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# On Almost Locally $h$-Pure QTAG-Modules

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Abstract. Following [5], A module $M_R$ is called a QTAG-module if every finitely generated submodule of every homomorphic image of $M_R$ is a direct sum of uniserial modules. This is a very fascinating structure and many mathematicians worked to generalize the results for abelian groups for these modules. Many interesting results have been surfaced, but there is a lot to explore.
The purpose of this paper is to introduce and investigate the concept of almost locally $h$-pure QTAG-modules. A QTAG-module $M$ is almost locally $h$-pure if every finitely generated submodule of $M$ may be embedded in a finitely generated $h$-pure submodule of $M$. It was found that a QTAG-module is almost locally $h$-pure if and only if it is $h$-reduced, a direct sum of almost locally $h$-pure submodules is almost locally $h$-pure and every submodule of an almost locally $h$-pure QTAG-module is almost locally $h$-pure.

## 1. Introduction

All the rings considered here are associative with identity $1 \neq 0$ and modules are unital QTAG-modules. An element $x \in M$ is uniform if $xR$ is a uniform module and $d(xR)$ is the decomposition length of $xR$. For a uniform element $x \in M$, height of $x$ in $M$ i.e., $H_M(x)$ or simply $H(x) = \sup\{d(U/xR)\}$, where $U$ runs through all the uniserial submodules of $M$ containing $x$. $H_k(M)$ denotes the submodule of $M$ generated by the elements of height at least $k$, and $M$ is $h$-divisible if $H_1(M) = M$. A module $M$ is bounded if there exists an integer $n$ such that $H(x) \leqslant n$, for every uniform $x \in M$. A submodule $N$ of a QTAG-module $M$ is called $h$-pure in $M$ if $N \cap H_k(M) = H_k(N)$ for all $k \geqslant 0$. A submodule $B$ of $M$ is a basic submodule of $M$ if $B = \bigoplus_{i=1}^{\infty} B_i$, where each $B_i$ is a direct sum of uniserial modules of length $i$, $B$ is $h$-pure in $M$ and $M/B$ is $h$-divisible. The submodule $M^1 \subset M$ is generated by the elements of infinite height, equivalently, $M^1 = \bigcap_{k=0}^{\infty} H_k(M) = H_\omega(M)$. A module $M$ is called $h$-reduced if it is free from the elements of infinite height, equivalently, it does not contain any $h$-divisible

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submodule. A module $M$ is uniserial if it has a unique composition series. For the other basic definitions and results we refer to [3,4].

We start with the following:

**Definition 1.1:** A QTAG-module $M$ is almost locally $h$-pure if every finitely generated submodule $\sum x_i R$ of $M$ can be embedded in a finitely generated $h$-pure submodule $N$ of $M$. Equivalently, for every finite set of elements $x_1, \cdots, x_n$ of $M$ there exists a finitely generated $h$-pure submodule $N$ of $M$ containing $x_i{}'s$.

**Remark:** Direct sums of uniserial modules are almost locally $h$-pure.

## 2. Main Results

In this section we prove the main results of the paper.

**Theorem 2.1:** A direct sum of almost locally $h$-pure QTAG-modules is almost locally $h$-pure.

**Proof:** Let $M = \sum_\alpha \bigoplus N_\alpha$, where $N_\alpha$ is almost locally $h$-pure for all $\alpha$. Now consider $x_1, \cdots, x_n \in M$. Since every $x \in M$ is a finite sum of uniform elements, each $x_i$ may be expressed as $x_i = x_{i_1} + x_{i_2} + \cdots + x_{i_n}$. Let $N_\beta$ be a summand having $x_{\beta_i} \neq 0$ for some $i$. Now consider the components $x_{\beta_1}, \cdots, x_{\beta_n}$ of $x_1, \cdots, x_n$ in $N_\beta$. Since each such $N_\beta$ is almost locally $h$-pure, there exists a finitely generated $h$-pure submodule $K_\beta$ containing $x_{\beta_1}, \cdots, x_{\beta_n}$. Then $\sum_\beta \bigoplus K_\beta$ is a finitely generated $h$-pure submodule containing $x_1, \cdots, x_n \in M$. Hence, $M$ is almost locally $h$-pure.

**Theorem 2.2:**

(i) Let $M$ be an almost locally $h$-pure QTAG-module and $N$ a submodule of $M$, and if for every finite set of elements $x_1, \cdots, x_n$ of $M$, there exists a $h$-pure submodule $K$ of $M$ such that the module generated by $N$ and $x_1, \cdots, x_n$ is a submodule of $K$ and $K/N$ is finitely generated, then $M/N$ is almost locally $h$-pure.

(ii) If $M$ and $M/N$ are almost locally $h$-pure, where $N$ is $h$-pure in $M$, then for every finite set of elements $x_1, \cdots, x_n$ of $M$, there exists a $h$-pure submodule $K$ of $M$ such that the module generated by $N$ and $x_1, \cdots, x_n$ is a submodule of $K$ and $K/N$ is finitely generated.

**Proof:** (i) Let us assume that there exists such a $h$-pure submodule $K$ of $M$ for every finite set of elements $x_1, \cdots, x_n$ of $M$. Then if $x_1 + N, \cdots, x_n + N$ are elements of $M/N$, there exists a $h$-pure submodule $K$ of $M$ such that the module generated by $N$ and $x_1, \cdots, x_n$ is a submodule of $K$ and $K/N$ is finitely generated. Now $K/N$ is $h$-pure in $M/N$, then $M/N$ is almost locally $h$-pure.

(ii) Suppose that $M/N$ is almost locally $h$-pure and $x_1, \cdots, x_n \in M$, then there exists a finitely generated $h$-pure submodule $K/N$ of $M/N$ which contains

$x_1 + N, \cdots, x_n + N$. Since we have a natural homomorphism from $K$ to $K/N$, so the inverse image $K$ of $K/N$ has the desired properties.

An immediate consequence of the above theorem is stated below:

**Corollary 2.3:** If $M$ is almost locally $h$-pure and $N$ is finitely generated submodule, then $M/N$ is almost locally $h$-pure.

**Corollary 2.4:** If $T$ is the submodule of an almost locally $h$-pure QTAG-module $M$, then $M/T$ is almost locally $h$-pure.

**Proof:** For $x_1, \cdots, x_n \in M$, let $N$ be a finitely generated $h$-pure submodule containing $x_1, \cdots, x_n$. The submodule $K$ generated by $N$ and $T$ is $h$-pure. Clearly, the submodule generated by $T$ and $x_1, \cdots, x_n$ is a submodule of $K$ and $K/T$ is finitely generated. Hence, by Theorem 2.2, $M/T$ is almost locally $h$-pure.

**Remark 2.5:** Since a QTAG-module $M$ does not contain elements of infinite height if and only if every finitely generated submodule of $M$ is contained in a finitely generated direct summand of $M$, therefore we may conclude that a QTAG-module $M$ is almost $h$-pure if and only if it is $h$-reduced, i.e., every element of $M$ is free from infinite height.

**Lemma 2.6:** The torsion submodule $T$ of an almost locally $h$-pure QTAG-module $M$ is almost locally $h$-pure.

**Proof:** Let $T$ be a torsion submodule of $M$ and $y_1, \cdots, y_n \in T$. Since $M$ is almost locally $h$-pure, then there exists a finitely generated $h$-pure submodule $K$ of $M$ which contains $y_1, \cdots, y_n$. Since $K \cap T$ is a finitely generated $h$-pure submodule of $T$, hence, $T$ is almost locally $h$-pure.

**Theorem 2.7:** Every submodule of an almost locally $h$-pure QTAG-module $M$ is almost locally $h$-pure.

**Proof:** Let $M$ be an almost locally $h$-pure QTAG-module, $K$ an arbitrary submodule of $M$ and $T$ the torsion submodule of $M$. By Corollary 2.4, $M/T$ is almost locally $h$-pure and by Lemma 2.6, $(K \cup T)/T$ is almost locally $h$-pure. Thus $K/(K \cap T)$ is almost locally $h$-pure. Now let $y_1, \cdots, y_n$ be elements of $K$. Since $K/(K \cap T)$ is almost locally $h$-pure, there exists a finitely generated $h$-pure submodule $N/(K \cap T)$ of $K/(K \cap T)$ such that $y_1 + (K \cap T), \cdots, y_n + (K \cap T)$ are elements of $N/(K \cap T)$. Since $N/(K \cap T)$ is finitely generated, so $N = (K \cap T) \oplus L$, where $L$ is finitely generated. Since $L$ is finitely generated, it is almost locally $h$-pure and it follows from lemma 2.6 and Remark 2.5, $K \cap T$ is almost locally $h$-pure. Hence by Theorem 2.1, $N$ is almost locally $h$-pure and $k_1, \cdots, k_n \in N$. Thus there exists a finitely generated $h$-pure submodule $P$ of $N$ containing $k_1, \cdots, k_n$. Since $K \cap T$ is a $h$-pure submodule of $K$, $N$ is a $h$-pure submodule of $K$. Hence $P$ is $h$-pure submodule of $K$ and $K$ is almost locally $h$-pure.

**Theorem 2.8:** If $K$ is $h$-pure in $M$ and if $K$ and $M/K$ are almost locally $h$-pure, then $M$ is almost locally $h$- pure.

**Proof:** Let $x_1, \cdots, x_n \in M$. Since $M/K$ is almost locally $h$-pure, there exists a finitely generated $h$-pure submodule $L/K$ of $M/K$ which contains $x_1+K, \cdots, x_n+K$. Since $K$ is $h$-pure and $L/K$ is finitely generated, $L = K \oplus N$, where $N$ is finitely generated. Now $x_i \in L$ for $i = 1, \cdots, n$. Let $x_i = y_i + z_i$ for $y_i \in K$ and $z_i \in N$. Since $K$ is almost locally $h$-pure, let $P$ be a finitely generated $h$-pure submodule of $K$ which contains $y_i's$. Now $x_i \in P \oplus N$ and $P \oplus N$ is $h$-pure in $L$, which is $h$-pure in $M$. Hence, $P \oplus N$ is a finitely generated $h$-pure submodule of $M$, which contains $x_i's$. Hence, $M$ is almost locally $h$-pure.

**Theorem 2.9:** Every QTAG-module $M$ has a maximal almost locally $h$-pure submodule $N$ (which may be 0) and 0 is the only almost locally $h$-pure submodule of $M/N$.

**Proof:** Using Zorn's lemma, we establish the existence of a maximal almost locally $h$-pure submodule $N$ of $M$. If $K/N$ were a non-zero almost locally $h$-pure submodule of $M/N$, then $K$ would be a $h$-pure submodule of $M$ and by Theorem 2.8, $K$ would be almost locally $h$-pure, which contradicts the maximality of $N$. Hence, 0 is the only almost locally $h$-pure submodule of $M/N$.

**Corollary 2.10:** If $M$ is a QTAG-module and $N$ is a maximal almost locally $h$-pure submodule of $M$, then $M/N$ is $h$-divisible.

**Proof:** Suppose that $M/N$ is not $h$-divisible, then by Theorem 4 in [3], $M/N = K/N \oplus L/N$, with $K/N$ is $h$-divisible and $L/N$ is $h$-reduced and $L/N$ has a submodule $P/N$, which is a summand of $L/N$ and bounded uniserial module. Therefore, $P/N$ is a almost locally $h$-pure submodule of $M/N$, contradicting the maximality of $N$.

**Remark 2.11:** From the above discussion, we may infer that for a QTAG-module $M$ and a countably generated maximal almost locally $h$-pure submodule $N$, $N$ is a basic submodule of $M$.

At the end, we would like to state a problem, which is still open.

Let $N$ be a submodule of an almost locally $h$-pure submodule of $M$ with $M/N$ is countably generated. When $N$ would be the summand of $M$?

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