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A note relating to Ramanujan's Bessel index integral

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Abstract. Ramanujan's Bessel index integral

$$\int_{-\infty}^{\infty} J_{\mu-\xi}(t)J_{\nu+\xi}(t)d\xi$$

and several extensions are evaluated by an alternative method.

1. Introduction

In 1920, Ramanujan [**3**] introduced his unusual, and potentially important [**4**], Bessel integral

$$\int_{-\infty}^{\infty} d\xi\, J_{\mu-\xi}(t)J_{\nu-\xi}(t) = J_{\mu+\nu}(2t). \tag{1.1}$$

In this note, (1.1) is re-derived by a more pedestrian route and the method applied to obtain several similar index-integrals, some alreadt tabulated [**1**, Chapter 17] and some apparantly new. The idea is simply to employ Nielsen's series [**4**]

$$J_a(z)J_b(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{(z/2)^{a+b+2m}(a+b+m+1)_m}{\Gamma(a+m+1)\Gamma(b+m+1)} \tag{1.2}$$

with an instance of

$$\int_{-\infty}^{\infty} \frac{F(\xi)}{\Gamma(a-\xi)\Gamma(b+\xi)} d\xi = \Phi(a,b) \tag{1.3}$$

a number of which were the principal subject of [**3**] and which can be found in [**1**], [**2**, Section 2.2]. Specifically, we shall examine the five examples, the first of which is known [**1, 2, 3**].

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	$F(z)$	$\Phi(a,b)$	Conditions
(i)	$\cos(cz)$	$\frac{[2\cos(c/2)]^{a+b-2}}{\Gamma(a+b-1)}\cos[\frac{1}{2}c(b-a)]$	$\lvert c\rvert<\pi,\ \mathrm{Re}\,(a+b)>1$
(ii)	$\sin(cz)$	$\frac{[2\cos(c/2)]^{a+b-2}}{\Gamma(a+b-1)}\sin[\frac{1}{2}c(b-a)]$	$\lvert c\rvert<\pi,\ \mathrm{Re}\,(a+b)>1$
(iii)	$\frac{\sin(n\pi z)}{\sin(\pi z)}\theta(z)$	$\frac{1-(-1)^n}{2}\frac{2^{2a-2}}{\Gamma(2a-1)}$	$b=a,\ \mathrm{Re}\,a>\frac{1}{2}$
(iv)	$\frac{\cos^2(\pi z)}{\Gamma(a-z)\Gamma(b+z)}$	$\frac{\Gamma(2a+2b-3)}{2\Gamma(2a-1)\Gamma(2b-1)\Gamma^2(a+b-1)}$	$\mathrm{Re}\,(a+b)>\frac{3}{2}$
(v)	$P(z)e^{(2n\pi+\phi)iz}$	$\frac{[2\cos(\phi/2)]^{a+b-2}}{\Gamma(a+b-1)}e^{i\phi(b-a)/2}I_n$	$\lvert\phi\rvert<\pi,\ \mathrm{Re}\,(a+b)>1$
		$P(x+1)=P(x)$	$I_n=\int_0^1 P(x)e^{2n\pi ix}\,dx$

In most cases the resulting series is hypergeometric and can be summed.

2. Calculations and results

From (1.2) and (1.3) one has

$$J=\int_{-\infty}^{\infty}F(\xi)J_{a-\xi}(t)J_{b+\xi}(t)\,d\xi= \left(\frac{t}{2}\right)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\frac{t}{2}\right)^{2m}\frac{\Gamma(a+b+2m+1)}{\Gamma(a+b+m+1)}\Phi(a+m+1,b+m+1). \tag{2.1}$$

For example (i), this easily reduces to

$$J_i = \left(\frac{t}{2}\right)^{a+b}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\frac{t}{2}\right)^{2m}\frac{[2\cos(c/2)]^{a+b+2m}}{\Gamma(a+b+1+m)}\cos[c(b-a)/2] \tag{2.2}$$

$$= \frac{(t\cos[\frac{1}{2}(b-a)]\cos(c/2))^{a+b}}{\Gamma(a+b+1)}\,{}_0F_1\left(\begin{matrix}-\\ a+b+1\end{matrix}\middle|-\left(t\cos\frac{c}{2}\right)^2\right). \tag{2.3}$$

But

$${}_0F_1\left(\begin{matrix}-\\ A\end{matrix}\middle|-z^2\right)=\Gamma(A)z^{1-A}J_{A-1}(2z), \tag{2.4}$$

so

$$\int_{-\infty}^{\infty}\cos(c\xi)J_{a-\xi}(t)J_{b+\xi}(t)d\xi=\cos[\tfrac{c}{2}(b-a)]J_{a+b}(2t\cos(\tfrac{c}{2})) \tag{2.5}$$

for $\mathrm{Re}\,(a+b)>1$. Setting $c=0$ in (2.5) one obtains Ramanujan's formula (1.1).

Similarly for (ii)

$$\int_{-\infty}^{\infty}\sin(c\xi)J_{a-\xi}(t)J_{b+\xi}(t)\,d\xi=\sin[\tfrac{c}{2}(b-a)]J_{a+b}(2t\cos(c/2)),\quad \mathrm{Re}\,(a+b)>1, \tag{2.6}$$

and for example (iii)

$$\int_0^{\infty}\frac{\sin(n\pi x)}{\sin(\pi x)}J_{a-x}(t)J_{a+x}(t)\,dx=\frac{1-(-1)^n}{2}J_{2a}(2t),\quad \mathrm{Re}\,a>\tfrac{1}{2}. \tag{2.7}$$

For example (iv) this procedure leads to

$$\int_0^{\infty}\frac{\cos^2(\pi x)J_{a-x}(2t)J_{a+x}(2t)}{\Gamma(b-x)\Gamma(b+x)}\,dx=\frac{t^{2a}}{2}\,{}_1F_1\left(\begin{matrix}2(a+b)+1\\ a+b\end{matrix}\middle|-t^2\right), \tag{2.8}$$

for $\mathrm{Re}\,(a+b) > \frac{3}{2}$. The right-hand side of (2.8) does not appear to reduce further.

Finally, for example (v) one has

(2.9)
$$\int_{\infty}^{\infty} P(x)e^{(2\pi n+\phi)ix} J_{a-x}(t)J_{b+x}(t)\,dx = e^{(b-a)\phi i/2} J_{a+b}(2t\cos(\phi/2)) \int_0^1 P(u)e^{2n\pi i u} du$$

where P has period 1.

In view of the unproven formula (10.2) in [**3**], expressing a four Bessel function index integral as a hypergeometric series, it is possible that Ramanujan was aware of the method used here, but chose not to use it.

References

[1] A. Erdélyi. *Tables of Integral Transforms*, volume II. McGraw-Hill, New York, 1st edition, 1954.
[2] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev. *Integrals and Series, volume 2: Special Functions*. Gordon and Breach Science Publishers, 1986.
[3] S. Ramanujan. A class of definite integrals. *Quart. J. Math. (Oxford)*, 48:294–310, 1920.
[4] G. N. Watson. *A treatise on the Theory of Bessel Functions*. Cambridge University Press, 1966.

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