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Upper Basic and Congruent Submodules of $QTAG$-Modules

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Abstract. The cardinality of the minimal generating set of a module M i.e $g(M)$ plays a very important role in the study of $QTAG$-Modules. Fuchs [1] mentioned the importance of upper and lower basic subgroups of primary groups. A need was felt to generalize these concepts for modules. An upper basic submodule B of a $QTAG$-Module M reveals much more information about the structure of M. We find that each basic submodule of M is contained in an upper basic submodule and contains a lower basic submodule.
Two submodules $N, K \subset M$ are congruent if there exists an automorphism of M which maps N onto K. In this case $M/N \cong M/K$ and $N \cong K$, but these conditions are not sufficient for the congruence. This motivates us to find the sufficiency conditions in terms of Ulm invariants and the extensions of height preserving isomorphism of submodules.

1. Introduction

A module M is a $QTAG$-Module if every submodule of a finitely generated homomorphic image of M is a direct sum of uniserial modules. All the rings considered here are associative with unity and the modules are unital $QTAG$-Module. For a uniform element $x \in M$, height of x in M i.e $H_M(x)$ or $H(x) = Sup\{d(U/xR)\}$ where U runs through all the uniserial modules containing x. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H_\omega(M) = \bigcap_{k=0}^{\infty} H_k(M)$.
A submodule $N \subset M$ is h-pure if $H_k(N) = N \cap H_k(M)$, $k = 0,1,...,\infty$ and N is isotype if for every ordinal σ ,$H_\sigma(N) = N \cap H_\sigma(M)$ where $H_\sigma(N) = \bigcap_{\rho<\sigma} H_\rho(N)$.
A module M is h-divisible if $H_1(M) = M$ and a submodule B of M is basic in M if B is a direct sum of uniserial modules, B is h-pure in M and M/B is h-divisible.
For M , $g(M)$ denotes the cardinality of the minimal generating subset of M, $fing(M) = Inf\ g(H_k(M))$ and σth Ulm invariant of M , $f_M(\sigma)$ is the cardinal

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number $g(Soc(H_\sigma(M))/Soc(H_{\sigma+1}(M)))$. M is σ-projective if $H_\sigma(Ext(M,C)) = 0$ for all $QTAG$-modules C.
M is totally projective if $M/H_\sigma(M)$ is σ-projective, for every ordinal σ.
In other words $H_\sigma(Ext(M/H_\sigma(M),C)) = 0$ for every module C and ordinal σ.

A submodule N of M is said to be balanced if it is an isotype and $H_\alpha(M/N) = (H_\alpha(M)+N)/N$ for all ordinals α and it is almost balanced if $H_\alpha(M)\cap N = H_\alpha(N)$ and $H_\alpha(M/N) = (H_\alpha(M)+N)/N$ for all ordinals $\alpha < \beta$ where β is the length of M.

2. Upper and Lower Basic Submodules

To generalize some results of [1] and [2] we proceed as follows :

Definition 2.1. A basic submodule B^u of a $QTAG$-module M is said to be an upper basic subodule if $g(M/B^u) = \min\{g(M/B) \mid B$ is a basic submodule of $M\}$.

Remark 2.2. If B and B' are upper basic submodules of $QTAG$-modules M and M' respectively then $B+B'$ is an upper basic submodule of $M+M'$. Furthermore if B is an upper basic submodule of a high submodule of M then B is also an upper basic submodule of M.

Definition 2.3. A basic submodule B^l is said to be a lower basic submodule of M if

$$g(M/B^l) \;=\; \operatorname{fin} g(M) \;=\; \min\{H_k(M)\}.$$

The existence of a lower basic submodule can be established by the following result:

Theorem 2.4. Every basic submodule B of a $QTAG$-module M contains a lower basic submodule.

Proof. If M is a $QTAG$-module such that $\operatorname{fin} g(M)$ is finite, then $\operatorname{fin} g(B) \leqslant \operatorname{fin} g(M)$, for every basic submodule B of M. Since B is a direct sum of uniserial modules $\operatorname{fin} g(B) = 0$,

$\Rightarrow$ B is bounded,

$\Rightarrow$ every basic submodule of M is bounded,

$\Rightarrow$ every basic submodule is a lower basic submodule.

If $\operatorname{fin} g(M)$ is not finite and the basic submodule B of M is not lower then $g(M/B) < \operatorname{fin} g(M)$ and for every integer k

$$M/B \;=\; (B+H_k(M))/B \;\cong\; H_k(M)/(B\cap H_k(M)) \;=\; H_k(M)/H_k(B)$$

$$\Rightarrow\; g\Big(\frac{H_k(M)}{H_k(B)}\Big) < \operatorname{fin} g(M) \;\leqslant\; g(H_k(M)),$$

$$\Rightarrow\; g(H_k(B)) \;=\; g(H_k(M)),$$

then the cardinality of the minimal generating set of uniserial direct summands of length $> k$ in B is at least $\operatorname{fin} g(M)$ for all k.

Now B may be expressed as the direct sum of uniserial modules of unbounded length i.e.

$$B \;=\; \bigoplus_{j\in J} C_j \text{ where } \#(J) \;=\; \operatorname{fin} g(M)$$

By Theorem 3. of [3] each C_j contains a basic submodule B'_j. Now B', the direct sum of these B_j's is a basic submodule of B, therefore it is a basic submodule of M. By construction

$$g\Big(\frac{M}{B'}\Big) \;\geqslant\; g\Big(\frac{B}{B'}\Big) \;=\; \sum_{j\in J} g\Big(\frac{C_j}{B_j}\Big) \;\geqslant\; \operatorname{fin} g(M)$$

then B' is a lower basic submodule of M.

The upper basic submodule of $QTAG$-module M tells much more about its structure than an ordinary basic submodule. For a basic submodule B of M, M/B is not an invariant. We investigate as follows:

Theorem 2.5. Let M be a $QTAG$-module and $M = M^0 + U$ where U is a direct sum of uniserial modules. For a basic submodule B^0 of M^0, suppose $B = B^0 + A$ and $B' = B^0 + A'$ be two basic submodules of M. Then there exists an automorphism f of M which is identity on M^0 and maps B onto B' if and only if

$$M/(M^0 + B) \;=\; M/(M^0 + B')$$

Proof. Since U is a direct sum of uniserial modules both A and A' are isomorphic to basic submodules of U and $A \cong U \cong A'$.

Let $U = \sum x_i R$, $A = \sum y_i R$, $A' = \sum z_i R$.

Since M/B is h-divisible, $M/(M^0 + B)$ is h-divisible, we may write $M/(M^0 + B) = \sum_{j\in J} D_j$ and $M/(M^0 + B') = \sum_{j\in J} D'_j$ where D_j's and D'_j 's are h-divisible.

Since $M^0 + B$ and $M^0 + B'$ are pure submodules of M we set,

$$I \;=\; \{\alpha \mid \alpha \text{ is an ordinal of cardinality less then } g(U)\},$$

$$J \;=\; \{\alpha \mid \alpha \text{ is an ordinal of cardinality less then } g(M/(M^0 + B))\}.$$

If J is empty then $M^0 + A = M = M^0 + A'$ and the required automorphism of M may be obtained by mapping A isomorphically onto A'. On comparing $U \cong M/M^0$ and $M/(M^0 + B)$, we may say that $J \subseteq I$. Since a bounded module cannot have a proper basic submodule, the finiteness of I implies that $M^0 + A = M^0 + A'$, therefore we assume that I is infinite.

Let f_0 denote the identity map of M^0. Suppose that for every $\gamma \in I$ and $\alpha < \gamma$ there exists a submodule M_α of M and a height preserving automorphism f_α of M_α

such that $M_\alpha \subset M_\beta$ and $f_\beta|M_\alpha = f_\alpha$ whenever $\alpha < \beta < \gamma$ and $f_\alpha(M_\alpha \cap B) = M_\alpha \cap B'$. With these assumptions the following conditions would be satisfied for suitable subsets $I_\alpha \subset I$ and $J_\alpha \subset J$. Here M_α etc. are not the α^{th} Ulm factors of M.

$$M_\alpha = M^0 + \sum_{I_\alpha} x_i R\ ,\ M_\alpha \cap B = B^0 + \sum_{I_\alpha} y_i R\ ,\ M_\alpha \cap B' = B^0 + \sum_{I_\alpha} z_i R,$$

$$(M_\alpha + B)/(M^0 + B) = \sum_{J_\alpha} D_j \text{ and } (M_\alpha + B')/(M^0 + B') = \sum_{J_\alpha} D'_j.$$

For any limit ordinal γ, $M_\gamma = \bigcup_{\alpha<\gamma} M_\alpha$ and f_γ denotes the automorphism of M_γ with $f_\gamma|M_\alpha = f_\alpha$ for all $\alpha < \gamma$.

Again $I_\gamma = \bigcup_{\alpha<\gamma} I_\alpha$ and $J_\gamma = \bigcup_{\alpha<\gamma} J_\alpha$.

If $\gamma - 1$ exists then put $\beta = \gamma - 1$ and consider the extensions K and L of M_β such that $g(K/M_\beta)$ and $g(L/M_\beta)$ are finite and $\psi : K \to L$ is a height preserving isomorphism such that $\psi(K \cap B) = L \cap B'$.

We have to extend ψ to a height preserving automorphism f_γ of a submodule M_γ containing K and L such that the above conditions hold for $\alpha \leqslant \gamma$ and $f_\gamma(M_\gamma \cap B) = M_\gamma \cap B'$. Moreover if $\alpha + 1 < \gamma$ then $\alpha \in I_{\alpha+1}$. Consider an element $x \notin K$. Now we may extend ψ to a height preserving isomorphism ψ' from $K' = K + xR$ into M such that $\psi'(K' \cap B) = \psi'(K') \cap B'$.

We may consider an element $x \notin K$ for which $\exists\ y \in K$ such that $d\Big(\dfrac{xR}{yR}\Big) = 1$ and x is a proper element with respect to K. To verify this suppose $H(x) = n$. If $H(y+k) > n+1$ for some $k \in K \cap H_n(M)$, we assume that $H(y) \geqslant n+1$. If $H(y) = n+1$ and $x + k \in B$ for some $k \in K$ we may choose $u \in H_n(M)$ such that $\psi(y) = v$ where $d\Big(\dfrac{uR}{vR}\Big) = 1$. If $u + \psi(k) \in B'$ we may set $\psi'(x) = u$, otherwise consider u', x' such that $d\Big(\dfrac{(u + \psi(k))R}{u'R}\Big) = 1$ and $d\Big(\dfrac{(x + k)R}{x'R}\Big) = 1$. Now $u' = \psi(x') \in H_1(B')$. Let $u + \psi(k) = b' + a'$ where $b' \in B'$ and $e(a') = 1$. Since B' is a basic submodule of M, we may express $a' = c' + w$ where $c' \in \mathrm{Soc}(B')$ and $w \in \mathrm{Soc}(H_{n+1}(M))$. Now $u - w + \psi(k) \in B'$. We extend ψ by mapping x onto $u - w$.

If $x + k \notin B$ for any $k \in K$ we choose $u \in H_n(M)$ such that $\psi(y) = v$ where $d\Big(\dfrac{uR}{vR}\Big) = 1$. If $u+\psi(k) \notin B'$ for all $k \in K$ we put $\psi'(x) = u$, othewrwise $u+\psi(k) \in B'$ for some $k \in K$. Consider x' such that $d\Big(\dfrac{(x + k)R}{x'R}\Big) = 1$. Now $\psi(x') = u' \in H_1(B')$, where $d\Big(\dfrac{(u + \psi(k))R}{u'R}\Big) = 1$. Hence $x' \in H_1(B)$.

We may express x' as $b + a$ where $b \in B$ and $e(a) = 1$. Now $a \notin B + K$ otherwise $x + k \in B$. Now we show that $\mathrm{Soc}(M)$ is not contained in $B' + L$. If $K = M_\beta = L$ then by induction hypothesis $M/(B + M_\beta) \cong M/(B' + M_\beta)$. Since $B + M_\beta$ and $B' + M_\beta$

are h-pure submodules of M, it follows that

$$(\mathrm{Soc}(M)+B+M_\beta)/(B+M_\beta) \;\cong\; (\mathrm{Soc}(M)+B'+M_\beta)/(B'+M_\beta)$$

and the isomorphism $\psi : K \to L$ induces an isomorphism

$$(B+K)/(B+M_\beta) \to (B'+L)/(B'+M_\beta).$$

Thus

$$(\mathrm{Soc}(M)+B+M_\beta)/(B+M_\beta+\mathrm{Soc}(B+K))$$
$$\cong (\mathrm{Soc}(M)+B'+M_\beta)/(B'+M_\beta+\mathrm{Soc}(B'+L)).$$

Since the left hand side is nonzero

$$(\mathrm{Soc}(M)+B'+M_\beta) \;\neq\; (\mathrm{Soc}(B'+L)+B'+M_\beta),$$

$\Rightarrow\ \exists\ z \in \mathrm{Soc}(M)$ such that $z \notin B'+L$. We may select $b_1 \in \mathrm{Soc}(B')$ so that $z-b_1 \in H_{n+1}(M)$. We now define $\psi'(x) = u+z-b_1 \notin B'+L$. Thus $u+z-b_1+\psi(k) \notin B'$ for every $k \in K$ and ψ' is the extension of ψ.

If $H(y) > n+1$ and $x+k \in B$ for some $k \in K$. We select $w \in H_{n+1}(M)$ such that $w' = \psi(y)$ where $d\Big(\dfrac{wR}{w'R}\Big) = 1$. This implies that $\exists\, z \in \mathrm{Soc}(M)$ which is proper with respect to L and $H(z) = n$. Put $u = w+z$. If $u+\psi(k) \in B'$ we may define $\psi'(x) = u$. If $u+\psi(k) \notin B'$ then $\psi'(x)$ is defined as in the previous case.

On the other hand if $x+k \notin B$ for all $k \in K$, then we define $u = w+z$ as in the preceeding case. If $u+\psi(k) \notin B'$ for any $k \in K$, we put $\psi'(x) = u$. If $u+\psi(k) \in B$ for some k, $\psi(x') \in H_1(B')$ where $d\Big(\dfrac{(x+k)R}{x'R}\Big) = 1$. Therefore $x' \in H_1(B)$. Now $x+k = b+c$ where $b \in B$, $e(c) = 1$. Since $c \notin K+B$, $\exists\ d \in \mathrm{Soc}(M)$ such that $d \notin L+B'$. We put $\psi'(x) = u+d-b'$, where $b' \in \mathrm{Soc}(B')$ and $d-b' \in \mathrm{Soc}(H_{n+1}(M))$.

In all the above four cases ψ' is an extension of ψ to $K+xR$ which is a height preserving isomorphism from $K+xR$ into M such that $\psi'((K+xR)\cap B) = \psi'(K+xR)\cap B'$.

Since B is a basic submodule, $g(B)$ is not uncountable. Moreover $I_{(\alpha)}$'s and $J_{(\alpha)}$'s are also finite or countable.

We may consider the ascending sequences

$$M_\beta \;=\; K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n \subseteq \cdots$$
$$M_\beta \;=\; L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n \subseteq \cdots$$

such that $g\Big(\dfrac{K_i}{M_\beta}\Big)$ and $g\Big(\dfrac{L_i}{M_\beta}\Big)$ are finite for all i, $\cup K_n = \cup L_n$ and there exists a sequence $\psi_n : K_n \to L_n$ of height preserving isomorphisms such that ψ_n is the extension of ψ_{n-1} and $\psi_n(K_n \cap B) = L_n \cap B'$. If we put $M_\gamma = UK_n = UL_n$, then we may choose the modules K_n and L_n so that the previously mentioned five conditions hold for $\alpha = \gamma$. Here I_γ and J_γ are the countable extensions of I_β and J_β

respectively. Now $\beta \in J_\gamma$. Let f_γ be the maximum extension of ψ_n i.e. $f_\gamma = \sup\{\psi_n\}$, $f_\gamma(M_\gamma \cap B) = M_\gamma \cap B'$ and we obtain an automorphism of M that maps B onto B'.

Corollary 2.6. Let M be a $QTAG$-module such that $M = M^0 + U$ where U is a direct sum of uniserial modules. Suppose B is a basic submodule of M such that $B \cap M^0$ is a basic submodule of M^0. Then $M = M^0 + K$ such that $B = (B \cap M^0) + (B \cap K)$.

Proof. Let $B^0 = B \cap M^0$. Now B/B^0 is isomorphic to a submodule of U therefore B/B^0 is also a direct sum of uniserial modules. Since B^0 is h-pure $B = B^0 + A$ for some submodule A of B. Again $B/B^0 \subseteq M/B^0 \cong (M^0/B^0) + U$, i.e. B/B^0 is isomorphic to a basic submodule U^0 of U such that $M/(B^0 + U^0) \cong M/B$. By Theorem 2.5 some automorphism f of M maps $B^0 + U^0$ onto $B = B^0 + A$ and f is idnetity on M^0. If $K = f(U)$ then we have $M = M^0 + K$. If $N = f(U^0)$ we have $B = B^0 + N$. Since $N \subseteq K$ the result follows.

Theorem 2.7. If M is a $QTAG$-module and B a basic submodule of M, then $M = M^0 + K$ and $B = (B \cap M^0) + (B \cap K)$ where $B \cap M^0$ is an upper basic submodule of M^0 and K is a direct sum of uniserial modules.

Proof. Let $g(B/B^u) = m$, where B^u is an upper basic submodule of M and B is an arbitrary basic submodule of M. Now $M = M^0 + U$ where U is a direct sum of uniserial modules, $g(M^0) \leqslant \max(\aleph_0, m)$ and $B \cap M^0$ is a basic submodule of M^0. By Corollary 2.6 there exists a decomposition $M = M^0 + K$ such that $B = (B \cap M^0) + (B \cap K)$. If $g(M^0) = m$, then each basic submodule of M^0, (hence $B \cap M^0$) is an upper basic submodule because $g(M/B^u) = m$ and K is a direct sum of uniserial modules. If m is finite $g(M^0)$ is countable. Suppose $g(M^0/(B \cap M^0)) = n \geqslant m$. Then $M^0 = N^0 + L$, where $B \cap M^0 = (B \cap N^0) + (B \cap L)$, L is a direct sum of uniserial modules and $g(N^0/(B \cap N^0)) = m$. We may express $M = N^0 + (L + K)$. Then $B \cap N^0$ is an upper basic submodule of N^0. Since $B = (B \cap N^0) + (B \cap (L + K))$, the result follows.

From the above discussion and results we immediately conclude that each basic submodule of a $QTAG$-module M is contained in an upper basic submodule of M.

3. Congruence submodules of totally projective $QTAG$-Modules

Two submodules $N, K \subset M$ are congruent modulo M if there exists an automorphism of M which maps N onto K i.e $N \cong K$ and $M/N \cong M/K$. These conditions are not sufficient for N, K being congruent modulo M. To find sufficiency conditions we proceed as follows:

Theorem 3.1. Let N and K be almost balanced submodules of the totally projective $QTAG$–module M of limit length. If N and K have same Ulm invariants and $M/N \cong M/K$ then N and K are congruent modulo M.

Proof. Consider the isomorphism $f : M/N \to M/K$ and direct decompositions $\mathrm{Soc}(H_\alpha(N)) = \mathrm{Soc}(H_{\alpha+1}(N)) \oplus S_\alpha$, $\mathrm{Soc}(H_\alpha(K)) = \mathrm{Soc}(H_{\alpha+1}(K)) \oplus T_\alpha$ for every ordinal $\alpha < \beta$, where β is the length of M. Since N and K have same Ulm invariants for all $\alpha < \beta$, there exists an isomorphism $\psi_\alpha : S_\alpha \to T_\alpha$. As N and K are nice submodules [4] of M we consider triples (ϕ, C, D) such that C, D are nice submodules of M and $\phi : C \to D$ is an isomorphism with the following conditions:

(a) ϕ preserves heights in M,
(b) $\phi(x) + K = f(x + N)$ for all $x \in C$.
(c) If $x_\lambda \in S_\lambda$ and $x \in C$ then $H_M(x + x_\lambda) > \lambda$ if and only if $H_M(\psi_\lambda(x_\lambda) + \phi(x)) > \lambda$.

Let $\mathcal{F}$ denote the class of all such triples satisfying (a) (b) and (c). Now we shall prove that if $(\phi, C, D) \in \mathcal{F}$ and if $x \in M$, then there is a triple $(\phi', C', D') \in \mathcal{F}$ where $C' = C + xR$ and $\phi'|C = \phi$. Since C is nice in M and x is the element of maximum height in the coset $x + C$ (x is proper with respect to C) we may consider $z \in C$ such that $d(xR/zR) = 1$. Let $H_M(x) = \alpha$. We have to find $y \in M$ such that,

(i) $H_M(y) = \alpha$,
(ii) $\phi(z) = y'$ where $d(yR/y'R) = 1$,
(iii) y is proper with respect to D and
(iv) $f(x + N) = y + K$.

Then we obtain an isomorphism $\phi' : C' = (C + xR) \to D' = (D + yR)$ which is an extension of ϕ and satisfies (a) and (b). Again C', D' are finite extensions of nice submodules hence they are nice submodules. If we can find such y, we have to verify (v), i.e. $H_M(s_\alpha + x + c) > \alpha$ if and only if $H_M(\psi(s_\alpha) + y + \phi(c)) > \alpha$ for all $c \in C$, $s_\alpha \in S_\alpha$.

Case (i). When $H_M(z) > \alpha + 1$ and $H_{M/N}(x + N) > \alpha$. Since $H_{\alpha+1}(M/K) = (H_{\alpha+1}(M) + K)/K$ and $H_{M/K}(f(x + N)) = H_{M/N}(x + N) > \alpha$, we may select $u \in H_{\alpha+1}(M)$ such that $u + K = f(x + N)$ and $f(x + N) = \phi(x) + K$. Consider u' such that $d(uR/u'R) = 1$, then by condition (b) $u' - \phi(z) \in K \cap H_{\alpha+2}(M) = H_{\alpha+2}(K)$ and therefore we have $v \in H_{\alpha+1}(K)$ such that $\phi(z) = u' + w'$ where $d\Big(\dfrac{(u + v)R}{(u' + v')R}\Big) = 1$. Again replacing u' by $u' + w'$ we have $u' = \phi(z)$. Since $H_{M/N}(x + N) > \alpha$ and $H_{\alpha+1}(M/N) = (H_{\alpha+1}(M) + N)/N$ we have $H_M(x + x_1) \geqslant \alpha + 1$ for some $x_1 \in N$. Then $H_M(x_1) = H_M(x) = \alpha$ and $H_M(x_1') > \alpha + 1$ where $d\Big(\dfrac{x_1R}{x_1'R}\Big) = 1$ and $H_M(z) > \alpha + 1$.

Now $H_{\alpha+2}(M) \cap N = H_{\alpha+2}(N)$ implies that $x_1' = x_2$ for some $x_2 \in H_{\alpha+1}(N)$,

$$\Rightarrow\ x_1 - x_2' \in \mathrm{Soc}(H_\alpha(N)) \text{ where } d\Big(\frac{x_2'R}{x_2R}\Big) = 1,$$

$$\Rightarrow\ x_1 - x_2' \in S_\alpha \oplus \mathrm{Soc}(H_{\alpha+1}(N)) = \mathrm{Soc}(H_\alpha(N)),$$

$$\Rightarrow\ \exists\ w \in H_{\alpha+1}(M) \text{ such that } x - w \in S_\alpha.$$

Put $y = u + \psi_\alpha(x - w)$. Now y satisfies conditions (i),(ii) and (iv). Again $x - w$ is proper with respect to C as $w \in H_{\alpha+1}(M)$. But the condition (c) implies that $\psi_\alpha(x - w)$ is proper with respect to $D = \phi(C)$, therefore y is proper with respect to D

because $u \in H_{\alpha+1}(M)$. Now $H_M(s_\alpha + x + c) > \alpha$ if and only if $H_M(s_\alpha + x - w + c) > \alpha$ and $H_M(\psi_\alpha(x - w) + y + \phi(c)) > \alpha$ if and only if $H_M(\psi_\alpha(s_\alpha + x - w) + \phi(c)) > \alpha$,

$\Rightarrow$ condition (iv) holds because ϕ satisfies (c).

Case (ii). When $H_M(z) = \alpha + 1$ or $H_{M/N}(x + N) = \alpha$ we have to show that if y satisfies (i),(ii) and (iv) and $s_\alpha \in S_\alpha$ and $c \in C$ such that atleast one of the inequalities in (v) holds then $x + c$ has height α, $H_M(x') > \alpha + 1$, $H_{M/N}(x + c + N) > \alpha$ where $d\Big(\dfrac{(x+c)R}{x'R}\Big) = 1$. The way ϕ and y are defined either of the inequalities in (v) implies that $H_M(c) \geqslant \alpha$ and $H_M(x') > \alpha + 1$ i.e. $x + C$ satisfies the first two conditions. Now,

$$\alpha = H_{M/N}(x+c+N) = H_{M/N}(f(x+N)) = H_{M/N}(y+\phi(c)+K) \geqslant H_M(\psi_\alpha(s_\alpha)+y+\phi(c)).$$

Since $H_\alpha(M/K) = (H_\alpha(M) + K)/K$ we may select $w \in H_\alpha(M)$ such that $w + K = f(x + N)$. Like case (i) there exists $v \in H_\alpha(K)$ such that $w' = \phi(z)$, here $d\Big(\dfrac{(w+v)R}{w'R}\Big) = 1$. We select $y = w + v$ so that (ii) and (iv) are satisfied. Since $H_M(z) = H_M(\phi(z)) = H_M(y')$, $\Big(d\Big(\dfrac{yR}{y'R}\Big) = 1\Big)$ and $H_{M/N}(x + N) = H_{M/K}(f(x + N)) = H_{M/K}(y + K)$. Either of the conditions defining this case implies, $H_M(y) = \alpha$. If y fails to be proper with respect to D then we may go to case (i). Because if $H_M(y + \phi(c)) > \alpha$ for some $c \in C$ we may replace x by $x + c$ and apply case (i) to $x + c$. It can be deduced now that y satisfies (iii). Again if either of the inequalities in (v) holds for some $s_\alpha \in S_\alpha$ and $c \in C$, then we replace x by $x + c$ and return to case (i). This makes us assume that neither of these inequalities hold for any $s_\alpha \in S_\alpha$ and $c \in C$. This implies y satisfies (v).

Let $\mathcal{A}$ be the subfamily of $\mathcal{F}$ consisting of triples (ϕ, C, D) such that C and D are equal and nice. Now $\mathcal{A}$ is partially ordered by the natural order. By Zorn's lemma we may select the maximal member $(\phi, C, C) \in \mathcal{A}$. We have to show that $C = M$, then ϕ would be desired automorphism. Applying the above arguments we may obtain a sequence (ϕ_i, C_i, D_i) in $\mathcal{F}$ which are extensions of (ϕ, C, C) such that $\phi_{i+1}|C_i = \phi_i$, $g(C_i|C)$, $g(D_i/C)$ are finite and for every i,

$$A = \bigcup_{i=1}^{\infty} C_i = \bigcup_{i=1}^{\infty} D_i = \bigcup_{i=1}^{\infty} A_i \text{ where } A_i \in \mathcal{B}$$

where $\mathcal{B}$ is a family of nice submodules of M such that $\{0\} \in \mathcal{B}$, $\mathcal{B}$ is closed under the union of chains and for every $M_1 \in \mathcal{B}$ and $M_2 \subset M_1$ such that M_1/M_2 is countably generated then $\exists\ M_3 \in \mathcal{B}$ with countably generated M_3/M_2.

If $A \in \mathcal{B}$ then (ψ, A, A) is a member of $\mathcal{A}$ (ψ being the extension of ϕ_i's) contradicting the maximality of (ϕ, C, C) as $\psi/A = \phi$.

This implies that ϕ is the required automorphism.

Remark If N, K are almost balanced submodules of totally projective modules M and M' with $M/N \cong M'/K$, we may obtain an isomorphism $\phi : M \to M'$ with

$\phi(N) = K$. Thus by taking trivial quotients, it is evident that totally projective modules are determined upto isomorphism by their Ulm invariants.

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