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Certain classes of Bilateral Generating Functions involving multiple series with arbitrary terms

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Abstract. In this paper we have established three theorems on bilateral generating relations for functions of several variables. In Theorem 1 the bilateral generating relation for two multiple series with essentially arbitrary terms is obtained while the Theorem 2 is its confluent case. The Theorem 3 aims at deriving a new generalization of the confluent form of the Theorem 2.

As applications of these theorems the bilateral generating functions for multivariable hypergeometric polynomials are obtained. Certain expansion formulae involving generalized Lauricella function of several variables are also derived. Some polynomial expansion for hypergeometric series of one and more variables and certain bilateral generating functions for celebrated hypergeometric polynomials are also mentioned as known special cases of our main findings.

1. Introduction

The Pochhammer symbol is defined by [3, p.16; eqns. (2),(3)]:

$$(\lambda)_n = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{for all } n \in \{1,2,3,\cdots\}. \end{cases} \tag{1.1}$$

so that [3, p.17. eqns. (9),(12)]

$$(\lambda)_{m+n} = (\lambda)_m \ (\lambda+m)_n \ , \tag{1.2}$$

and

$$(-n)_k = \frac{(-1)^k \ n!}{(n-k)!}, \ 0 \leqslant k \leqslant n. \tag{1.3}$$

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The generalized Lauricella function of several variables is defined and represented as follows [3, p.37, eqns.(21)-(23)];

$$F^{A:B';\cdots;B^{(n)}}_{C:D';\cdots,D^{(n)}}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\equiv F^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}\left[\begin{array}{ll} [(a):\theta',\cdots,\theta^{(n)}]:[(b'):\phi'];\cdots;[(b^{(n)}):\phi^{(n)}]; & \\ & x_1,\cdots,x_n \\ [(c):\psi',\cdots,\psi^{(n)}]:[(d'):\delta'];\cdots;[(d^{(n)}):\delta^{(n)}]; & \end{array}\right]$$

$$(1.4) \quad = \sum_{m_1,\cdots,m_n=0}^{\infty} \Omega(m_1,\cdots,m_n)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!},$$

where

$$\Omega(m_1,\cdots,m_n) = \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+\cdots+m_n\theta_j^{(n)}}\prod_{j=1}^{B'}(b'_j)_{m_1\phi'_j}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+\cdots+m_n\psi_j^{(n)}}\prod_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}.$$

The coefficients

$$\theta_j^{(k)},\ j=1,2,\cdots,A;\ \phi_j^{(k)},\ j=1,2,\cdots,B^{(k)};\ \psi_j^{(k)},\ j=1,2,\cdots,C;$$

$$\delta_j^{(k)},\ j=1,2,\cdots,D^k \text{ for all } k\in\{1,\cdots,n\}$$

are real and positive, (a) abbreviates an array of A parameters $a_1,\cdots,a_A$, (b^k) abbreviates the array of $B^{(k)}$ parameters $b_j^{(k)}$, $j=1,2,,\cdots,B^{(k)}$, for all $k\in\{1,2,\cdots,n\}$ with similar interpretations for (c) and $(d^{(k)})$, $k=1,2,\cdots,n$ etc.

The Legendre's duplication formula [3, p.17 eqn.(13)]:

$$(1.5) \qquad (\lambda)_{2n} = 2^{2n}\left(\frac{\lambda}{2}\right)_n\left(\frac{\lambda+1}{2}\right)_n$$

and following series transformations are also required [4, pp.100-102 eqns. (1),(17)]:

$$(1.6) \qquad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,\ n-k)$$

and

$$(1.7) \qquad \sum_{n=0}^{\infty}\sum_{k_1,\cdots,k_r=0}^{M\leqslant n}\phi(k_1,\cdots,k_r;n) = \sum_{n=0}^{\infty}\sum_{k_1,\cdots,k_r=0}^{\infty}\phi(k_1,\cdots,k_r;n+M)$$

where $M=m_1k_1+\cdots+m_rk_r$.

Motivated by the results on bilateral generating functions due to Srivastava and Pathan [6,7] and the polynomial expansion due to Srivastava [2], we have established our main results. Our proposed generalizations include the multivariable hypergeometric polynomials as well as the hypergeometric functions of several variables.

2. Main Theorems

For bounded complex coefficients $\Lambda(k_1,\cdots,k_r)$ and $\Omega_n^{l_2,\ldots,l_s}$ for all $n,k_i\in\{0,1,2\cdots\}$, $l_j\in\{0,1,\cdots\}$, $i=1,\cdots,r$, $j=2,\cdots,s$, and let

$$\phi(t_1^{m_1}x_1,\cdots,t_1^{m_r}x_r;\,t_2,\cdots,t_s) \;=\; \sum_{k_1,\cdots,k_r=0}^{\infty}\;\sum_{l_2,\cdots,l_s=0}^{\infty}\Lambda(k_1,\cdots,k_r)$$

$$\Omega_M^{l_2,\cdots,l_s}(t_1^{m_1}x_1)^{k_1}\cdots(t_1^{m_r}x_r)^{k_r}\frac{t_2^{l_2}}{l_2!}\cdots\frac{t_s^{l_s}}{l_s!};\; M \;=\; \sum_{i=1}^{r}m_ik_i \tag{2.1}$$

the following three theorems are established in this section.

Theorem 1. Let $\phi(t_1^{m_1}x_1,\cdots,t_1^{m_r}x_r;t_2,\cdots,t_s)$ be defined by (**??**), then

$$\phi(t_1^{m_1}x_1,\cdots,t_1^{m_r}x_r;\,t_2,\cdots,t_s) \;=\; \sum_{n=0}^{\infty}\frac{(-t_1)^n}{n!\;(\lambda+n)_n}\sum_{k_1,\cdots,k_r=0}^{M\leqslant n}(-n)_M(\lambda+n)_M$$

$$\times\;\Lambda(k_1,\cdots,k_r)x_1^{k_1}\cdots x_r^{k_r}\sum_{l_1,\cdots,l_s=0}^{\infty}\frac{\Omega_{n+l_1}^{l_2,\cdots,l_s}}{(\lambda+2n+1)_{l_1}}\frac{t_1^{l_1}}{l_1!}\cdots\frac{t_s^{l_s}}{l_s!} \tag{2.2}$$

where $M=\sum\limits_{i=1}^{r}m_ik_i$, provided that for every complex number $\lambda\neq 0,-1,-2,\cdots$ the result in (**??**) exists.

Theorem 2. Let $\phi(t_1^{m_1}x_1,\cdots,t_1^{m_r}x_r;t_2,\cdots,t_s)$ be defined by (**??**), then

$$\phi(t_1^{m_1}x_1,\cdots,t_1^{m_r}x_r;\,t_2,\cdots,t_s) \;=\; \sum_{n=0}^{\infty}\frac{(-t_1)^n}{n!}\sum_{k_1,\cdots,k_r=0}^{M\leqslant n}(-n)_M$$

$$\times\;\Lambda(k_1,\cdots,k_r)x_1^{k_1}\cdots x_r^{k_r}\sum_{l_1,\cdots,l_s=0}^{\infty}\Omega_{n+l_1}^{l_2,\cdots,l_s}\frac{t_1^{l_1}}{l_1!}\cdots\frac{t_s^{l_s}}{l_s!} \tag{2.3}$$

where $M=\sum\limits_{i=1}^{r}m_ik_i$, provided that the result in (**??**) exists.

Theorem 3. Let $\phi(t_1^{m_1}x_1,\cdots,t_1^{m_r}x_r;t_2,\cdots,t_s)$ be defined by (**??**), then for arbitrary α and β, $\beta\neq 0$;

$$\phi(t_1^{m_1}x_1,\cdots,t_1^{m_r}x_r;\,t_2,\cdots,t_s) \;=\; \sum_{n=0}^{\infty}\frac{(-t_1)^n}{n!}\sum_{k_1,\cdots,k_r=0}^{M\leqslant n}\frac{(-n)_M}{(\beta-\alpha n)_M}\Lambda(k_1,\cdots,k_r)$$

$$\times\;x_1^{k_1}\cdots x_r^{k_r}\frac{[(1-\alpha)M+\beta]}{[M-\alpha n+\beta]}\sum_{l_1,\cdots,l_s=0}^{\infty}(\beta-\alpha n)_{n+l_1}\Omega_{n+l_1}^{l_2,\cdots,l_s}\frac{t_1^{l_1}}{l_1!}\cdots\frac{t_s^{l_s}}{l_s!} \tag{2.4}$$

where $M=\sum\limits_{i=1}^{r}m_ik_i$, provided that the result in (**??**) exists.

Outlines of Proofs. To prove the Theorem 1 we denote the R.H.S. of (**??**) by Δ_1, i.e.,

$$\Delta_1 = \sum_{n=0}^{\infty} \frac{(-t_1)^n}{n!\ (\lambda+n)_n} \sum_{k_1,\cdots,k_r=0}^{M\leqslant n} (-n)_M(\lambda+n)_M$$
$$\times\ \Lambda(k_1,\cdots,k_r)x_1^{k_1}\cdots x_r^{k_r} \sum_{l_1,\cdots,l_s=0}^{\infty} \frac{\Omega_{n+l_1}^{l_2,\cdots,l_s}}{(\lambda+2n+1)_{l_1}} \frac{t_1^{l_1}}{l_1!}\cdots\frac{t_s^{l_s}}{l_s!}.$$

Then with the help of (**??**) and (**??**), we have

$$\Delta_1 = \sum_{n,l_1,\cdots,l_s=0}^{\infty} \sum_{k_1,\cdots,k_r=0}^{M\leqslant n} \frac{(-1)^{n+M}(\lambda)_{n+M}(\lambda+1)_{2n}}{(n-M)!\ (\lambda)_{2n}\ (\lambda+1)_{2n+l_1}}$$

$$\times\ \Lambda(k_1,\cdots,k_r)\ \Omega_{n+l_1}^{l_2,\cdots,l_s} x_1^{k_1}\cdots x_r^{k_r}\ \frac{t_1^{l_1+n}}{l_1!}\ \frac{t_2^{l_2}}{l_2!}\cdots\frac{t_s^{l_s}}{l_s!}. \tag{2.5}$$

Now on making use of series rearrangements (**??**) and (**??**) respectively, it reduces to:

$$\Delta_1 = \sum_{l_1,\cdots,l_s=0}^{\infty} \sum_{k_1,\cdots,k_r=0}^{\infty} \Lambda(k_1,\cdots,k_r)\ \Omega_{M+l_1}^{l_2,\cdots,l_s}\ (t_1^{m_1}x_1)^{k_1}\cdots(t_1^{m_r}x_r)^{k_r}$$

$$\times\ \frac{t_2^{l_2}}{l_2!}\cdots\frac{t_s^{l_s}}{l_s!}\sum_{n=0}^{l_1} \frac{(-1)^{l_1}(\lambda)_{n+2M}(\lambda+1)_{2(n+M)}\, t_1^{l_1}}{n!\ (l_1-n)!\ (\lambda)_{2(n+M)}\ (\lambda+1)_{2(n+M)+l_1-n}}. \tag{2.6}$$

On applying (**??**) and (**??**) respectively and then on using the formula (**??**) therein we have

$$\Delta_1 = \sum_{l_1,\cdots,l_s=0}^{\infty} \sum_{k_1,\cdots,k_r=0}^{\infty} \Lambda(k_1,\cdots,k_r)\ \Omega_{M+l_1}^{l_2,\cdots,l_s}\ (t_1^{m_1}x_1)^{k_1}\cdots(t_1^{m_r}x_r)^{k_r}$$

$$\times\ \frac{t_1^{l_1}}{l_1!}\cdots\frac{t_s^{l_s}}{l_s!}\ {}_3\mathrm{F}_2\left[\begin{matrix}\lambda+2M,\ \frac{\lambda}{2}+1+M,\ -l_1 & ; \\ & & 1 \\ \frac{\lambda}{2}+M,\ \lambda+2M+l_1+1 & ;\end{matrix}\right]. \tag{2.7}$$

The above hypergeometric ${}_3\mathrm{F}_2$ series is well poised and therefore by applying Dixon's summation theorem [1, p.243, eq.(III-9)]

$${}_3\mathrm{F}_2\left[\begin{matrix}\lambda+2M,\ \frac{\lambda}{2}+1+M,\ -l_1 & ; \\ & & 1 \\ \frac{\lambda}{2}+M,\ \lambda+2M+l_1+1 & ;\end{matrix}\right] = \begin{cases}1 & ,\ \text{as } l_1=0 \\ 0 & ,\ \text{for all } l_1=1,2,3,\cdots\ .\end{cases}$$

we at once arrive at the desired result in (**??**).

Theorem 2 is a confluent case of Theorem 1 and is obtained on replacing x_i by $\frac{x_i}{\lambda_i}$, for all $i \in \{1, 2, \cdots, r\}$ and t_1 by λt_1 in (**??**) and making use of the principle of confluence [3, p.20, eq. (26)].

$$\lim_{|\lambda|\to\infty} \left\{ (\lambda)^n \left(\frac{z}{\lambda}\right)^n \right\} = \lim_{|\mu|\to\infty} \left\{ \frac{(\mu z)^n}{(\mu)_n} \right\} = z^n$$

for bounded z and $n = 0, 1, 2 \cdots$.

To prove the Theorem 3, we denote the R.H.S of (**??**) by Δ_3, i.e.,

$$\Delta_3 = \sum_{n=0}^{\infty} \frac{(-t_1)^n}{n!} \sum_{k_1,\cdots,k_r=0}^{M\leqslant n} \frac{(-n)_M}{(\beta-\alpha n)_M} \Lambda(k_1,\cdots,k_r) x_1^{k_1} \cdots x_r^{k_r}$$
$$\times \frac{[(1-\alpha)M+\beta]}{[M-\alpha n+\beta]} \sum_{l_1,\cdots,l_s=0}^{\infty} (\beta-\alpha n)_{n+l_1} \Omega_{n+l_1}^{l_2,\cdots,l_s} \frac{t_1^{l_1}}{l_1!} \cdots \frac{t_s^{l_s}}{l_s!}.$$

Then with the help of (**??**) and (**??**), we have

$$\Delta_3 = \sum_{n,l_1,\cdots,l_s=0}^{\infty} \sum_{k_1,\cdots,k_r=0}^{M\leqslant n} \frac{(-1)^{n+M}(\beta-\alpha n)_{n+l_1}[(1-\alpha)M+\beta]}{(n-M)!\,[M-\alpha n+\beta](\beta-\alpha n)_M} \Lambda(k_1,\cdots,k_r)$$
$$\times \Omega_{n+l_1}^{l_2,\cdots,l_s} x_1^{k_1} \cdots x_r^{k_r} \frac{t_1^{l_1+n}}{l_1!} \frac{t_2^{l_2}}{l_2!} \cdots \frac{t_s^{l_s}}{l_s!}. \tag{2.8}$$

Now on making series rearrangements (**??**) and then (**??**) respectively it reduces to

$$\Delta_3 = \sum_{l_1,\cdots,l_s=0}^{\infty} \sum_{k_1,\cdots,k_r=0}^{\infty} \Lambda(k_1,\cdots,k_r)\, \Omega_{M+l_1}^{l_2,\cdots,l_s} x_1^{k_1} \cdots x_r^{k_r} t_1^{l_1+M} \frac{t_2^{l_2}}{l_2!} \cdots \frac{t_s^{l_s}}{l_s!}$$
$$\times \sum_{n=0}^{l_1} \frac{(-1)^n [(1-\alpha)M+\beta][\beta-\alpha(M+n)]_{M+l_1}}{n!\,[M(1-\alpha)-\alpha n+\beta][\beta-\alpha(M+n)]_M (l_1-n)!}. \tag{2.9}$$

On applying (**??**) we have

$$\Delta_3 = \sum_{l_1,\cdots,l_s=0}^{\infty} \sum_{k_1,\cdots,k_r=0}^{\infty} \Lambda(k_1,\cdots,k_r)\, \Omega_{M+l_1}^{l_2,\cdots,l_s} x_1^{k_1} \cdots x_r^{k_r} \frac{t_1^{l_1+M}}{l_1!} \frac{t_2^{l_2}}{l_2!} \cdots \frac{t_s^{l_s}}{l_s!}$$
$$\times \sum_{n=0}^{l_1} \frac{(-1)^n \binom{l_1}{n} [(1-\alpha)M+\beta][\beta-\alpha n+M(1-\alpha)]_{l_1}}{[M(1-\alpha)+\beta-\alpha n]}. \tag{2.10}$$

Now in (**??**) on using the following generalization of an elementary combinatorial identity (corresponding to the special case $\alpha = 0$)

$$\sum_{n=0}^{l_1} (-1)^n \binom{l_1}{n} \frac{\zeta(\zeta-\alpha n)_{l_1}}{(\zeta-\alpha n)} = \begin{cases} 1 & , \text{ as } l_1 = 0 \\ 0 & , \text{ for all } l_1 = 1, 2, \cdots \end{cases}$$

at $\zeta = (1-\alpha)M + \beta$, $\beta \neq 0$; we atonce arrive at the desired result in (**??**).

3. Applications

(i) If in (**??**) and (**??**) we take $r = 1$ and set

$$\Lambda(k_1, 0, \cdots, 0) \to \frac{c_k}{k!},\ \Omega_{n+l_1}^{l_2,\cdots,l_s} = A_{n+l_1+l_2+\cdots+l_s} B_{n+l_1} \Delta(l_2, \cdots, l_s); \tag{3.1}$$

$\{A_n\}$, $\{B_n\}$, $\{C_n\}$ are arbitrary sequences, then on replacing x_i by ω these reduce to the known results [6, pp.26-27, eq.(3.1); p.29, eq.(3.9)] respectively, which in turn at $s = 2$ provides the known results [6, p.25, eq.(2.1);p.26,eq.(2.3)].

(ii) On similar setting at $r = 1$, as in (**??**) the result in (**??**) reduces to the known results [7, p.110, eq.(2.2); pp.108-109, eq.(1.18)].

(iii) If in (**??**), (**??**) and (**??**) we take $s = 1$ i.e. $l_2 = \cdots = l_s = 0$ and set $\Omega_{n+l_1}^{0,\cdots,0} \to \Omega_{n+l_1}$ these reduce to known results [3; pp.339-340, eqs.(251)-(253)] respectively.

(iv) If in (**??**) we take $\alpha = 0$ and set as in (**??**) it reduces to the known result [6, p.29, eq.(3.9)].

(v) If in (**??**), (**??**) and (**??**) we take $s = 1$, i.e., $t_2, t_3, \cdots, t_s \to 0$, $t_1 \to \omega$ and set

$$\Lambda(k_1, \cdots, k_r) = \frac{\prod_{j=1}^{A} (a_j)_{k_1\theta'_j+\cdots+k_r\theta_j^{(r)}} \prod_{j=1}^{V} (v_j)_M \prod_{j=1}^{B'} (b'_j)_{k_1\phi'_j} \cdots \prod_{j=1}^{B^{(r)}} (b_j^{(r)})_{k_r\phi_j^{(r)}}}{\prod_{j=1}^{C} (c_j)_{k_1\psi'_j+\cdots+k_r\psi_j^{(r)}} \prod_{j=1}^{U} (u_j)_M \prod_{j=1}^{D'} (d'_j)_{k_1\delta'_j} \cdots \prod_{j=1}^{D^{(r)}} (d_j^{(r)})_{k_r\delta_j^{(r)}}},$$

$$\Omega_M^{0,\cdots,0} = \frac{\prod_{j=1}^{E} (e_j)_M \prod_{j=1}^{U} (u_j)_M}{\prod_{j=1}^{G} (g_j)_M \prod_{j=1}^{V} (v_j)_M}$$

then we obtain the known expansion formulae [3, pp.341-342, eqs.(254)-(256)].

(vi) If in (**??**) we set

$$r = 1,\ x_1 \to \omega\ \Lambda(k_1, 0, \cdots, 0) = \frac{\beta_{k_1}}{k_1!} \text{ and } s = 1,\ t_1 \to x,\ \Omega_{m_1k_1}^{0,\cdots,0} = A_{m_1k_1} \tag{3.2}$$

then it reduces to the known result [6, p.26, eq.(2.3)], which in turn at $m_1 = 1$ provides another known result [8, p.348, §4, eq.(1)].

(vii) On similar setting as in (**??**) the result in (**??**) reduces to the known result [7, p.111 eq.(3.1)] which in turn at $m_1 = 1$ provides another known result [9, p.75, eq.(3.1)] {see also [5, p.20, eq.(9)]}.

(viii) If in (**??**) we take $r = 1$, $x_1 \to \omega$, $m_1 = 1$, $\Lambda(k_1, 0, \cdots, 0) = \dfrac{\prod_{j=1}^{l}(u_j)_{k_1}}{\prod_{j=1}^{m}(v_j)_{k_1}\ k_1!}$

and

$$\Omega_{k_1}^{l_2,\cdots,l_s} = \frac{\prod_{j=1}^{p}(\alpha_j)_{k_1+l_2+\cdots+l_s}\prod_{j=1}^{q_1}(\beta'_j)_{k_1}\prod_{j=1}^{q_2}(\beta''_j)_{l_2}\cdots\prod_{j=1}^{q_s}(\beta_j^{(s)})_{l_s}}{\prod_{j=1}^{p}(\gamma_j)_{k_1+l_2+\cdots+l_s}\prod_{j=1}^{Q_1}(\delta'_j)_{k_1}\prod_{j=1}^{Q_2}(\delta''_j)_{l_2}\cdots\prod_{j=1}^{Q_s}(\delta_j^{(s)})_{l_s}} \tag{3.3}$$

$t_i \to n_i$, for all $i \in 1, \cdots, s$, it reduces to the known result [6, p.28, eq.(3.6)].

(ix) On similar setting as in (**??**) the result in (**??**) reduces to the known result [7, pp.112-113, eq.(3.7)].

(x) If in (**??**) we take

$$\Lambda(k_1, \cdots, k_r) = \frac{\prod_{j=1}^{q_1}(b'_j)_M\prod_{j=1}^{D'}(d'_j)_{k_1\phi'_j}\cdots\prod_{j=1}^{D^{(r)}}(d_j^{(r)})_{k_r\phi_j^{(r)}}}{\prod_{j=1}^{p}{}_1(a'_j)_M\prod_{j=1}^{E'}(e'_j)_{k_1\delta'_j}\cdots\prod_{j=1}^{E^{(r)}}(e_j^{(r)})_{k_r\delta_j^{(r)}}\left[\prod_{i=1}^{r}k_i!\right]}$$

and

$$\Omega_M^{l_2,\cdots,l_s} = \frac{\prod_{j=1}^{L}(a_j)_{M+l_2+\cdots+l_s}\prod_{j=1}^{p_1}(a'_j)_M\prod_{j=1}^{p_2}(a''_j)_{l_2}\cdots\prod_{j=1}^{p_s}(a_j^{(s)})_{l_s}}{\prod_{j=1}^{l}(b_j)_{M+l_2+\cdots+l_s}\prod_{j=1}^{q_1}(b'_j)_M\prod_{j=1}^{q_2}(b''_j)_{l_2}\cdots\prod_{j=1}^{q_s}(b_j^{(s)})_{l_s}}$$

then we get the following result:

$$\mathrm{F}^{L:D';\cdots;D^{(r)};p_2;\cdots;p_r}_{l:E';\cdots;E^{(r)};q_2;\cdots;q_s}\left[\begin{array}{l}[(a);\,m_1,\cdots,m_r,\,1,\cdots,1];\,[(d'),\phi'];\cdots;[(d^{(r)}),\phi^{(r)}]\ ;\\ [(b);\,m_1,\cdots,m_r,\,1,\cdots,1];\,[(e'),\delta'];\cdots;[(e^{(r)}),\delta^{(r)}]\ ;\end{array}\right.$$
$$\left.\begin{array}{l}[(a''),1];\cdots;[(a^{(s)}),1]\ ;\\ [(b''),1];\cdots;[(b^{(s)}),1]\ ;\end{array} t_1^{m_1}x_1,\cdots,t_1^{m_r}x_r,\,t_2,\cdots,t_s\right]$$

$$= \sum_{n=0}^{\infty}\frac{(-t_1)^n\prod_{j=1}^{L}(a_j)_n\prod_{j=1}^{p_1}(a'_j)_n}{n!\ (\lambda+n)_n\prod_{j=1}^{l}(b_j)_n\prod_{j=1}^{q_1}(b'_j)_n}$$

$$\times\,\mathrm{F}^{q_1+2:D';\cdots;D^{(r)}}_{p_1:E';\cdots;E^{(r)}}\left[\begin{array}{ll}[(b');\,m_1,\cdots,m_r],(-n;m_1,\cdots,m_r),(\lambda+n;m_1,\cdots,m_r) & :\\ [(a');\,m_1,\cdots,m_r] & :\end{array}\right.$$

$$\left.\begin{array}{ll} [(d'),\phi'];\cdots;[(d^{(r)}),\phi^{(r)}] & ; \\ & x_1,\cdots,x_r \\ [(e'),\delta'];\cdots;[(e^{(r)}),\delta^{(r)}] & ; \end{array}\right]$$

(3.4)

$$\mathrm{F}^{L;p_1;p_2;\cdots;p_s}_{l:q_1+1;q_2;\cdots;q_s}\left[\begin{array}{llll} [(a)+n]: & [(a')+n] & :(a'');\cdots;(a^{(s)}) & ; \\ & & & t_1,\cdots,t_s \\ [(b)+n]: & [(b')+n],(\lambda+2n+1) & :(b'');\cdots;(b^{(s)}) & ; \end{array}\right]$$

where $F[t_1,\cdots,t_s]$ is the generalized Kampé-de-Fériet function of r-variables [3, p.38].

(xi)If in (**??**) we take

$$\Lambda(k_1,\cdots,k_r) = \frac{\prod_{i=1}^{r}(\beta_i)_{k_i}}{(\gamma)_M\; k_1!,\cdots k_r!}$$

and

$$\Omega_M^{l_2,\cdots,l_s} = \frac{(\gamma)_M \prod_{j=1}^{L}(a'_j)_{M+l_2+\cdots+l_s}\prod_{j=1}^{p_2}(a''_j)_{l_2}\cdots\prod_{j=1}^{p_s}(a^{(s)}_j)_{l_s}}{\prod_{j=1}^{l}(b'_j)_{M+l_2+\cdots+l_s}\prod_{j=1}^{q_2}(b''_j)_{l_2}\cdots\prod_{j=1}^{q_s}(b^{(s)}_j)_{l_s}} \tag{3.5}$$

then we get the following result:

$$\mathrm{F}^{L:1;\cdots;1;p_2;\cdots;p_s}_{l:0;\cdots;0;q_2;\cdots;q_s}\left[\begin{array}{ll} [(a_L);\,m_1,\cdots,m_r,\,1,\cdots,1];\,(\beta_1,1);\cdots;(\beta_r,1) & ; \\ [(b_l);\,m_1,\cdots,m_r,\,1,\cdots,1];\,\_;\cdots;\_ & ; \end{array}\right.$$

$$\left.\begin{array}{ll} [(a''),1];\cdots;[(a^{(s)}),1] & ; \\ & t_1^{m_1}x_1,\cdots,t_1^{m_r}x_r,\,t_2,\cdots,t_s \\ [(b''),1];\cdots;[(b^{(s)}),1] & ; \end{array}\right]$$

$$= \sum_{n=0}^{\infty}\frac{(-t_1)^n\prod_{j=1}^{L}(a_j)_n(\gamma)_n}{n!\prod_{j=1}^{l}(b_j)_n}F_D^{(r)}[(-n,m_i);\,(\beta_i,1);\,(\gamma;m_i);\,x_1,\cdots,x_r]$$

$$\times\,\mathrm{F}^{L:1;p_2;\cdots;p_s}_{l:0;q_1;\cdots;q_s}\left[\begin{array}{lll} (a)+n;\,(\gamma+n) & ;(a'');\cdots;(a^{(s)}) & ; \\ & & t_1,\cdots,t_s \\ (b)+n; & ;(b'');\cdots;(b^{(s)}) & ; \end{array}\right] \tag{3.6}$$

where $F_D^{(r)}[\cdot]$ is the Lauricella polynomial [4, pp.462-463, eq.(4)] and $F[t_1,\cdots,t_s]$ is the Kampé-de-Fériet function [3, p.38, eq.(24)].

(xii) If in (**??**) we take $\alpha=0$ and set as in (**??**) then it also reduces to the result (**??**) which in turn provides at $L=l=1$, $p_i=1$, $q_i=0$, for all $i\in\{2,\cdots,s\}$ and in view of [4, p.462, eqns.(1),(2),(4)] we get the following results:

$F_D^{r+s-1}[(a_1;m_1,\cdots,m_r,1,\cdots,1);(\beta_1,1),\cdots,(\beta_r,1),(a'',1),\cdots,(a^{(r)},1);$

$$(b_1; m_1, \cdots, m_r, 1, \cdots, 1);\, t_1^{m_1} x_1, \cdots, t_1^{m_r} x_r, t_2, \cdots, t_s] = \sum_{n=0}^{\infty} \frac{(-t_1)^n\, (a_1)_n\, (\gamma)_n}{n!\, (b_1)_n}$$

$$\times F_D^{(r)}[(-n, m_i);\, (\beta_i, 1); (\gamma, m_i); x_1, \cdots, x_r]$$

$$\times F_D^{(s)}[a_1 + n; \gamma + n,\, a'', \cdots, a^{(s)};\, b_1 + n;\, t_1, \cdots, t_s]. \tag{3.7}$$

where $F_D^{(r+s-1)}[\cdot]$ is the generalized Lauricella function [4,p.462] and $F_D^{(s)}[\cdot]$ is the Lauricella function [4, p.462-463, eq.4].

Due to general nature of sequences considered in the main results many new and known results involving generalized Lauriclla function as well as Kampé-de-Fériet function of several variables may obtained but due to lack of space these are not recorded here.

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