SCIENTIA
Series A: *Mathematical Sciences*, Vol. 19 (2010), 105–111
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

On spaces with locally countable weak-bases

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ABSTRACT. In this paper, we discuss the relationships between spaces with locally countable weak-bases and spaces with various locally countable networks, establish the relationships between spaces with locally countable weak-bases and locally separable metric spaces, and show that spaces have locally countable weak-bases if and only if they are locally Lindelöf, g-metrizable spaces. These are improvement of the results in [5,6].

1. Introduction

Weak-bases were introduced by A.V.Arhangel'skii [1]. Spaces with locally countable weak-bases were introduced and discussed in [5,6], and some results were showed. For example:

Theorem A [5, 6] The following are equivalent for a space X:
(1) X has a locally countable weak-base.
(2) X is a g-first countable space with a locally countable k-network.
(3) X is a topological sum of g-second countable spaces.
Theorem B [5] A space has a locally countable weak-base if and only if it is a quotient, π (or compact), ss-image of a metric space.

In this paper, we further discuss spaces with locally countable weak-bases. In section 2, we discuss the relationships between spaces with locally countable weak-bases and spaces with various locally countable networks. In section 3, we establish the relationships between spaces with locally countable weak-bases and locally separable metric spaces. In section 4, we show that spaces have locally countable weak-bases if and only if they are locally Lindelöf, g-metrizable spaces.

Throughout this paper, all spaces are regular and T_1, all mappings are continuous and surjective. N denotes the set of all natural numbers. ω denotes $N \cup \{0\}$.

2000 *Mathematics Subject Classification.* 54E99; 54C10.
Key words and phrases. weak-bases; sn-networks; k-networks; cs-networks; sn-first spaces; α_4-space; compact-covering mappings; ss-mappings; g-metrizable spaces.
The work supported by the NSFC (No.11061004, 10771056), the NSF of Hunan Province (No. 09JJ6005) and the Science Research Project of Guangxi University for Nationalities (No. 2010ZD009).

2. The relationships between spaces with locally countable weak-bases and spaces with various locally countable networks

Definition 2.1 Let $\mathcal{P}$ be a cover of a space X.

(1) $\mathcal{P}$ is a network X if, whenever $x \in V$ with V open in X, then $x \in P \subset V$ for some $P \in \mathcal{P}$.

(2) $\mathcal{P}$ is a k-network [17] for X if for each compact subset K of X and its open neighborhood V in X, there exists a finite subfamily $\mathcal{P}'$ of $\mathcal{P}$ such that $K \subset \cup\mathcal{P}' \subset V$.

(3) $\mathcal{P}$ is a cs-network [18] for X if for each $x \in X$, its open neighborhood V in X and a sequence $\{x_n\}$ converging to x in X, there exists $P \in \mathcal{P}$ such that $\{x_n : n \geqslant m\} \cup \{x\} \subset P \subset V$ for some $m \in N$.

(4) $\mathcal{P}$ is a cs^*-network [19] for X if for each $x \in X$, its open neighborhood V in X and a sequence $\{x_n\}$ converging to x in X, there exists a subsequence $\{x_{n_i}\}$ such that $\{x_{n_i} : i \in N\} \cup \{x\} \subset P \subset V$ for some $P \in \mathcal{P}$.

A space X is an $\aleph$-space[5] if X has a σ-locally finite k-network.

Definition 2.2 [12] For a space X and $x \in P \subset X$, P is a sequential neighborhood of x in X if, whenever $\{x_n\}$ is a sequence converging to x in X, then $x_n \in P$ for all but finitely many $n \in N$. P is a sequential open set of X if for each $x \in P$, P is a sequential neighborhood of x in X.

A space X is a sequential space if each sequential open set of X is open in X.

Definition 2.3 Let $\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ be a family of subsets of a space X satisfying that for each $x \in X$,

(1) $\mathcal{P}_x$ is a network of x in X.

(2) If $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

$\mathcal{P}$ is a weak-base [1] for X if $G \subset X$ such that for each $x \in G$, there exists $P \in \mathcal{P}_x$ satisfying $P \subset G$, then G is open in X. $\mathcal{P}$ is an sn-network [10] (i.e., an sequential neighborhood network) for X if each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X, here $\mathcal{P}_x$ is an sn-network of x in X.

A space X is g-first countable [1] (resp. sn-first countable [7]) if X has a weak-base (resp. an sn-network) $\mathcal{P}$ such that each $\mathcal{P}_x$ is countable.

A space X is g-second countable [1] if X has a countable weak-base.

A space X is g-metrizable [4] (resp. sn-metrizable [23])if X has a σ-locally finite weak-base (resp. sn-network) .

For a space, weak-base $\Rightarrow$ sn-network $\Rightarrow$ cs-network $\Rightarrow$ cs^*-network. An sn-network for a sequential space is a weak-base [10].

Definition 2.4 Call a subspace of a space a fan (at a point x) if it consists of a point x, and a countably infinite family of disjoint sequences converging to x. Call a subset of a fan a diagonal if it is a convergent sequence meeting infinitely many of the sequences converging to x and converges to some point in the fan.

(1) A space X is an α_1-space [2, 3] if $T = \{x\} \cup (\cup\{T_n : n \in N\})$ is a fan at x of X, where each sequence T_n converges to x, then there exists a sequence S converging to x such that $T_n \setminus S$ is finite for each $n \in N$.

(2) A space X is an α_4-space [2, 3] if every fan at x of X has a diagonal converging to x.

We have the following implications for a space X[4, 7, 20].

$$\begin{array}{c} g\text{-second countable} \\ \Downarrow \\ \text{metrizable} \Rightarrow g\text{-metrizable} \iff g\text{-first countable} + \aleph\text{-space.} \\ \Downarrow \end{array}$$

k-space $\Leftarrow$ sequential space $\Leftarrow$ g-first countable $\Rightarrow$ sn-first countable $\Rightarrow$ α_1-space $\Rightarrow$ α_4-space.

Lemma 2.5 [15] The following are equivalent for a space X:
(1) X has a locally countable k-network.
(2) X has a locally countable cs-network.
(3) X has a locally countable cs^*-network.

Lemma 2.6 The following are equivalent for a space X:
(1) X has a locally countable sn-network.
(2) X is an sn-first countable space with a locally countable cs-network (k-network, cs^*-network).
(3) X is an α_1-space with a locally countable cs-network (k-network, cs^*-network).
(4) X is an α_4-space with a locally countable cs-network (k-network, cs^*-network).

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4) are clear.

(4)$\Rightarrow$(2) holds by Theorem 3.13 in [7].

(2)$\Rightarrow$(1). Suppose X is an sn-first countable space with a locally countable cs-network. Let $\mathcal{P}$ be a locally countable cs-network for X which is closed under finite intersections. For each $x \in X$, let $\{B(n,x) : n \in N\}$ be a decrease sn-network at x in X. Put

$$\mathcal{F}_x = \{P \in \mathcal{P} : B(n,x) \subset P \text{ for some } n \in N\}.$$
$$\mathcal{F} = \cup\{\mathcal{F}_x : x \in X\}$$

Obviously, $x \in \cap\mathcal{F}_x$ and $\mathcal{F}_x$ is closed under finite intersections. Then $\mathcal{F}$ satisfies Definition 2.3 (1),(2). We claim that each element of $\mathcal{F}_x$ is a sequential neighborhood at x in X. Otherwise, there exists $P \in \mathcal{F}_x$ such that P is not a sequential neighborhood at x in X. Then there exists a sequence $\{x_n\}$ converging to x such that for each $k \in N$, $\{x_n : n > k\} \not\subset P$. Take $x_{n_1} \in \{x_n : n > 1\} \setminus P$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that each $x_{n_{k+1}} \in \{x_n : n > n_k\} \setminus P$. Obviously, x_{n_k} converges to x. Since $P \in \mathcal{F}_x$, then $B(m,x) \subset P$ for some $m \in N$. Because $B(m,x)$ is a sequential neighborhood at x in X, then $\{x\} \cup \{x_{n_k} : k \geqslant j\} \subset B(m,x)$ for some $j \in N$, and so $\{x_{n_k} : k \geqslant j\} \subset P$, a contradiction. Hence $\mathcal{F}$ is an sn-network for X. Obviously, $\mathcal{F} \subset \mathcal{P}$. Therefore $\mathcal{F}$ is a locally countable sn-network for X.

Theorem 2.7 The following are equivalent for a regular space X:
(1) X has a locally countable weak-base.
(2) X is a k-space with a locally countable sn-network.

(3) X is a k-and sn-first countable space with a locally countable cs-network (k-network, cs^*-network).

(4) X is a k-and α_1-space with a locally countable cs-network (k-network, cs^*-network).

(5) X is a k-and α_4-space with a locally countable cs-network (k-network, cs^*-network).

Proof. (1)⇒(2) is obvious.

(2)⇒(1). Suppose X is a k-space with a locally countable sn-network $\mathcal{P}$, then $\mathcal{P}$ is a locally countable cs-network for X. By Lemma 2.5, X has a locally countable k-network. Since a k-space with a point countable k-network is sequential (see [14, Corollary 3.4]), then X is a sequential space. Thus $\mathcal{P}$ is a weak-base for X. Hence X has a locally countable weak-base.

(2) ⇔ (3) ⇔ (4) ⇔ (5) hold by Lemma 2.6.

Corollary 2.8[5, 6] The following are equivalent for a space X:

(1) X has a locally countable weak-base.

(2) X is a g-first countable space with a locally countable k-network.

3. The relationships between spaces with a locally countable weak-base and locally separable metric spaces

Definition 3.1 Let $f : X \to Y$ be a mapping.

(1) f is a compact-covering mapping [16] if each compact subset of Y is the image of some compact subset of X.

(2) f is a compact mapping if for each $y \in Y$, $f^{-1}(y)$ is compact in X.

(3) f is a π-mapping[13] if (X, d) is a metric space and for each $y \in Y$ and its open neighborhood V in $Y, d(f^{-1}(y), X \setminus f^{-1}(V)) > 0$.

(4) f is an ss-mapping [5] if for each $y \in Y$, there exists a open neighborhood V of y in Y such that $f^{-1}(V)$ is separable in X.

Every compact mapping of a metric space is a π-mapping.

Theorem 3.2 The following are equivalent for a space X:

(1) X has a locally countable weak-base.

(2) X is a compact-covering, quotient, compact, ss-image of a locally separable metric space.

(3) X is a quotient, compact, ss-image of a locally separable metric space.

(4) X is a quotient, π, ss-image of a locally separable metric space.

Proof. (1) ⇒ (2). Suppose X has a locally countable weak-base. By Theorem A, X is a topological sum of g-second countable spaces. Let $X = \bigoplus_{\alpha\in\bigwedge} X_\alpha$, where each X_α is a g-second countable space. By Corollary 4.7 in [8], there are a separable metric space M_α and a compact-covering, quotient, compact mapping f_α from M_α onto X_α. Put

$$M = \bigoplus_{\alpha\in\bigwedge} M_\alpha \quad \text{and} \quad f = \bigoplus_{\alpha\in\bigwedge} f_\alpha : M \to X.$$

Then M is a locally separable metric space and f is a quotient, compact, ss-mapping. It suffices to show that f is compact-covering.

For each compact subset K of X, $K \subset \bigcup_{i=1}^{n} X_{\alpha_i}$ for some finitely many $\alpha_i \in \bigwedge$. Since every X_{α_i} is both open and closed in X, $K \cap X_{\alpha_i}$ is compact in X_{α_i}, and so $f_{\alpha_i}(L_i) = K \cap X_{\alpha_i}$ for some compact subset L_i of M_{α_i} for each $i \leqslant n$. Let $L = \bigoplus_{i=1}^{n} L_i$. Then L is compact in M with $f(L) = K$. Hence f is compact-covering.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1) holds by Theorem B.

Remark 3.3 Let Z be the topological sum of the unite interval [0,1], and the family $\{S(x) : x \in [0,1]\}$ of 2^ω convergent sequence $S(x)$. Let X be the space obtained from Z by identifying the limit point of $S(x)$ with $x \in [0,1]$, for each $x \in [0,1]$. Then, from example 2.9.27 in [11] or see example 9.8 in [14], we have the following facts:

(1) X is a compact-covering, quotient, compact image of a locally compact metric space.

(2) X has no point-countable cs-network.

(3) X has no locally countable weak-base.

From the facts above, we have that the condition “ss-” in Theorem 3.2 cannot be omitted.

4. The relationships between spaces with locally countable weak-bases and g-metrizable spaces

Theorem 4.1. Spaces have locally countable weak-bases if and only if they are locally Lindelöf, g-metrizable spaces.

Proof ”if” part is obvious, because every σ-locally finite cover in any locally Lindelöf space is locally countable. The ”only if” part: Suppose a space X has a locally countable weak-base. Then X is a g-first countable space with a locally countable k-network by Theorem A, and so X is a k-space with a locally countable k-network. By Theorem 1 in [9], X is an $\aleph$ -space. Thus X is g-metrizable by Theorem 2.4 in [20]. By Theorem A, X is a topological sum of g-second countable spaces. Since g-second countable spaces is Lindelöf, then X is locally Lindelöf.

Remark 4.2 Let X be the space in [11, Example 2.8.17], then X is not an $\aleph$-space, which has a locally countable k-network. From Lemma 2.5, X has a locally countable cs-network (or cs^*-network). Note that a space is an $\aleph$-space if and only if it has a σ-locally finite cs-network (or cs^*-network)(see [21, Theorem 4]). Thus, X has a locally countable cs-network (k-network, cs^*-network)$\not\Rightarrow$ X has a σ-locally finite cs-network (k-network, cs^*-network).

From Theorem 4.1 and Theorem 1.13 in [4], we have

Corollary 4.3 Let X be a space with a locally countable weak-base. If (1) or (2) below holds, then X is metrizable.

(1) X is a Fréchet space.

(2) X is a q-space.

Corollary 4.4 For a separable space X, the following are equivalent.
(1) X is a g-second countable space.
(2) X has a locally countable weak-base.
(3) X is a g-metrizable space.
Proof. (1) $\Rightarrow$ (2) is obvious.
(2) $\Rightarrow$ (3) holds by Theorem 4.1.
(3) $\Rightarrow$ (1). Suppose X is a separable space with a locally countable weak-base $\mathcal{P}$, then X is a sequential space with a locally countable k-network by Theorem A. Since a sequential space with a σ-locally countable k-network is meta-Lindelöf (see [9, Proposition 1]), and since a separable, meta-Lindelöf space is Lindelöf, then X is Lindelöf. Note that a locally countable family of a Lindelöf space is countable, $\mathcal{P}$ is a countable weak-base for X. This implies that X is g-second countable.

Remark 4.5. It is well-known that a separable, metrizable space is Lindelöf. From the proof of Corollary 4.4, we get that a separable, g-metrizable space is Lindelöf. But, X is a separable, sn-metrizable space $\not\Rightarrow$ X is Lindelöf. In fact, let X be the space in [22, Example 2.3]. Then X is a separable, sn-metrizable space, which has not any countable sn-network. Since a σ-locally finite family of a Lindelöf space is countable, then X is not Lindelöf.

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Received 05 10 2009, revised 12 07 2010

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