10}\big).$$

For the fourth limit, we have $a=1$, $b=1$, $n=9$, $c=16$, $d=16$ in Theorem 2:

$$3\lim_{s\to 0^+}\frac{e^{\partial_s}\sin(\partial_s)}{\partial_s^{10}}\circ\frac{1}{s}=\frac{3}{9!}\big(16\log\sqrt{2}+16\arctan 1-16h_9\big).$$

For the fifth limit, use $a=1$, $b=3$, $n=9$, $c=7696$, $d=-30672$ in Theorem 2:

$$-\lim_{s\to 0^+}\frac{e^{\partial_s}\sin(3\partial_s)}{\partial_s^{10}}\circ\frac{1}{s}=-\frac{1}{9!}\big(-30672\log\sqrt{10}+7696\arctan 3+30672h_9\big).$$

For the sixth limit, we have $a=1$, $n=8$ in Corollary 4:

$$-3\lim_{s\to 0^+}\frac{e^{\partial_s}}{\partial_s^{9}}\circ\frac{1}{s}=-\frac{3}{8!}\big(\log 1-h_8\big)=\frac{3h_8}{8!}.$$

For the seventh limit, we have $a=1$, $b=2$, $n=8$, $c=-527$, $d=336$ in Theorem 2:

$$3\lim_{s\to 0^+}\frac{e^{\partial_s}\cos(2\partial_s)}{\partial_s^{9}}\circ\frac{1}{s}=\frac{3}{8!}\big(-527\log\sqrt{5}-336\arctan 2+527h_8\big).$$

For the eighth limit, we have $a = 1$, $b = 1$, $n = 7$, $c = 8$, $d = -8$ in Theorem 2:

$$4 \lim_{s\to 0^+} \frac{e^{\partial_s}\sin(\partial_s)}{\partial_s^8} \circ \frac{1}{s} = \frac{4}{7!}\big(-8\log\sqrt{2} + 8\arctan 1 + 8h_7\big).$$

For the ninth limit, we have $a = 1$, $n = 6$ in Corollary 4:

$$-\lim_{s\to 0^+} \frac{e^{\partial_s}}{\partial_s^7} \circ \frac{1}{s} = -\frac{1}{6!}\big(\log 1 - h_6\big) = \frac{h_6}{6!}.$$

We can then use these values to find the coefficients for each of the terms.

$$\ln 17:\quad -\frac{1093425}{16\cdot 10!} = -\frac{14579}{774144}$$

$$\ln 5:\quad \frac{237}{4\cdot 10!} + \frac{15336}{9!} - \frac{1581}{2\cdot 8!} = \frac{109699}{4838400}$$

$$\ln 2:\quad \frac{24}{9!} + \frac{15336}{9!} - \frac{16}{7!} = \frac{37}{945}$$

$$\arctan 4:\quad \frac{905768}{8\cdot 10!} = \frac{113221}{3628800}$$

$$\arctan 3:\quad -\frac{7696}{9!} = -\frac{481}{22680}$$

$$\arctan 2:\quad \frac{1558}{10!} - \frac{1008}{8!} = -\frac{44581}{1814400}$$

$$\arctan 1:\quad \frac{48}{9!} + \frac{32}{7!} = \frac{7}{1080}$$

$$1:\quad \frac{3h_{10}}{8\cdot 10!} - \frac{237h_{10}}{2\cdot 10!} + \frac{1093425h_{10}}{8\cdot 10!} - \frac{48h_9}{9!} - \frac{30672h_9}{9!} + \frac{3h_8}{8!} + \frac{1581h_8}{8!} + \frac{32h_7}{7!} + \frac{h_6}{6!} = -\frac{227}{86400}$$

Putting these values together yields the exact value of $\displaystyle\int_0^\infty e^{-x}\frac{\big(\sin x - x\big)^4}{x^{11}}\,dx$.

## 5. Conclusion

We hope that this introduction to a recently developed technique for evaluating improper integrals allows readers to recognize the merits of the LTO method. In particular, the limits that appear in Theorem 2 and its corollaries greatly streamline the details that arise when using this method. The number and variety of the examples presented in this paper illustrate a wide range of integrals that can be evaluated with the LTO method. We encourage the reader to apply this method to other appropriate improper integrals as well as to add this option to the tool kit for evaluating integrals of the form $\int_0^\infty f(t)\,dt$.

## References


[1] J. Arias-de-Reyna, On the Theorem of Frullani, *Proc. A.M.S.*, **99** (1990) 165–175.
[2] D. Borwein, J. Borwein, Some Remarkable Properties of Sinc and Related Integrals, *The Ramanujan Journal* **5** (2001) 73–89.
[3] D. M. Bradley, Note on Dirichlet's sinc integral, *Amer. Math. Monthly* **128** (2021) 273–274.
[4] R. A. Gordon, Integrating the tails of two Maclaurin series, *J. Class. Anal.* **18** (2021) 83–95.
[5] R. A. Gordon, Integrating sine and cosine Maclaurin remainders, *Math. Gaz.* **107** (2023) 96–102.
[6] R. A. Gordon, S. M. Stewart, Evaluating improper integrals using Laplace transforms, *Real Analysis Exchange* **48** (2023) 201–221.
[7] I. S. Gradshteyn, I. M. Ryzhik, *Tables of Integrals, Series, and Products*, 8th ed. 2015, Academic Press, New York.
[8] G. J. O. Jameson, *Sine, cosine and exponential integrals*, Math. Gaz. **99** (2015) 276–289.
[9] D. Jia, E. Tang, A. Kempf, Integration by differentiation: new proofs, methods and examples, *J. Phys. A: Math. Theor.* **50** (2017) 235201. see https://arxiv.org/abs/1610.09702
[10] A. Kempf, D. M. Jackson, A. H. Morales, New Dirac delta function based methods with applications to perturbative expansions in quantum field theory, *J. Phys. A: Math. Theor.* **47** (2014) 415204.
[11] A. Kempf, D. M. Jackson, A. H. Morales, How to (path-)integrate by differentiating, *J. Phys. Conf. Ser.* 626 (2015) 012015. see https://arxiv.org/pdf/1507.04348.pdf
[12] S. M. Stewart, Some improper integrals involving the square of the tail of the sine and cosine functions, *J. Class. Anal.* **16** (2020) 91–99.
[13] S. M. Stewart, R. A. Gordon, On some improper integrals involving the cube of the tails of two Maclaurin series, under review at *J. Class. Anal.*


*Received 27 05 2022, revised 23 01 2023*


*Department of Mathematics and Statistics,
Whitman College,
Walla Walla, WA 99362,
USA.
*E-mail address*: `gordon@whitman.edu`

†Physical Science and Engineering Division,
King Abdullah University of Science and Technology (KAUST),
Thuwal 23955-6900,
Saudi Arabia
*E-mail address*: `sean.stewart@physics.org`