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Summation formulae involving Voigt functions and generalized hypergeometric functions

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Abstract. The Voigt functions play an important role in several diverse fields of physics and engineering. Motivated by the contributions toward the unification (and generalization) of these functions, in this paper, we establish several explicit representations and summation formulae involving Voigt functions and generalized hypergeometric function. Further, we derive various other interesting results as applications of these connections.

1. Introduction.

Let

$$\mathrm{J}_\nu(\mathrm{z}) = \sum_{\mathrm{n}=0}^{\infty} \frac{(-1)^{\mathrm{n}} (\mathrm{z}/2)^{\nu+2\mathrm{n}}}{\Gamma(\mathrm{n}+1)\,\Gamma(\nu+\mathrm{n}+1)}\ , \ |\mathrm{z}| < \infty, \tag{1.1}$$

be the Bessel function [14] of the first kind of order ν. We note that $J_\nu(z)$ is the defining oscillatory kernel of Hankel's integral transform

$$(\mathrm{H}_\nu \mathrm{f})(\mathrm{x}) = \int_0^\infty \mathrm{f}(\mathrm{t})\,\mathrm{J}_\nu(\mathrm{xt})\mathrm{dt}. \tag{1.2}$$

Furthermore, we have

$$\mathrm{J}_{-1/2}(\mathrm{z}) = \sqrt{\frac{2}{\pi \mathrm{z}}}\cos\mathrm{z}, \quad \mathrm{J}_{1/2}(\mathrm{z}) = \sqrt{\frac{2}{\pi \mathrm{z}}}\sin\mathrm{z}. \tag{1.3}$$

Motivated by these relationships, Srivastava and Miller [15], Klusch [7] and Srivastava and Chen [13], studied rather systematically a unification (and generalization) of the Voigt functions

$$\mathrm{K}(\mathrm{x},\mathrm{y}) = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp\left(-\mathrm{yt} - \frac{\mathrm{t}^2}{4}\right) \cos(\mathrm{xt})\,\mathrm{dt} \tag{1.4}$$

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and

$$L(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp\left(-yt - \frac{t^2}{4}\right) \sin(xt)\, dt \quad (x \in R; y \in R^+) \tag{1.5}$$

in the form

$$V_{\mu,\nu}(x,y,z) = \sqrt{\frac{x}{2}} \int_0^\infty t^\mu \exp\left(-yt - zt^2\right) J_\nu(xt)\, dt \quad (x,y,z \in R^+; Re(\mu+\nu) > -1). \tag{1.6}$$

Now from equations (1.3) - (1.6), it follows that

$$K(x,y) = V_{1/2,-1/2}(x,y) \quad \text{and} \quad L(x,y) = V_{1/2,1/2}(x,y) \tag{1.7}$$

where $V_{\mu,\nu}(x,y) = V_{\mu,\nu}(x,y,1/4)$.

The above two functions K(x,y) and L(x,y) were introduced by Voigt in 1899. Furthermore, the function K(x,y) + i L(x,y) is, except for a numerical factor, identical to the so called plasma dispersion function, which is tabulated by Fried and Conte [3] and by Fettis et.al. [2]. In any physical problem, a numerical or analytical evaluation of the Voigt functions K(x,y) and L(x,y) (or of their aforementioned variants) is required. For a review of various mathematical properties and computational methods concerning the Voigt function, see (for example) Rieche [12], Haubold and John [5], Armstrong and Nicholls [1], Klush [7], Srivastava and Miller [15], Srivastava and Chen [13] and Yang [16]. These function occur in great diversity in astrophysical spectroscopy, neutron physics, plasma physics and statistical communication theory, as well as in some areas in mathematical physics and engineering associated with multi-dimensional analysis of spectral harmonics.

Pathan et.al. [9] introduced and studied the multivariable Voigt functions by means of the integral

$$V_{\mu,\nu_1,\ldots,\nu_n}(x_1,x_2,\ldots,x_n,y) = \left(\frac{x_1}{2}\right)^{1/2}\left(\frac{x_2}{2}\right)^{1/2}\cdots\left(\frac{x_n}{2}\right)^{1/2} \int_0^\infty t^\mu \exp\left(-yt - \frac{t^2}{4}\right)$$

$$\prod_{j=1}^{n} (J_{\nu_j}(x_j t))\, dt \left(\mu, y, x_1, x_2, \ldots, x_n \in R^+; \ Re\left(\mu + \sum_{j=1}^{n} \nu_j\right) > -1\right). \tag{1.8}$$

For n = 1, Equation (1.8) reduces to (1.6) (for z = 1/4).
We note the following relation [9, p.253-254(2.7)]

$$V_{\mu,\nu_1,\ldots,\nu_n}(x_1,x_2,\ldots,x_n,y) = \frac{(2)^{\mu-1/2}(x_1)^{\nu_1+1/2}(x_2)^{\nu_2+1/2}\cdots(x_n)^{\nu_n+1/2}}{\Gamma(\nu_1+1)\,\Gamma(\nu_2+1)\cdots\Gamma(\nu_n+1)} \times$$

$$\{\Gamma(\sigma)\ \psi_2^{(n+1)}\left[\sigma\,;\nu_1+1,\,\nu_2+1,.,\nu_n+1,\,\frac{1}{2};\,-x_1^2,-x_2^2,.,-x_n^2,y^2\right]$$

$$-2y\Gamma\left(\sigma+\frac{1}{2}\right)\ \psi_2^{(n+1)}\left[\sigma+\frac{1}{2}\,;\nu_1+1,\nu_2+1,\ldots,\nu_n+1\,,3/2\,;\,-x_1^2,-x_2^2,.,-x_n^2,\,y^2\right]\Bigg\} \tag{1.9}$$

where $\sigma = \frac{1}{2}\left(\mu + \sum_{j=1}^{n} \nu_j + 1\right)$, $\left(x_1, x_2, ..., x_n \in R\,; \mu, y \in R^+\,; \mathrm{Re}\left(\mu + \sum_{j=1}^{n} \nu_j\right) > -1\right)$ and $\psi_2^{(n)}$ denotes Humbert's confluent hypergeometric function of n variable [14, p.62 (11)].

$\psi_2^{(n)}[a\,; c_1, c_2, ..., c_n;\, x_1, x_2, ..., x_n]$

$$= \sum_{m_1, m_2, ..., m_n = 0}^{\infty} \frac{(a)_{m_1+m_2+...+m_n}\, x_1^{m_1} x_2^{m_2} ... x_n^{m_n}}{(c_1)_{m_1} (c_2)_{m_2} ... (c_n)_{m_n} (m_1)\,!\,(m_2)\,! ... (m_n)\,!}\ (\max\{|x_1|,\, |x_2|\,, ...,\, |x_n|\} < \infty). \tag{1.10}$$

Following the work [7], [9], [13], [15] and [16] closely, Pathan and Shahwan [10, p.77] gave a generalization of Voigt functions which is recalled here in its modified form

$$\Omega_{\mu,\alpha,\beta,\nu}(x, y, z) = \sqrt{\frac{x}{2}} \int_0^{\infty} t^{\mu}\, e^{-yt - zt^2}\, {}_1F_2\left(\alpha, \beta\,; 1 + \nu\,; -\frac{x^2 t^2}{4}\right) dt$$

$$(\mu,\, y,\, z \in R^+,\, x \in R\, and\ \mathrm{Re}(\mu + \nu) > -1). \tag{1.11}$$

Denote $\Omega_{\mu,\alpha,\beta,\nu}(x,\, y,\, 1/4) = \Omega_{\mu,\alpha,\beta,\nu}(x, y)$ and note that for $z = 1/4$, (1.11) reduces to
[9, p.76 (2.1)]
Also

$$\Omega_{\mu,\alpha,\alpha,\nu}(x, y) = \Gamma(\nu + 1)\left(\frac{2}{x}\right)^{\nu} V_{\mu-\nu,\nu}(x\,,\, y) \tag{1.12}$$

when $\alpha = \beta$. In fact, $J_\nu(x)$ defined by (1.1), ${}_1F_2$ and ${}_0F_1$ are contained as special cases, in the generalized hypergeometric function [15, p.42(1)].

We aim here at presenting a new hypergeometric representation of $\Omega_{\mu,\alpha,\beta,\nu}(x, y)$. Connections of the various Voigt functions and their representations are presented and several summation formulae involving Voigt functions and Kampe de Feriet functions are obtained.

2. Representation of $\Omega_{\mu,\alpha,\beta,\nu}(x, y, z)$.

To obtain the representation for the generalized Voigt function $\Omega_{\mu,\alpha,\beta,\nu}(x, y, z)$, we first expand e^{-yt} and ${}_1F_2$ in series in (1.11) and integrate the resulting series term by term with the help of the integral

$$\int_0^{\infty} t^{\lambda}\, e^{-zt^2}\, dt = \frac{1}{2}\,\Gamma\left(\frac{\lambda + 1}{2}\right) z^{-\left(\frac{\lambda+1}{2}\right)}\ ,\ \mathrm{Re}(z) > 0,\ \mathrm{Re}(\lambda) > -1.$$

We thus obtain

$$\Omega_{\mu,\alpha,\beta,\nu}(x, y, z) = \frac{1}{2} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-y)^s\, (-x^2/4)^r\, (\alpha)_r}{s\,!\ \ r\,!\ (\beta)_r\ (\nu + 1)_r}\ \frac{\Gamma\left(\frac{\mu+s+1}{2}\right)\ \left(\frac{\mu+s+1}{2}\right)_r}{z^{\left(\frac{\mu+s+1}{2}\right)+r}}. \tag{2.1}$$

Separating the s-series into its even and odd terms and using $(a + r)_s = (a)_{r+s}\,/\,(a)_r$, we get

$$\Omega_{\mu,\alpha,\beta,\nu}(x,y,z) = \frac{x^{1/2}\ \Gamma(\mu+1)/2}{2^{3/2}\ z^{(\mu+1)/2}}\ F^{1:1;0}_{0:2;1}\left[\begin{matrix} \frac{1+\mu}{2} : & \alpha & ; --- ; \\ -- : & \beta, \nu+1 ; & 1/2 ; \end{matrix} -\frac{x^2}{4z}, \frac{y^2}{4z}\right]$$

$$-\frac{x^{1/2}\ y\ \Gamma(\mu+2)/2}{2^{3/2}\ z^{(\mu+2)/2}}\ F^{1:1;0}_{0:2;1}\left[\begin{matrix} \frac{2+\mu}{2} : & \alpha & ; --- ; \\ -- : & \beta, \nu+1 ; & 3/2 ; \end{matrix} -\frac{x^2}{4z}, \frac{y^2}{4z}\right]$$

$$(x,\ y,\ z\ \in\ R^+\ and\ \mathrm{Re}(\mu)\ >\ -1) \tag{2.2}$$

where $F^{p:q;r}_{l:m;n}$ denotes Kampede Feriet hypergeometric function of two variables [14, p.63].
Setting y = 0, (2.2) would reduce to

$$\Omega_{\mu,\alpha,\beta,\nu}(x,0,z) = \frac{x^{1/2}\ \Gamma(\mu+1)/2}{2^{3/2}\ z^{(\mu+1)/2}}\ {}_2F_2\left[\begin{matrix}\alpha, & \frac{1+\mu}{2} \\ \beta, & \nu+1\end{matrix} ; -\frac{x^2}{4z}\right]. \tag{2.3}$$

In its special case when $\alpha = \beta$, (2.2) would obviously correspond to a result, though in a slightly different form, given by Pathan [8, p.13(4.3)]

$$V_{\mu,\nu}(x,y,z) = \frac{x^{\nu+1/2}\ \Gamma(\mu+\nu+1)}{2^{\nu+1/2}\ y^{(\mu+\nu+1)}\,\Gamma(\nu+1)}\ F^{2:0;0}_{0:1;0}\left[\begin{matrix}\frac{1+\mu+\nu}{2}, \frac{2+\mu+\nu}{2} : & --- & ; --- ; \\ --- : & \nu+1 & ; --- ; \end{matrix} -\frac{x^2}{y^2}, -\frac{4z}{y^2}\right]. \tag{2.4}$$

3. Connection between $\Omega_{\mu,\alpha,\beta,\nu}$ and $V_{\mu,\nu}$

Using standard technique of replacing x by (xt)/2, multiplying by $t^{\mu}\, e^{-yt-zt^2}$ and integrating with respect to t from 0 to ∞, we can apply results [11, p.608].

$${}_1F_2\left(1\,; b, 2-b\,; -x^2\right) - \frac{2(b-1)}{(1-2b)\,(3-2b)}\ {}_1F_2\left(1\,; \frac{1}{2}+b,\ \frac{5}{2}-b\,; -x^2\right)$$

$$= \frac{2\pi\,(b-1)}{\sin\, 2b\pi}\ J_{2b-2}(2x) \tag{3.1}$$

and

$${}_1F_2\left(1\,; b, 1-b\,; -x^2\right) = \frac{\pi\,(b-1)}{\sin\, 2b\pi}\ \left[J_{2b-1}(2x) + J_{2-2b}(2x)\right] \tag{3.2}$$

to obtain connections between $\Omega_{\mu,\alpha,\beta,\nu}$ and $V_{\mu,\nu}$.
Among the results which can be obtained in this way are

$$\Omega_{\mu,1,b,1-b}(x,y,z) - \frac{2(b-1)}{(1-2b)\,(3-2b)}\,\Omega_{\mu,1,1/2+b,3/2-b}(x,y,z) = \frac{2\pi\,(b-1)}{\sin\,2b\pi}\ V_{\mu,2b-2}(x,y,z) \tag{3.3}$$

and

$$\Omega_{\mu,1,b,1-b}(x,y,z) = \frac{\pi\,(b-1)}{\sin\,2b\pi}\ \left[\, V_{\mu,2b-1}(x,y,z) + V_{\mu,2-2b}(x,y,z)\right]. \tag{3.4}$$

4. Summation Formulae.

Now, we shall obtain four summation formulae with the help of the following results recorded in the well known work by Prudnikov et al [11, p.421, eqs. (2), (3), (4) and p.410, eq.(9)].

(i) $$\sum_{k=0}^{\infty} (2k+\nu)\,\frac{(\nu)_k}{k!}\, J_{2k+\nu}(x)\ {}_{p+2}F_q\left(\begin{matrix}-k, k+\nu, (a_p)\,; \\ (b_q) \quad ;\end{matrix}\ y\right) = \frac{(x/2)^{\nu}}{\Gamma(\nu)}\ {}_pF_q\left(\begin{matrix}(a_p)\,; \\ (b_q)\,;\end{matrix}\ -\frac{x^2 y}{4}\right). \tag{4.1}$$

(ii) $\sum_{k=0}^{\infty} (2k+\mu+\nu)\frac{(\mu+\nu)_k}{k!} J_{k+\mu}(x)J_{k+\nu}(x)\,{}_{p+2}F_q\left(\begin{matrix}-k,k+\mu+\nu,(a_p)\,; \\ (b_q)\,;\end{matrix}\ y\right)$

$$= \frac{\mu+\nu}{\Gamma(\mu+1)\,\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\mu+\nu} {}_{p+2}F_{q+2}\left(\begin{matrix}\frac{\mu+\nu+1}{2},\frac{\mu+\nu+2}{2},(a_p)\,; \\ \mu+1,\nu+1,(b_q)\,;\end{matrix}\ -x^2y\right). \tag{4.2}$$

(iii) $\sum_{k=0}^{\infty} \frac{(x/2)^k}{k!} J_{k+\nu}(x)\,{}_{p+1}F_q\left(\begin{matrix}-k,(a_p)\,; \\ (b_q)\,;\end{matrix}\ y\right) = \frac{(x/2)^{\nu}}{\Gamma(\nu+1)}\,{}_pF_{q+1}\left(\begin{matrix}(a_p)\,; \\ (b_q),\nu+1\,;\end{matrix}\ -\frac{x^2y}{4}\right).$ (4.3)

(iv) $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!\,(2-a)_k}\,{}_1F_1\left(\begin{matrix}2-2a\,; \\ 2-a+k\,;\end{matrix}\ x\right)$

$$= {}_1F_2\left(\begin{matrix}a/2\,; \\ 1/2\,,\,(3-a)/2\,;\end{matrix}\ -x^2\right) + x\left(\frac{1-a}{2-a}\right)\ {}_1F_2\left(\begin{matrix}(a+1)/2\,; \\ 3/2\,,\,(2-a)/2\,;\end{matrix}\ -x^2\right). \tag{4.4}$$

Formula 1.

$\sum_{k=0}^{\infty} (2k+\nu)\frac{(\nu)_k}{k!}\,{}_{p+2}F_q\left[\begin{matrix}-k,k+\nu,(a_p)\,; \\ (b_q)\,;\end{matrix}\ y\right] V_{\sigma,2k+\nu}(x,w)$

$$= \frac{(x/2)^{\nu+1/2}}{\Gamma(\nu)}\Gamma(1+\sigma+\nu)\,\Gamma(1/2)\left\{\frac{1}{\Gamma\left(\frac{2+\sigma+\nu}{2}\right)}\ F^{1:p;0}_{0:q;1}\left[\begin{matrix}\frac{1+\sigma+\nu}{2}:(a_p)\,;\ ---\,; \\ ----\ :(b_q)\,;\,1/2\,;\end{matrix}\ -x^2y,w^2\right]\right.$$

$$\left.-\frac{2w}{\Gamma\left(\frac{1+\sigma+\nu}{2}\right)}F^{1:p;0}_{0:q;1}\left[\begin{matrix}\frac{2+\sigma+\nu}{2}\,;(a_p)\,;\,------\,; \\ -----\,;(b_q)\,;\,3/2\,;\end{matrix}\ -x^2y,w^2\right]\right\}. \tag{4.5}$$

Formula 2.

$\sum_{k=0}^{\infty} \frac{(x/2)^k}{k!}\,{}_{p+1}F_q\left[\begin{matrix}-k,(a_p)\,; \\ (b_q)\,;\end{matrix}\ y\right] V_{\sigma+k,k+\nu}(x,w)$

$$= \left(\frac{x}{2}\right)^{\nu+1/2}\frac{\Gamma(1+\sigma+\nu)\,\Gamma(1/2)}{\Gamma(\nu+1)}\left\{\frac{1}{\Gamma\left(\frac{2+\sigma+\nu)}{2}\right)}\ F^{1:p;0}_{0:q+1;1}\left[\begin{matrix}\frac{1+\sigma+\nu}{2}:(a_p)\,;\ ---\,; \\ ----\ :(b_q),\nu+1\,;\,1/2\,;\end{matrix}\ -x^2y,w^2\right]\right.$$

$$\left.-\frac{2w}{\Gamma\left(\frac{1+\sigma+\nu}{2}\right)}F^{1:p;0}_{0:q+1;1}\left[\begin{matrix}\frac{2+\sigma+\nu}{2}\,;(a_p)\,;\,------\,; \\ -----\,;(b_q),\nu+1\,;\,3/2\,;\end{matrix}\ -x^2y,w^2\right]\right\}. \tag{4.6}$$

Formula 3.

$\sum_{k=0}^{\infty} (2k+\mu+\nu)\frac{(\mu+\nu)_k}{k!}\,{}_{p+2}F_q\left[\begin{matrix}-k,k+\mu+\nu,(a_p)\,; \\ (b_q)\,;\end{matrix}\ y\right] V_{\sigma,k+\mu,k+\nu}(x,x,w)$

$$= \frac{(\mu+\nu)\,\Gamma(1+\mu+\nu+\sigma)\,\Gamma(1/2)}{\Gamma(\mu+1)\,\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\mu+\nu+1}\times$$

$$\left\{\frac{1}{\Gamma\left(\frac{2+\mu+\nu+\sigma}{2}\right)}F^{1:p+2;0}_{0:q+2;1}\left[\begin{matrix}\frac{1+\mu+\nu+\sigma}{2}:\frac{1+\mu+\nu}{2},\frac{2+\mu+\nu}{2},(a_p)\,;\,----\,; \\ ----\ :\mu+1,\nu+1,\ (b_q)\ ;\ 1/2\ ;\end{matrix}\ -4x^2y,w^2\right]\right.$$

$$\left.-\frac{2w}{\Gamma\left(\frac{1+\mu+\nu+\sigma}{2}\right)}F^{1:p+2;0}_{0:q+2;1}\left[\begin{matrix}\frac{2+\mu+\nu+\sigma}{2}:\frac{1+\mu+\nu}{2},\frac{2+\mu+\nu}{2},(a_p)\,;\,----\,; \\ -----\ :\mu+1,\nu+1,\ (b_q)\ ;3/2\ ;\end{matrix}\ -4x^2y,w^2\right]\right\}. \tag{4.7}$$

Formula 4.

$$\sum_{k\,,\,r=0}^{\infty} \frac{(-1)^k (x/2)^{2k+2r} (2-2a)_r\, \Gamma(\sigma)}{k!\ r!\ (2-a)_{k+r}\ 2^{-\sigma/2}}\ D_{-\mu-2r-2k-1}(\sqrt{2}\ y)$$

$$= \frac{2^{1/2}}{x^{1/2}}\ e^{-y^2/2}\ \left[\Omega_{\mu\,,\,a/2\,,\,1/2\,,\,(1-a)/2}(x\,,y) + \frac{x\,(1-a)}{2(2-a)}\ \Omega_{\mu+1\,,\,(a+1)/2\,,\,3/2\,,\,(2-a)/2}(x\,,y)\right] \tag{4.8}$$

where $\sigma \ = \ \mu + 2r + 2k + 1$.

Method of Derivation : In the process of establishing the formulae, we use the following

operation ($\wp$) : Replacement of x by xt, multiplication by $t^{\sigma}\exp\left(-wt - \frac{t^2}{4}\right)$ and then integrating with respect to t from 0 to ∞.

Derivation of Formulae 1 to 4. To establish the formula 1, performing the operation ($\wp$) on equation (4.1) and using definition (1.6) (for z =1/4), we obtain

$$\sum_{k=0}^{\infty} (2k+\nu)\, \frac{(\nu)_k}{k!}\ {}_{p+2}F_q \left[\begin{matrix} -k, k+\nu, (a_p)\ ; \\ (b_q)\quad ; \end{matrix}\ y\right]\ V_{\sigma,2k+\nu}\,(x,w)$$

$$= \frac{(x/2)^{\nu+1/2}}{\Gamma(\nu)} \int_0^{\infty} t^{\sigma+\nu}\, e^{-wt-\frac{t^2}{4}}\ {}_pF_q \left[\begin{matrix} (a_p)\ ; \\ (b_q)\ ; \end{matrix}\ -\frac{x^2yt^2}{4}\right]\ dt. \tag{4.9}$$

Now, expanding ${}_pF_q$ in series and interchanging the order of summation and integration, the right hand side of above equation takes the following form

$$\frac{(x/2)^{\nu+1/2}}{\Gamma(\nu)} \sum_{n=0}^{\infty} \frac{(a_p)_n}{(b_q)_n}\, \frac{\left(-\frac{x^2y}{4}\right)^n}{n!} \int_0^{\infty} t^{\sigma+\nu+2n}\, e^{-wt-\frac{t^2}{4}}\, dt.$$

Further, we evaluate the $t-$integral by using the following integral representation [7, p.231 (12)]

$$\int_0^{\infty} t^{\sigma-1} \exp\left(-wt - \frac{t^2}{4}\right) dt = (2)^{\sigma/2}\, \Gamma(\sigma) \exp\left(\frac{w^2}{2}\right)\ D_{-\sigma}(\sqrt{2}w)(w \in\ C; Re(\sigma)\ > 0)$$

$$= \Gamma(1/2)\, \Gamma(\sigma) \left\{ \frac{1}{\Gamma\left(\frac{1+\sigma}{2}\right)}\ {}_1F_1 \left[\begin{matrix} \sigma/2\ ; \\ 1/2\ ; \end{matrix} w^2\right]\ -\ \frac{2w}{\Gamma(\sigma/2)}\ {}_1F_1 \left[\begin{matrix} \frac{1+\sigma}{2}\ ; \\ 3/2\ ; \end{matrix}\ w^2\right] \right\} \tag{4.10}$$

where $D_{-\sigma}(x)$is parabolic cylinder function [14].

Finally, expanding confluent hypergeometric function ${}_1F_1$ in its well known series and using the relations [14, p.22, eqns.(15), (20)] and Legendres duplication formula [14, p.23, eqn.(24)], we arrive at the required result after a little simplification.

The Formulae 2, 3 and 4 can be established on the same lines by performing the operation ($\wp$) on equations (4.2), (4.3) and (4.4) and using definition (1.6) (for $z = 1/4$) respectively.

5. Special Cases.

I. By substituting p = 1, q = 2, $a_1 = \alpha$, $b_1 = \beta$ and $y = 1$ in (4.7), we get an interesting expansion for Voigt function

$$\Omega_{\mu,\alpha,\beta,\lambda}(x,y) \;=\; \left(\frac{x}{2}\right)^{-\nu}\sum_{\mathrm{k}=0}^{\infty}\frac{(2\mathrm{k}+\nu)\,\Gamma(\nu+k)}{\mathrm{k}!}\;{}_3\mathrm{F}_2\left[\begin{matrix}-\mathrm{k}\,,\,\mathrm{k}+\nu\,,\,\alpha\,; \\ \beta\,,\,1+\lambda\quad ;\end{matrix}\;1\right]\;V_{\mu-\nu\,,\,2k+\nu}(x,y) \tag{5.1}$$

which further for $\alpha=\beta$ reduces to

$$V_{\mu-\lambda\,,\,\lambda}(x,y) \;=\; \frac{1}{\Gamma(\lambda+1)}\sum_{\mathrm{k}=0}^{\infty}\frac{(2\mathrm{k}+\nu)\,\Gamma(\nu+k)}{\mathrm{k}!}\;\frac{(\lambda+1-\nu)_k}{(\lambda+1)_k}\;V_{\mu-\nu\,,\,2k+\nu}(x,y) \tag{5.2}$$

by means of the result (1.12) and Gauss Summation Theorem

$${}_2\mathrm{F}_1\left[\begin{matrix}-\mathrm{k}\,,\,\mathrm{k}+\nu\quad ; \\ 1+\lambda\quad ;\end{matrix}\;1\right] \;=\; \frac{(\lambda+1-\nu)_k}{(\lambda+1)_k}.$$

II. An immediate consequence of (4.5) for $y=1$ is another expansion for Voigt function

$$\Omega_{\mu,\alpha,\beta,\lambda}(x,y) \;=\; \Gamma(\nu+1)\left(\frac{x}{2}\right)^{-\nu}\sum_{\mathrm{k}=0}^{\infty}\frac{x^k\,(\beta-\alpha)_k}{2^k\;\mathrm{k}!\;(\beta)_{\mathrm{k}}}\;V_{k+\mu-\nu\,,\,k+\nu}(x,y). \tag{5.3}$$

If $y=0$, it follows from (4.5) that

$$\sum_{\mathrm{k}=0}^{\infty}\frac{(\mathrm{x}/2)^{\mathrm{k}}}{\mathrm{k}!}\,\mathrm{V}_{\sigma+\mathrm{k},\mathrm{k}+\nu}(\mathrm{x},\mathrm{w}) \;=\; \left(\frac{\mathrm{x}}{2}\right)^{\nu+1/2}\frac{\Gamma(1+\sigma+\nu)}{\Gamma(\nu+1)}$$

$$\left\{\frac{1}{\Gamma\left(\frac{2+\sigma+\nu}{2}\right)}\;{}_1F_1\left(\frac{1+\sigma+\nu}{2}\;;\;\frac{1}{2}\;;\;w^2\right)-\frac{2w}{\Gamma\left(\frac{1+\sigma+\nu}{2}\right)}\;{}_1F_1\left(\frac{2+\sigma+\nu}{2}\;;\;\frac{3}{2}\;;\;w^2\right)\right\}. \tag{5.4}$$

III. Taking y = 0 in Formula 3 and using [8, p.212 (5)], we get

$$\sum_{\mathrm{k}=0}^{\infty}(2\mathrm{k}+\mu+\nu)\,\frac{(\mu+\nu)_{\mathrm{k}}}{\mathrm{k}!}\,\mathrm{V}_{\sigma,\mathrm{k}+\mu,\mathrm{k}+\nu}(\mathrm{x},\mathrm{x},\mathrm{w}) = \frac{\Gamma(1+\mu+\nu+\sigma)\,(\mu+\nu)}{\Gamma(\mu+1)\,\Gamma(\nu+1)}\,(x)^{\mu+\nu+1}\times$$

$$2^{\frac{-1-\mu-\nu+\sigma}{2}}\;\mathrm{e}^{\mathrm{w}^2/2}\,\mathrm{D}_{-\mu-\nu-\sigma-1}(\sqrt{2}\,\mathrm{w}). \tag{5.5}$$

IV. Taking y = 0 and using (4.8) in Formula 1, it reduces to the known result [6, p.62, eqn. (3.11)].

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