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Weighted estimates for multilinear Marcinkiewicz Operators in the endpoint cases

Zhou Xiaosha

ABSTRACT. In this paper, we establish weighted endpoint estimates for multilinear Marcinkiewicz operators.

1. Introduction

As the development of singular integral operators, their commutators and multilinear operators have been well studied(see [2-8]). In [11], authors obtain the boundedness properties of the commutators for the extreme values of p. The purpose of this paper is to introduce some multilinear operator associated to the Marcinkiewicz integral operator and prove the weighted boundedness properties of the multilinear operators for the extreme cases.

2. Preliminaries and Results

First, let us introduce some preliminaries. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and any locally integrable function f on R^n, we denote that $f(Q) = \int_Q f(x)dx$, $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^\#(x) = \sup_{Q\ni x} |Q|^{-1}\int_Q |f(y) - f_Q|dy$. For a weight functions $w \in A_1$(see [10, p.389]), f is said to belong to $BMO(w)$ if $f^\# \in L^\infty(w)$ and define $||f||_{BMO(w)} = ||f^\#||_{L^\infty(w)}$, if $w = 1$, we denote that $BMO(R^n) = BMO(w)$. Also, we give the concepts of the atom and weighted H^1 space. A function a is called a $H^1(w)$ atom if there exists a cube Q such that a is supported on Q, $||a||_{L^\infty(w)} \leqslant w(Q)^{-1}$ and $\int_{R^n} a(x)dx = 0$. It is well known that, for $w \in A_1$, the weighted Hardy space $H^1(w)$ has the atomic decomposition characterization(see [1]).

Suppose that S^{n-1} is the unit sphere of $R^n (n \geqslant 2)$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let Ω be homogeneous of degree zero and satisfy the following two conditions:

(i) $\Omega(x)$ is continuous on S^{n-1} and satisfies the Lip_γ condition on $S^{n-1} (0 < \gamma \leqslant 1)$, i.e.,

$$|\Omega(x') - \Omega(y')| \leqslant M|x' - y'|^\gamma, \quad x', y' \in S^{n-1};$$

(ii) $\int_{S^{n-1}} \Omega(x')dx' = 0$.

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Fix $\lambda > \max(1, 2n/(n+2))$. Let m be a positive integer and A be a function on R^n. We denote that $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The multilinear Marcinkiewicz operator is defined by

$$\mu_\lambda^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^A(f)(x,y) = \int_{|y-z|\leqslant t} \frac{\Omega(y-z)}{|y-z|^{n-1}} \frac{R_{m+1}(A;x,z)}{|x-z|^m} f(z)dz$$

and

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\leqslant m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha.$$

Set

$$F_t(f)(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

We also define that

$$\mu_\lambda(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2},$$

which is the Marcinkiewicz integral operator(see [9][17]).

Let H be the Hilbert space $H = \left\{h : ||h|| = \left(\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+3}\right)^{1/2} < \infty\right\}$, then for each fixed $x \in R^n$, $F_t^A(f)(x,y)$ may be viewed as a mapping from $(0,+\infty)$ to H, and it is clear that

$$\mu_\lambda^A(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t^A(f)(x,y)\right\|,$$

$$\mu_\lambda(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t(f)(y)\right\|.$$

We also consider the variant of μ_λ^A, which is defined by

$$\tilde{\mu}_\lambda^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$\tilde{F}_t^A(f)(x,y) = \int_{|y-z|\leqslant t} \frac{\Omega(y-z)}{|y-z|^{n-1}} \frac{Q_{m+1}(A;x,z)}{|x-z|^m} f(z)dz$$

and

$$Q_{m+1}(A;x,z) = R_m(A;x,z) - \sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha A(x)(x-z)^\alpha.$$

Noticing that when $m=0$, μ_λ^A is just the commutator of Marcinkiewicz integral operator (see [17]). It is well known that multilinear operators, as the extension of commutators, are of great interest in harmonic analysis and have been widely studied by many authors (see [3-8][12]). In [11], the endpoint boundedness of commutators generated by the Calderón-Zygmund operator and BMO functions are obtained. In [13-15], the multilinear Littlewood-Paley operators are studied. The main purpose of this paper is to discuss the weighted boundedness properties of the multilinear Marcinkiewicz operators for the extreme cases of p. Some works in this aspect have done(see [16]).

We shall prove the following theorems in Section 3.

Theorem 1. Let $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$ and $w \in A_1$. Then μ_λ^A is bounded from $L^\infty(w)$ to $BMO(w)$.

Theorem 2. Let $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$ and $w \in A_1$. Then $\tilde{\mu}_\lambda^A$ is bounded from $H^1(w)$ to $L^1(w)$.

Theorem 3. Let $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$ and $w \in A_1$. Then μ_λ^A is bounded from $H^1(w)$ to weak $L^1(w)$.

Remark. In general, μ_λ^A is not bounded from $H^1(w)$ to $L^1(w)$.

3. Proofs of Theorems

To prove the theorems, we need the following lemmas.

Lemma 1.(see [6]) Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then

$$|R_m(A; x, y)| \leqslant C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Lemma 2.(see [9]) Let $w \in A_1$ and $1 < p < \infty$. Then μ_λ is bounded on $L^p(w)$, that is

$$||\mu_\lambda(f)||_{L^p(w)} \leqslant C||f||_{L^p(w)}.$$

Lemma 3.(see [2][3]) Let T_b be the commutator defined by

$$T_b(f)(x) = \int_{R^n} \frac{b(x) - b(y)}{|x-y|^n} f(y) dy.$$

If $w \in A_1$, $1 < p < \infty$ and $b \in BMO(R^n)$. Then

(i) T_b is bounded on $L^p(w)$, that is

$$||T_b(f)||_{L^p(w)} \leqslant C||f||_{L^p(w)};$$

(ii) T_b is weak type bounded of $(H^1(w), L^1(w))$, that is, for any $\eta > 0$,

$$w\left(\{x \in R^n : |T_b(f)(x)| > \eta\}\right) \leqslant C||f||_{H^1(w)}/\eta.$$

Proof of Theorem 1. It is only to prove that there exists a constant C_Q such that

$$\frac{1}{w(Q)} \int_Q |\mu_\lambda^A(f)(x) - C_Q| w(x) dx \leqslant C||f||_{L^\infty(w)}$$

holds for any cube Q.
Fix a cube $Q = Q(x_0, l)$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}} x^\alpha$. Then

$R_m(A;x,y)=R_m(\tilde{A};x,y)$ and $D^\alpha\tilde{A}=D^\alpha A-(D^\alpha A)_{\tilde{Q}}$ for $|\alpha|=m$.

We write $F_t^A(f)=F_t^A(f_1)+F_t^A(f_2)$ for $f_1=f\chi_{\tilde{Q}}$ and $f_2=f\chi_{R^n\setminus\tilde{Q}}$,

$$F_t^A(f)(x,y)=\int_{|y-z|\leqslant t}\frac{\Omega(y-z)}{|y-z|^{n-1}}\frac{R_{m+1}(\tilde{A};x,z)}{|x-z|^m}f_2(z)dz$$
$$+\int_{|y-z|\leqslant t}\frac{\Omega(y-z)}{|y-z|^{n-1}}\frac{R_m(\tilde{A};x,z)}{|x-z|^m}f_1(z)dz-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{|y-z|\leqslant t}\frac{\Omega(y-z)}{|y-z|^{n-1}}\frac{(x-z)^\alpha}{|x-z|^m}D^\alpha\tilde{A}(z)f_1(z)dz,$$

then

$$\begin{aligned}
&\left|\mu_\lambda^A(f)(x)-\mu_\lambda^{\tilde{A}}(f_2)(x_0)\right|\\
=&\left|\left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}F_t^A(f)(x,y)\right|\right|-\left|\left|\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2}F_t^{\tilde{A}}(f_2)(x_0,y)\right|\right|\right|\\
\leqslant&\left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}F_t^A(f)(x,y)-\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2}F_t^{\tilde{A}}(f_2)(x_0,y)\right|\right|\\
\leqslant&\left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}F_t\left(\frac{R_m(\tilde{A};x,\cdot)}{|x-\cdot|^m}f_1\right)(y)\right|\right|\\
&+\sum_{|\alpha|=m}\frac{1}{\alpha!}\left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}F_t\left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m}D^\alpha\tilde{A}f_1\right)(y)\right|\right|\\
&+\left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}F_t^{\tilde{A}}(f_2)(x,y)-\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2}F_t^{\tilde{A}}(f_2)(x_0,y)\right|\right|=I(x)+II(x)+III(x),
\end{aligned}$$

thus

$$\begin{aligned}
&\frac{1}{w(Q)}\int_Q\left|\mu_\lambda^A(f)(x)-\mu_\lambda^{\tilde{A}}(f_2)(x_0)\right|w(x)dx\\
\leqslant&\frac{1}{w(Q)}\int_Q I(x)w(x)dx+\frac{1}{w(Q)}\int_Q II(x)w(x)dx+\frac{1}{w(Q)}\int_Q III(x)w(x)dx=I+II+III.
\end{aligned}$$

Now, let us estimate I and II. First, for $x\in Q$ and $y\in\tilde{Q}$, using Lemma 1, we get

$$R_m(\tilde{A};x,y)\leqslant C|x-y|^m\sum_{|\alpha|=m}||D^\alpha A||_{BMO},$$

thus, by the $L^p(w)$-boundedness of μ_λ for $1<p<\infty$(see Lemma 2), we gain

$$\begin{aligned}
I &\leqslant \frac{C}{w(Q)}\int_Q |\mu_\lambda(\sum_{|\alpha|=m}||D^\alpha A||_{BMO}f_1)(x)|w(x)dx \\
&\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\left(\frac{1}{w(Q)}\int_{R^n}|\mu_\lambda(f_1)(x)|^p w(x)dx\right)^{1/p} \\
&\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\left(\frac{1}{w(Q)}\int_{R^n}|f_1(x)|^p w(x)dx\right)^{1/p} \\
&\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\left(\frac{w(\tilde{Q})}{w(Q)}\right)^{1/p}||f||_{L^\infty(w)} \\
&\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}.
\end{aligned}$$

Secondly, since $w\in A_1$, w satisfies the reverse of Hölder's inequality:

$$\left(\frac{1}{|Q|}\int_Q w(x)^q dx\right)^{1/q}\leqslant \frac{C}{|Q|}\int_Q w(x)dx$$

for all cube Q and some $1<q<\infty$(see [10, p.396]), thus, taking $p>1$, by the $L^p(w)$-boundedness of μ_λ and the Hölder's inequality, we gain

$$\begin{aligned}
II &\leqslant \frac{C}{w(Q)}\int_Q |\mu_\lambda(\sum_{|\alpha|=m}(D^\alpha A-(D^\alpha A)_{\tilde{Q}})f_1)(x)|w(x)dx \\
&\leqslant C\sum_{|\alpha|=m}\left(\frac{1}{w(Q)}\int_Q|\mu_\lambda((D^\alpha A-(D^\alpha A)_{\tilde{Q}})f_1)(x)|^p w(x)dx\right)^{1/p} \\
&\leqslant C\sum_{|\alpha|=m}\left(\frac{1}{w(Q)}\int|(D^\alpha A(x)-(D^\alpha A)_{\tilde{Q}})f_1(x)|^p w(x)dx\right)^{1/p} \\
&\leqslant C\sum_{|\alpha|=m}w(Q)^{-1/p}\left(\int_{\tilde{Q}}|D^\alpha A(x)-(D^\alpha A)_{\tilde{Q}}|^{pq'}dx\right)^{1/pq'}\left(\int_{\tilde{Q}}w(x)^q dx\right)^{1/pq}||f||_{L^\infty(w)} \\
&\leqslant C\sum_{|\alpha|=m}\left(\frac{1}{|Q|}\int_{\tilde{Q}}|D^\alpha A(x)-(D^\alpha A)_{\tilde{Q}}|^{pq'}dx\right)^{1/pq'}\left(\frac{1}{|Q|}\int_{\tilde{Q}}w(x)^q dx\right)^{1/pq}\left(\frac{|Q|}{w(Q)}\right)^{1/p}||f||_{L^\infty(w)} \\
&\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\left(\frac{1}{|Q|}\int_{\tilde{Q}}w(x)dx\right)^{1/p}\left(\frac{|Q|}{w(Q)}\right)^{1/p}||f||_{L^\infty(w)} \\
&\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}.
\end{aligned}$$

To estimate III, we write

$$
\begin{aligned}
&\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t^{\tilde{A}}(f_2)(x,y)-\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2} F_t^{\tilde{A}}(f_2)(x_0,y)\\
= \quad & \int_{|y-z|\leqslant t}\left[\frac{1}{|x-z|^m}-\frac{1}{|x_0-z|^m}\right]\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}\frac{\Omega(y-z)R_m(\tilde{A};x,z)f_2(z)}{|y-z|^{n-1}}dz\\
&+\int_{|y-z|\leqslant t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}\frac{\Omega(y-z)f_2(z)}{|y-z|^{n-1}|x_0-z|^m}[R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)]dz\\
&+\int_{|y-z|\leqslant t}\left[\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}-\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2}\right]\frac{\Omega(y-z)R_m(\tilde{A};x_0,z)f_2(z)}{|y-z|^{n-1}|x_0-z|^m}dz\\
&-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{|y-z|\leqslant t}\left[\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}\frac{(x-z)^\alpha}{|x-z|^m}-\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2}\frac{(x_0-z)^\alpha}{|x_0-z|^m}\right]\\
&\times\frac{\Omega(y-z)D^\alpha\tilde{A}(z)f_2(z)}{|y-z|^{n-1}}dz\\
:= \quad & III_1^t(x)+III_2^t(x)+III_3^t(x)+III_4^t(x).
\end{aligned}
$$

Note that $|x-z|\sim|x_0-z|$ for $x\in\tilde{Q}$ and $z\in R^n\setminus\tilde{Q}$. Also Note that $|x-z|\leqslant 2t$, $|y-z|\geqslant|x-z|-|x-y|\geqslant|x-z|-t\geqslant|x-z|-3t$ when $|x-y|\leqslant t$, $|y-z|\leqslant t$, and $|x-z|\leqslant t(1+2^{k+1})\leqslant 2^{k+2}t$, $|y-z|\geqslant|x-z|-2^{k+3}t$ when $|x-y|\leqslant 2^{k+1}t$, $|y-z|\leqslant t$, we get, by Minkowski inequality,

$$
\begin{aligned}
&||III_1^t(x)||\\
\leqslant \quad & \int_{R^n}\left[\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\left(\frac{|x-x_0||\Omega(y-z)||R_{m+1}(\tilde{A};x,z)||f_2(z)|}{|y-z|^{n-1}|x-z|^{m+1}}\right)^2\chi_{\Gamma(z)}(y,t)\frac{dydt}{t^{n+3}}\right]^{1/2}dz\\
\leqslant \quad & C\int_{R^n}\frac{|R_{m+1}(\tilde{A};x,z)||f_2(z)||x-x_0|}{|x-z|^{m+1}}\left[\int_0^\infty\int_{|x-y|\leqslant t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\frac{\chi_{\Gamma(z)}(y,t)}{(|x-z|-3t)^{2n-2}}\frac{dydt}{t^{n+3}}\right]^{1/2}dz\\
&+C\int_{R^n}\frac{|R_{m+1}(\tilde{A};x,z)||f_2(z)||x-x_0|}{|x-z|^{m+1}}\\
&\times\left[\int_0^\infty\sum_{k=0}^\infty\int_{2^kt<|x-y|\leqslant 2^{k+1}t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\frac{\chi_{\Gamma(z)}(y,t)}{(|x-z|-2^{k+3}t)^{2n-2}}\frac{dydt}{t^{n+3}}\right]^{1/2}dz
\end{aligned}
$$

$$\begin{aligned}
&\leqslant C\int_{R^n}\frac{|R_{m+1}(\tilde{A};x,z)||f_2(z)||x-x_0|}{|x-z|^{m+1+1/2}}\left[\int_{|x-z|/2}^{\infty}\frac{dt}{(|x-z|-3t)^{2n}}\right]^{1/2}dz\\
&\quad +C\int_{R^n}\frac{|R_{m+1}(\tilde{A};x,z)||f_2(z)||x-x_0|}{|x-z|^{m+1+1/2}}\left[\sum_{k=0}^{\infty}\int_{2^{-2-k}|x-z|}^{\infty}2^{-kn\lambda}(2^kt)^nt^{-n}\frac{2^kdt}{(|x-z|-2^{k+3}t)^{2n}}\right]^{1/2}dz\\
&\leqslant C\int_{R^n}\frac{|R_{m+1}(\tilde{A};x,z)||f_2(z)||x-x_0|}{|x-z|^{m+n+1}}dz+C\int_{R^n}\frac{|R_{m+1}(\tilde{A};x,z)||f(z)|}{|x-z|^{m+n+1}}dz\left[\sum_{k=0}^{\infty}2^{kn(1-\lambda)}\right]^{1/2}\\
&= C\int_{R^n}\frac{|R_{m+1}(\tilde{A};x,z)||x-x_0|}{|x-z|^{m+n+1}}|f_2(z)|dz,
\end{aligned}$$

by Lemma 1, we have

$$\begin{aligned}
|R_m(\tilde{A};x,z)| &\leqslant C|x-z|^m\sum_{|\alpha|=m}(||D^\alpha A||_{BMO}+|(D^\alpha A)_{\tilde{Q}(x,z)}-(D^\alpha A)_{\tilde{Q}}|)\\
&\leqslant Ck|x-z|^m\sum_{|\alpha|=m}||D^\alpha A||_{BMO},
\end{aligned}$$

thus, for $x\in Q$,

$$\begin{aligned}
&||III_1^t(x)||\leqslant C\int_{R^n\setminus\tilde{Q}}\frac{|R_{m+1}(\tilde{A};x,z)||x-x_0|}{|x-z|^{m+n+1}}|f(z)|dz\\
&\leqslant C\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|R_{m+1}(\tilde{A};x,z)||x-x_0|}{|x-z|^{m+n+1}}|f(z)|dz\\
&\leqslant C\sum_{k=1}^{\infty}\frac{kl}{(2^kl)^{n+1}}\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\int_{2^k\tilde{Q}}|f(z)|dz\\
&\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}\sum_{k=1}^{\infty}k2^{-k}\\
&\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}.
\end{aligned}$$

For $III_2^t(x)$, by the formula (see [6]):

$$R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)=R_m(\tilde{A};x,x_0)+\sum_{0<|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};x_0,z)(x-x_0)^\beta$$

and Lemma 1, we get

$$|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)|\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}(|x-x_0|^m+\sum_{0<|\beta|<m}|x_0-z|^{m-|\beta|}|x-x_0|^{|\beta|}),$$

thus, similar to the proof of $III_1^t(x)$, we get, for $x \in Q$,

$$||III_2^t(x)|| \leqslant C\int_{R^n}\frac{|f_2(z)|}{|x-z|^{m+n}}|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)|dz$$

$$\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\int_{R^n}\frac{|x-x_0|^m+\sum_{0<|\beta|<m}|x_0-z|^{m-|\beta|}|x-x_0|^{|\beta|}}{|x_0-z|^{m+n}}|f_2(z)|dz$$

$$\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=0}^{\infty}\frac{kl^m}{(2^kl)^{m+n}}\int_{2^{k+1}Q}|f(z)|dz$$

$$\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}\sum_{k=1}^{\infty}k2^{-km}\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}.$$

For $III_3^t(x)$, by the inequality: $a^{1/2}-b^{1/2}\leqslant(a-b)^{1/2}$ for $a\geqslant b>0$, we obtain, similar to the estimate of $III_1^t(x)$,

$$||III_3^t(x)||\leqslant$$

$$C\int_{R^n}\left(\int_{R_+^{n+1}}\left[\frac{t^{n\lambda/2}|x-x_0|^{1/2}}{(t+|x-y|)^{(n\lambda+1)/2}}\frac{|f_2(z)||\Omega(y-z)|\chi_{\Gamma(z)}(y,t)}{|y-z|^{n-1}|x_0-z|^m}|R_m(\tilde{A};x_0,z)|\right]^2\frac{dydt}{t^{n+3}}\right)^{1/2}dz$$

$$\leqslant C\int_{R^n}\frac{|f_2(z)||R_m(\tilde{A};x_0,z)||x-x_0|^{1/2}}{|x_0-z|^m}\left[\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\lambda+1}\frac{t^{-n}\chi_{\Gamma(z)}(y,t)}{|y-z|^{2n+2}}dydt\right]^{1/2}dz$$

$$\leqslant C\int_{R^n}\frac{|f_2(z)||R_m(\tilde{A};x_0,z)||x-x_0|^{1/2}}{|x_0-z|^{m+n+1/2}}dz$$

$$\leqslant C\sum_{k=1}^{\infty}\frac{kl^{1/2}}{(2^kl)^{n+1/2}}\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\int_{2^k\tilde{Q}}|f(z)|dz$$

$$\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}\sum_{k=1}^{\infty}k2^{-k/2}\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}.$$

For $III_4^t(x)$, similar to the estimates of $III_1^t(x)$ and $III_3^t(x)$, we have

$$||III_4^t(x)||\leqslant C\int_{R^n\setminus\tilde{Q}}\left[\frac{|x-x_0|}{|x-z|^{n+1}}+\frac{|x-x_0|^{1/2}}{|x-z|^{n+1/2}}\right]\sum_{|\alpha|=m}|D^\alpha\tilde{A}(z)||f(z)|dz$$

$$\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}\sum_{k=1}^{\infty}k(2^{-k}+2^{-k/2})\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}.$$

Thus

$$III\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}.$$

Combining these estimates, we complete the proof of Theorem 1.

Proof of Theorem 2. It suffices to show that there exists a constant $C > 0$ such that for every H^1-atom a (that is that a satisfy: supp$a \subset Q = Q(x_0, r)$, $||a||_{L^\infty(w)} \leqslant w(Q)^{-1}$ and $\int_{R^n} a(y)dy = 0$(see [1])), we have

$$||\tilde{\mu}_\lambda^A(a)||_{L^1(w)} \leqslant C.$$

We write

$$\int_{R^n} \tilde{\mu}_\lambda^A(a)(x)w(x)dx = \left[\int_{|x-x_0|\leqslant 2r} + \int_{|x-x_0|>2r}\right] \tilde{\mu}_\lambda^A(a)(x)w(x)dx := J + JJ.$$

For J, by the following equality

$$Q_{m+1}(A;x,y) = R_{m+1}(A;x,y) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha(D^\alpha A(x) - D^\alpha A(y)),$$

we have

$$\tilde{\mu}_\lambda^A(a)(x) \leqslant \mu_\lambda^A(a)(x) + C\sum_{|\alpha|=m}\int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n}|a(y)|dy,$$

thus, $\tilde{\mu}_\lambda^A$ is $L^p(w)$-bounded by Lemma 2 and 3. We see that

$$J \leqslant ||\tilde{\mu}_\lambda^A(a)||_{L^p(w)} w(2Q)^{1-1/p} \leqslant C||a||_{L^p(w)} w(Q)^{1-1/p} \leqslant C.$$

To obtain the estimate of JJ, we denote that $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{2B}x^\alpha$, then $Q_m(A;x,y) = Q_m(\tilde{A};x,y)$. We write, by the vanishing moment of a and $Q_{m+1}(A;x,y) = R_m(A;x,y) - \sum_{|\alpha|=m}\frac{1}{\alpha!}(x-y)^\alpha D^\alpha A(x)$, for $x \in (2Q)^c$,

$$\begin{aligned}
\tilde{F}_t^A(a)(x,y) &= \int_{|y-z|\leqslant t} \frac{\Omega(y-z)R_m(\tilde{A};x,z)}{|y-z|^{n-1}|x-z|^m}a(z)dz \\
&\quad - \sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{|y-z|\leqslant t} \frac{\Omega(y-z)D^\alpha\tilde{A}(x)(x-z)^\alpha}{|y-z|^{n-1}|x-z|^m}a(z)dz \\
&= \int_{R^n}\left[\frac{\chi_{\Gamma(y)}(z,t)\Omega(y-z)R_m(\tilde{A};x,z)}{|y-z|^{n-1}|x-z|^m} - \frac{\chi_{\Gamma(y)}(x_0,t)\Omega(y-x_0)R_m(\tilde{A};x,x_0)}{|y-x_0|^{n-1-\delta}|x-x_0|^m}\right]a(z)dz \\
&\quad - \sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\left[\frac{\chi_{\Gamma(y)}(z,t)\Omega(y-z)(x-z)^\alpha}{|y-z|^{n-1}|x-z|^m} - \frac{\chi_{\Gamma(y)}(x_0,t)\Omega(y-x_0)(x-x_0)^\alpha}{|y-x_0|^{n-1}|x-x_0|^m}\right] \\
&\quad \times D^\alpha\tilde{A}(x)a(z)dz,
\end{aligned}$$

thus, similar to the proof of III in Theorem 1, we obtain

$$||\tilde{F}_t^A(a)(x,y)|| \leqslant C\frac{|Q|^{1+1/n}}{w(Q)}\left(\sum_{|\alpha|=m}||D^\alpha A||_{BMO}|x-x_0|^{-n-1} + |x-x_0|^{-n-1}|D^\alpha\tilde{A}(x)|\right),$$

note that if $w \in A_1$, then $\frac{w(Q_2)}{|Q_2|}\frac{|Q_1|}{w(Q_1)} \leqslant C$ for all cubes Q_1, Q_2 with $Q_1 \subset Q_2$. Thus, by Hölder's inequality and the reverse of Hölder's inequality for $w \in A_1$ and some $p > 1$ with

$1/p+1/p'=1$, we obtain

$$\begin{aligned} JJ &\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=1}^\infty 2^{-k}\left(\frac{|Q|}{w(Q)}\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\right)\\ &\quad +C\sum_{|\alpha|=m}\sum_{k=1}^\infty 2^{-k}\frac{|Q|}{w(Q)}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|D^\alpha\tilde{A}(x)|^{p'}dx\right)^{1/p'}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}w(x)^p dx\right)^{1/p}\\ &\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=1}^\infty k2^{-k}\left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\frac{|Q|}{w(Q)}\right)\leqslant C, \end{aligned}$$

which together with the estimate for J yields the desired result. This finishes the proof of Theorem 2.

Proof of Theorem 3. By the equality

$$R_{m+1}(A;x,y)=Q_{m+1}(A;x,y)+\sum_{|\alpha|=m}\frac{1}{\alpha!}(x-y)^\alpha(D^\alpha A(x)-D^\alpha A(y))$$

we get

$$\mu_\lambda^A(f)(x)\leqslant\tilde{\mu}_\lambda^A(f)(x)+C\sum_{|\alpha|=m}\int_{R^n}\frac{|D^\alpha A(x)-D^\alpha A(y)|}{|x-y|^n}|f(y)|dy,$$

by Theorem 2 and Lemma 3, we obtain

$$\begin{aligned} &w(\{x\in R^n:\mu_\lambda^A(f)(x)>\eta\})\\ \leqslant\ & w(\{x\in R^n:\tilde{\mu}_\lambda^A(f)(x)>\eta/2\})+w\left(\left\{x\in R^n:\sum_{|\alpha|=m}\int_{R^n}\frac{|D^\alpha A(x)-D^\alpha A(y)|}{|x-y|^n}|f(y)|dy>C\eta\right\}\right)\\ \leqslant\ & C||f||_{H^1(w)}/\eta. \end{aligned}$$

This completes the proof of Theorem 3.

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College of Mathematics,
Changsha University of Science and Technology,
Changsha 410077,
P.R. of China
E-mail address: zhouxiaosha57@126.com