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The integrals in Gradshteyn and Ryzhik. Part 1: An Addendum

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Abstract. We present another generalization of a logarithmic integral studied by V. H. Moll in 2007. The family of integrals contains three free parameters and its evaluation involves the harmonic numbers.

1. Motivation

The classical "Table of Integrals, Series and Products" by Gradshteyn and Ryzhik [**3**] contains a huge range of values of definite integrals. In a series of papers beginning in 2007, Moll, Amdeberhan, Medina, Boyadzhiev, Vignat and others established, corrected and generalized many of these formulas. Part 30 [**1**] is probably one of the most recent papers in this series, although Boros and Moll [**2**] expressed the desire to prove all the formulas from [**3**], which is a hard and tortuous task. Moll [**5, 6**] has written excellent books dealing with special integrals of Gradshteyn and Ryzhik [**3**].

Formula 4.232.3 in [**3**] states that

$$\int_0^\infty \frac{\ln x\, dx}{(x+a)(x-1)} = \frac{\pi^2 + \ln^2 a}{2(a+1)}, \quad a > 0. \tag{1.1}$$

This formula is interesting as it allows to derive some related values as well. For instance, with $a = \alpha^2$ and $a = \alpha^{-2}$ ($\alpha = \frac{1+\sqrt{5}}{2}$) upon combining we get, the formulas

$$\int_0^\infty \frac{\ln x\, dx}{(x-1)(x^2+3x+1)} = \frac{1}{5}\int_0^\infty \frac{(2x+3)\ln x\, dx}{(x-1)(x^2+3x+1)} = \frac{\pi^2 + 4\ln^2\alpha}{10},$$

$$\int_0^\infty \frac{(x+1)\ln x\, dx}{(x-1)(x^2+3x+1)} = \frac{\pi^2 + 4\ln^2\alpha}{5}.$$

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Also, with $a = \alpha$ and $a = -\beta = \alpha^{-1}$, in turn, from (1.1) we have the following interesting integrals:

$$\int_0^\infty \frac{(x+1-\alpha)\ln x\,dx}{(x^2+x-1)(x-1)} = \frac{\pi^2+\ln^2\alpha}{2\alpha^2},$$
$$\int_0^\infty \frac{(x-\alpha)\ln x\,dx}{(x^2-x-1)(x-1)} = \frac{\pi^2+\ln^2\alpha}{2\alpha}.$$

In the very first paper of the above series [**4**], Moll generalized (1.1) by considering the family of logarithmic integrals

$$f_n(a) = \int_0^\infty \frac{\ln^{n-1} x\,dx}{(x+a)(x-1)}, \quad n \geqslant 2,\ a > 0.$$

Moll proved that

$$\begin{aligned} f_n(a) &= \frac{(-1)^n(n-1)!}{a+1}\Big((1-(-1)^{n-1})\zeta(n) - \operatorname{Li}_n\left(-\frac{1}{a}\right) + (-1)^{n-1}\operatorname{Li}_n(-a)\Big) \\ &= \frac{(-1)^n(n-1)!}{a+1}\Big((1-(-1)^{n-1})\zeta(n) \\ &\quad - \frac{1}{n(a+1)}\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\binom{n}{2j}(2^{2j}-2)\pi^{2j}B_{2j}\ln^{n-2j} a\Big), \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function, $\operatorname{Li}_n(z)$ is the polylogarithm and B_n are the Bernoulli numbers.

In this paper we provide an addendum to Moll's paper by considering the different family of integrals

$$F(m,k,a) = \int_0^\infty \frac{x^m \ln x\,dx}{(x-1)(x+a)^{k+m+1}}, \tag{1.2}$$

where the three parameters satisfy $m, k \in \mathbb{N}_0$ and $a > 0$. We require the following lemma in the sequel.

LEMMA 1. *If c is an arbitrary constant and $s \in \mathbb{R}$, then*

(1.3)
$$\frac{d^k}{da^k}\left(\frac{a+c}{(ax+1)^s}\right) = \frac{(-1)^k k! x^{k-1}}{(ax+1)^{k+s}}\left((a+c)x\binom{s+k-1}{s-1} - (ax+1)\binom{s+k-2}{s-1}\right).$$

PROOF. Leibniz rule gives

$$\frac{d^k}{da^k}\left(\frac{a+c}{(ax+1)^s}\right) = (a+c)\frac{d^k}{da^k}\left(\frac{1}{(ax+1)^s}\right) + k\frac{d}{da}(a+c)\frac{d^{k-1}}{da^{k-1}}\left(\frac{1}{(ax+1)^s}\right),$$

from which (1.3) follows, since

$$\frac{d^k}{da^k}\left(\frac{1}{(ax+1)^s}\right) = (-1)^k\frac{(s+k-1)!x^k}{(s-1)!(ax+1)^{k+s}}.$$

□

2. The evaluations of $F(m,k,a)$ for $m = 0, 1, 2$

Before deriving the general expression for $F(m,k,a)$ we study in detail some special cases. First we prove the following formula for $F(0,k,a)$.

THEOREM 2. *For $k \in \mathbb{N}_0$ and $a > 0$, we have*

$$F(0,k,a) = \frac{1}{(a+1)^{k+1}} \left(\frac{\pi^2 + \ln^2 a}{2} + \sum_{j=0}^{k-1} \frac{\left(1 + 1/a\right)^{j+1}}{j+1} (H_j - \ln a) \right) \tag{2.1}$$

with $H_n = \sum_{k=1}^{n} \frac{1}{k}$, $H_0 = 0$, being the harmonic numbers.

PROOF. Starting with (1.1) we differentiate both sides k times with respect to a to get

$$\begin{aligned} (-1)^k k! \int_0^\infty \frac{\ln x \, dx}{(x+a)^{k+1}(x-1)} &= \frac{1}{2} \frac{d^k}{da^k} \left(\frac{\pi^2 + \ln^2 a}{a+1} \right) \\ &= \frac{1}{2} \sum_{j=0}^{k} \binom{k}{j} ((a+1)^{-1})^{(j)} (\pi^2 + \ln^2 a)^{(k-j)}, \end{aligned}$$

where we have used the Leibniz rule for derivatives. We have

$$\frac{d^k}{da^k} \left(\frac{1}{x+a} \right) = \frac{(-1)^k \, k!}{(x+a)^{k+1}}, \quad k \geqslant 0. \tag{2.2}$$

Now, assuming that

$$\frac{d^k}{da^k} (\pi^2 + \ln^2 a) = \frac{X_k}{a^k} + \frac{Y_k \, \ln a}{a^k}$$

we get the recurrences, for $k \geqslant 1$,

$$X_{k+1} = Y_k - kX_k \quad \text{and} \quad Y_{k+1} = -kY_k,$$

with $X_1 = 0$ and $Y_1 = 2$. The recurrence for Y_k is solved straightforwardly and the result is $Y_k = 2(-1)^{k-1} \, (k-1)!$. This gives

$$\begin{aligned} X_{k+1} &= \sum_{j=0}^{k-1} (-1)^j j! \binom{k}{j} Y_{k-j} \\ &= 2(-1)^{k-1} k! \sum_{j=0}^{k-1} \frac{1}{k-j} = 2(-1)^{k-1} k! \, H_k, \end{aligned}$$

and finally, for $k \geqslant 1$

$$\frac{d^k}{da^k} (\pi^2 + \ln^2 a) = \frac{2(-1)^k (k-1)!}{a^k} (H_{k-1} - \ln a). \tag{2.3}$$

The formula (2.1) follows upon simplifications. □

For $k = 0$ in (2.1) we get (1.1). The next two cases are

$$F(0,1,a) = \frac{a(\pi^2 + \ln^2 a) - 2(a+1) \ln a}{2a^2(a+1)^2},$$

and

$$F(0,2,a)=\frac{a^2(\pi^2+\ln^2 a)-(a+1)(3a+1)\ln a+(a+1)^2}{2a^2(a+1)^3}.$$

COROLLARY 3. *For $k\in\mathbb{N}_0$, we have*

$$\begin{aligned}\int_0^\infty \frac{\ln x\sum\limits_{j=0}^{k+1}\binom{k+1}{j}L_{2(k+1-j)}x^j}{(x-1)(x^2+3x+1)^{k+1}}\,dx &= \frac{\pi^2+4\ln^2\alpha}{2\cdot 5^{k/2}}\begin{cases}\frac{L_{k+1}}{\sqrt5}, & \text{if } k \text{ is odd;}\\ F_{k+1}, & \text{if } k \text{ is even}\end{cases}\\ &+\frac{(-1)^k}{5^{k/2}}\sum_{j=0}^{k-1}\frac{(-1)^j5^{j/2}}{j+1}\Big(H_j\big(L_{k+2+j}+\alpha^{k+2+j}((-1)^{k-j}-1)\big)\\ &-2\ln\alpha\big(L_{k+2+j}-\alpha^{k+2+j}((-1)^{k-j}+1)\big)\Big)\end{aligned}\tag{2.4}$$

and

$$\begin{aligned}\int_0^\infty \frac{\ln x\sum\limits_{j=0}^{k+1}\binom{k+1}{j}F_{2(k+1-j)}x^j}{(x-1)(x^2+3x+1)^{k+1}}\,dx &= \frac{\pi^2+4\ln^2\alpha}{2\cdot 5^{(k+1)/2}}\begin{cases}F_{k+1}, & \text{if } k \text{ is odd;}\\ \frac{L_{k+1}}{\sqrt5}, & \text{if } k \text{ is even}\end{cases}\\ &-\frac{(-1)^k}{5^{(k+1)/2}}\sum_{j=0}^{k-1}\frac{(-1)^j5^{j/2}}{j+1}\Big(H_j\big(L_{k+2+j}-\alpha^{k+2+j}((-1)^{k-j}+1)\big)\\ &-2\ln\alpha\big(L_{k+2+j}+\alpha^{k+2+j}((-1)^{k-j}-1)\big)\Big)\end{aligned}\tag{2.5}$$

with F_n (L_n) being the Fibonacci (Lucas) numbers and where $\alpha=(1+\sqrt5)/2$ is the golden ratio.

PROOF. To get (2.4) insert $a=\alpha^2$ and $a=\beta^2=\alpha^{-2}$, in turn, in (2.1) and add the expressions. When simplifying use the relations $\alpha^2+1=\sqrt5\alpha$ and $\beta^2+1=-\sqrt5\beta$ as well as

$$\alpha^{k+1}+(-1)^{k+1}\beta^{k+1}=\begin{cases}L_{k+1}, & \text{if } k \text{ is odd;}\\ \sqrt5F_{k+1}, & \text{if } k \text{ is even.}\end{cases}$$

Identity (2.5) is obtained by subtraction using

$$\alpha^{k+1}-(-1)^{k+1}\beta^{k+1}=\begin{cases}L_{k+1}, & \text{if } k \text{ is even;}\\ \sqrt5F_{k+1}, & \text{if } k \text{ is odd.}\end{cases}$$

□

When $k = 0$ and $k = 1$ then Corollary 3 yields the following results as particular cases:

$$\int_0^\infty \frac{(2x+3)\ln x\,dx}{(x-1)(x^2+3x+1)} = \frac{\pi^2}{2} + 2\ln^2\alpha, \tag{2.6}$$

$$\int_0^\infty \frac{\ln x\,dx}{(x-1)(x^2+3x+1)} = \frac{\pi^2}{10} + \frac{2}{5}\ln^2\alpha, \tag{2.7}$$

$$\int_0^\infty \frac{(2x^2+6x+7)\ln x\,dx}{(x-1)(x^2+3x+1)^2} = \frac{3\pi^2}{10} + \frac{6}{5}\ln^2\alpha + \frac{8}{\sqrt{5}}\ln\alpha \tag{2.8}$$

and

$$\int_0^\infty \frac{(2x+3)\ln x\,dx}{(x-1)(x^2+3x+1)^2} = \frac{\pi^2}{10} + \frac{2}{5}\ln^2\alpha + \frac{4}{\sqrt{5}}\ln\alpha. \tag{2.9}$$

Since, as is easily shown from (2.6) and (2.7),

$$\int_0^\infty \frac{x\ln x\,dx}{(x-1)(x^2+3x+1)} = \frac{\pi^2+4\ln^2\alpha}{10},$$

it follows that

$$\int_0^\infty \frac{(sx+q)\ln x\,dx}{(x-1)(x^2+3x+1)} = (s+q)\frac{\pi^2+4\ln^2\alpha}{10},$$

for arbitrary s and q. Similarly, from (2.8) and (2.9) we have

$$\int_0^\infty \frac{(sx^2+qx+r)\ln x\,dx}{(x^2+3x+1)^2} = \frac{2(s-r)}{\sqrt{5}}\ln\alpha,$$

for arbitrary s, q and r.

COROLLARY 4. *If $k \in \mathbb{N}_0$ and r is an even integer, then*

(2.10)

$$\int_0^\infty \frac{\ln x \sum_{j=0}^{k+1}\binom{k+1}{j} L_{2rj}x^{k+1-j}}{(x-1)(x^2+L_{2r}x+1)^{k+1}}\,dx = \frac{L_{r(k+1)}}{2L_r^{k+1}}(\pi^2+4r^2\ln^2\alpha) + \frac{1}{L_r^k}\sum_{j=0}^{k-1}\frac{L_r^j}{j+1}\left(L_{r(k+2+j)}H_j + 2\sqrt{5}rF_{r(k+2+j)}\ln\alpha\right),$$

(2.11)

$$\int_0^\infty \frac{\ln x \sum_{j=0}^{k+1}\binom{k+1}{j} F_{2rj}x^{k+1-j}}{(x-1)(x^2+L_{2r}x+1)^{k+1}}\,dx = \frac{F_{r(k+1)}}{2L_r^{k+1}}(\pi^2+4r^2\ln^2\alpha) + \frac{1}{L_r^k}\sum_{j=0}^{k-1}\frac{L_r^j}{j+1}\left(F_{r(k+j+2)}H_j + \frac{2rL_{r(k+2+j)}}{\sqrt{5}}\ln\alpha\right).$$

PROOF. Consider $F(0,k,\alpha^{2r}) \pm F(0,k,\beta^{2r})$, using (1.2) and (2.1); and the fact that if r is an even integer, then $\alpha^{2r}+1 = \alpha^r L_r$ and $\beta^{2r}+1 = \beta^r L_r$. □

THEOREM 5. *If $a > 0$ and $k \in \mathbb{N}$, then*

$$\int_0^\infty \frac{\ln x\, dx}{(x+a)^{k+1}} = \frac{\ln a - H_{k-1}}{ka^k}. \tag{2.12}$$

PROOF. Write (1.1) as

$$2\int_0^\infty \frac{(a+1)\ln x\, dx}{(ax+1)(x-1)} = \pi^2 + \ln^2 a;$$

differentiate both sides k times with respect to a, making use of (1.3) and (2.3). Write $1/a$ for a. □

Note that (2.12) is equivalent to Gradshteyn and Ryzhik [**3**, 4.253.6]; in which case the harmonic number is expressed in terms of the digamma function, thereby removing the restriction on k.

COROLLARY 6. *If $a, k > 0$ and $m > 1$, then*

$$\int_0^\infty \frac{x^{k-1}\ln x}{(x+a)^{k+m}}\left(x\binom{m+k-2}{m-2} - a\binom{m+k-2}{m-1}\right)dx = \frac{1}{(m-1)ka^{m-1}}.$$

PROOF. Write $1/a$ for a and $m-1$ for k in (2.12) to obtain

$$\int_0^\infty \frac{a\ln x\, dx}{(ax+1)^m} = \frac{\ln a + H_{m-2}}{1-m}.$$

Differentiate the above expression k times with respect to a, using (1.3) and (2.15). Finally, write a for $1/a$. □

The integral $F(1,k,a)$ is evaluated in the next theorem.

THEOREM 7. *For $k \in \mathbb{N}_0$ and $a > 0$, we have*

$$\begin{aligned} F(1,k,a) &= \frac{\pi^2 + \ln^2 a}{2(a+1)^{k+2}} + \frac{\ln a}{(k+1)(a+1)^{k+1}} \\ &\quad + \frac{1}{(k+1)(a+1)^{k+1}}\sum_{j=0}^{k-1}\frac{\left(1+1/a\right)^{j+1}}{j+1}\left(\frac{k-j}{a+1}\left(H_j - \ln a\right) - 1\right). \end{aligned} \tag{2.13}$$

PROOF. We start with the observation that

$$\int_0^\infty \frac{x\ln x\, dx}{(x+a)^2(x-1)} = \frac{\pi^2+\ln^2 a}{2(a+1)^2} + \frac{\ln a}{a+1}. \tag{2.14}$$

This is true since we have

$$\int \frac{x\ln x\, dx}{(x+a)^2(x-1)} = g(x,a)$$

with (the constant C is not displayed)

$$\begin{aligned} g(x,a) = -\frac{1}{(a+1)^2}\Big(&\mathrm{Li}_2\left(-\frac{x}{a}\right) + \mathrm{Li}_2(1-x) \\ &+ \ln x \ln\left(1+\frac{x}{a}\right) + \frac{a(a+1)\ln x}{x+a} + (a+1)\ln\left(1+\frac{a}{x}\right)\Big). \end{aligned} \tag{2.15}$$

Now, taking the limits $\lim_{x\to\infty} g(x,a)$ and $\lim_{x\to 0} g(x,a)$ leads us to (2.14). The remainder of the proof is the same as in Theorem 2 using (2.2) and

$$\frac{d^k}{da^k}\ln a = \frac{(-1)^{k-1}\,(k-1)!}{a^k}, \quad k \geqslant 1.$$

□

For $k=0$ in (2.13) we get (2.14). The next two cases are

$$F(1,1,a) = \frac{a(\pi^2+\ln^2 a)+(a^2-1)\ln a-(a+1)^2}{2a(a+1)^3}$$

and

$$F(1,2,a) = \frac{3a^2(\pi^2+\ln^2 a)+(a+1)(2a^2-5a-1)\ln a-3a(a+1)^2}{6a^2(a+1)^4}.$$

To derive a formula for $F(2,k,a)$ we need the next lemma.

LEMMA 8. *For $k \in \mathbb{N}_0$, the following formula holds*

$$\frac{d^k}{da^k}\left(\frac{a+3}{(a+1)^2}\right) = (-1)^k\, k!\,\frac{a+3+2k}{(a+1)^{k+2}}\,.$$

PROOF. Use $c=3$, $s=2$ and $x=1$ in (1.3). □

The integral $F(2,k,a)$ admits the following evaluation.

THEOREM 9. *For $k \in \mathbb{N}_0$ and $a>0$, we have*

$$F(2,k,a) = \frac{\pi^2+\ln^2 a}{2(a+1)^{k+3}} + \frac{1}{(k+1)(k+2)(a+1)^{k+1}}\left(\frac{a+3+2k}{a+1}\ln a + 1 + \frac{1}{a}\sum_{j=0}^{k-1}\frac{\big(1+1/a\big)^j}{j+1}\left(\frac{(k-j)(k+1-j)}{a+1}\big(H_j-\ln a\big)-a-1-2(k-j)\right)\right).$$

PROOF. The proof is similar to the previous two proofs. □

When $k=0$ then we get

$$F(2,0,a) = \frac{\pi^2+\ln^2 a}{2(a+1)^3} + \frac{a+3}{2(a+1)^2}\ln a + \frac{1}{2(a+1)}.$$

3. The general case

Here we state a general formula for $F(m,k,a)$. The structure of such a formula is indicated in the above analysis. Our main argument is not to try to derive an explicit expression for the indefinite integral

$$\int \frac{x^m \ln x\,dx}{(x-1)(x+a)^{m+1}}$$

but instead using the results from the first part of the paper.

THEOREM 10. *For $m,k\in\mathbb{N}_0$ and $a>0$, we have*

$$\begin{aligned}F(m,k,a)&=\int_0^\infty \frac{x^m\ln x\,dx}{(x-1)(x+a)^{k+m+1}}\\&=\frac{\pi^2}{2(a+1)^{k+m+1}}+\frac{(-1)^m}{2m!\binom{k+m}{m}a^{k+m+1}}\frac{d^m}{db^m}\left(\frac{\ln^2 b}{(b+1)^{k+1}}\right)\Big|_{b=1/a}\\&\quad+\sum_{j=0}^{k-1}\frac{\binom{j+m}{m}}{\binom{k+m}{m}}\frac{H_{k-j-1}}{k-j}\frac{a^{j-k}}{(a+1)^{j+m+1}}\\&\quad+\frac{(-1)^m}{m!\binom{k+m}{m}a^{k+m+1}}\sum_{j=0}^{k-1}\frac{1}{k-j}\frac{d^m}{db^m}\left(\frac{\ln b}{(b+1)^{j+1}}\right)\Big|_{b=1/a}.\end{aligned}\tag{3.1}$$

PROOF. Using $F(0,k,1/a)$ from Theorem 2 we find

$$\int_0^\infty\frac{\ln x\,dx}{(x-1)(ax+1)^{k+1}}=\frac{\pi^2+\ln^2 a}{2(a+1)^{k+1}}+\sum_{j=0}^{k-1}\frac{H_{k-j-1}+\ln a}{(k-j)(a+1)^{j+1}}.\tag{3.2}$$

Differentiating (3.2) m times with respect to a and replacing a with $1/a$ gives (3.1). □

In particular, $F(m,0,a)$ equals

$$\begin{aligned}\int_0^\infty\frac{x^m\ln x\,dx}{(x-1)(x+a)^{m+1}}&=\frac{\pi^2}{2(a+1)^{m+1}}+\frac{(-1)^m}{2m!a^{m+1}}\frac{d^m}{db^m}\left(\frac{\ln^2 b}{(b+1)}\right)\Big|_{b=1/a}\\&=\frac{1}{2(a+1)^{m+1}}\left(\pi^2+\ln^2 a+2\sum_{j=0}^{m-1}\frac{(a+1)^{j+1}}{j+1}\big(H_j+\ln a\big)\right),\end{aligned}$$

since

$$\frac{d^m}{db^m}\left(\frac{\ln^2 b}{b+1}\right)\Big|_{b=1/a}=(-1)^m m!\frac{a^{m+1}\ln^2 a}{(a+1)^{m+1}}+(-1)^m 2m!a^{m+1}\sum_{j=0}^{m-1}\frac{H_{m-j-1}+\ln a}{(m-j)(a+1)^{j+1}}.$$

THEOREM 11. *If $m\in\mathbb{N}_0$ and r is an even integer, then*

$$\begin{aligned}&\int_0^\infty\frac{\ln x\sum_{j=0}^{m+1}\binom{m+1}{j}L_{2rj}\,x^{2m+1-j}}{(x-1)(x^2+L_{2r}x+1)^{m+1}}\,dx\\&\quad=\frac{L_{r(m+1)}}{2L_r^{m+1}}(\pi^2+4r^2\ln^2\alpha)+\frac{1}{L_r^m}\sum_{j=0}^{m-1}\frac{L_r^j}{j+1}\left(L_{r(m-j)}H_j-2\sqrt{5}r\ln\alpha F_{r(m-j)}\right),\end{aligned}\tag{3.3}$$

(3.4)

$$\int_0^\infty \frac{\ln x \sum_{j=0}^{m+1} \binom{m+1}{j} F_{2rj}\, x^{2m+1-j}}{(x-1)(x^2+L_{2r}x+1)^{m+1}}\,dx = \frac{F_{r(m+1)}}{2L_r^{m+1}}(\pi^2+4r^2\ln^2\alpha) + \frac{1}{L_r^m}\sum_{j=0}^{m-1}\frac{L_r^j}{j+1}\left(F_{r(m-j)}H_j - \frac{2\sqrt{5}}{5}r\ln\alpha L_{r(m-j)}\right).$$

PROOF. Evaluate $F(m,0,\alpha^{2r}) \pm F(m,0,\beta^{2r})$. □

COROLLARY 12. *If $m,k \in \mathbb{N}_0$ and r is an even integer, then*

(3.5)

$$\int_0^\infty \frac{x^{k+1}(x^k-1)\ln x \sum_{j=0}^{k+1}\binom{k+1}{j}\frac{L_{2rj}}{x^j}}{(x-1)(x^2+L_{2r}x+1)^{k+1}}\,dx = -\frac{5}{L_r^k}\sum_{j=0}^{k-1}\frac{L_r^jF_{r(k+1)}}{j+1}\Big(F_{r(j+1)}H_j + \frac{2\sqrt{5}r}{5}\ln\alpha L_{r(j+1)}\Big),$$

(3.6)

$$\int_0^\infty \frac{x^{k+1}(x^k+1)\ln x \sum_{j=0}^{k+1}\binom{k+1}{j}\frac{L_{2rj}}{x^j}}{(x-1)(x^2+L_{2r}x+1)^{k+1}}\,dx = \frac{L_{r(k+1)}}{L_r^{k+1}}(\pi^2+4r^2\ln^2\alpha) + \frac{L_{r(k+1)}}{L_r^k}\sum_{j=0}^{k-1}\frac{L_r^j}{j+1}\left(L_{r(j+1)}H_j + 2\sqrt{5}r\ln\alpha F_{r(j+1)}\right),$$

(3.7)

$$\int_0^\infty \frac{x^{k+1}(x^k-1)\ln x \sum_{j=0}^{k+1}\binom{k+1}{j}\frac{F_{2rj}}{x^j}}{(x-1)(x^2+L_{2r}x+1)^{k+1}}\,dx = -\frac{L_{r(k+1)}}{L_r^k}\sum_{j=0}^{k-1}\frac{L_r^j}{j+1}\Big(F_{r(j+1)}H_j + \frac{2\sqrt{5}r}{5}\ln\alpha L_{r(j+1)}\Big),$$

(3.8)

$$\int_0^\infty \frac{x^{k+1}(x^k+1)\ln x \sum_{j=0}^{k+1}\binom{k+1}{j}\frac{F_{2rj}}{x^j}}{(x-1)(x^2+L_{2r}x+1)^{k+1}}\,dx = \frac{F_{r(k+1)}}{L_r^{k+1}}(\pi^2+4r^2\ln^2\alpha) + \frac{F_{r(k+1)}}{L_r^k}\sum_{j=0}^{k-1}\frac{L_r^j}{j+1}\left(L_{r(1+j)}H_j + 2\sqrt{5}r\ln\alpha F_{r(1+j)}\right).$$

PROOF. Set $m=k$ in (3.3); subtract/add (2.10) to obtain (3.5)/(3.6). Similarly, (3.7) and (3.8) follow from (2.11) and (3.4). Note the use of the following identities that are valid for all integers u and v having the same parity:

$$F_u + F_v = \begin{cases} L_{(u-v)/2}F_{(u+v)/2}, & \text{if } (u-v)/2 \text{ is even;} \\ F_{(u-v)/2}L_{(u+v)/2}, & \text{if } (u-v)/2 \text{ is odd,} \end{cases}$$

$$F_u - F_v = \begin{cases} L_{(u-v)/2}F_{(u+v)/2}, & \text{if } (u-v)/2 \text{ is odd;} \\ F_{(u-v)/2}L_{(u+v)/2}, & \text{if } (u-v)/2 \text{ is even,} \end{cases}$$

$$L_u + L_v = \begin{cases} L_{(u-v)/2}L_{(u+v)/2}, & \text{if } (u-v)/2 \text{ is even;} \\ 5F_{(u-v)/2}F_{(u+v)/2}, & \text{if } (u-v)/2 \text{ is odd,} \end{cases}$$

$$L_u - L_v = \begin{cases} L_{(u-v)/2}L_{(u+v)/2}, & \text{if } (u-v)/2 \text{ is odd;} \\ 5F_{(u-v)/2}F_{(u+v)/2}, & \text{if } (u-v)/2 \text{ is even.} \end{cases}$$

□

Differentiating (3.3) k times with respect to a gives the following alternative to (3.1).

THEOREM 13. *For $m, k \in \mathbb{N}_0$ and $a > 0$, we have*

$$\begin{aligned} F(m,k,a) &= \int_0^\infty \frac{x^m \ln x\, dx}{(x-1)(x+a)^{k+m+1}} \\ &= \frac{\pi^2}{2(a+1)^{k+m+1}} + \frac{(-1)^k}{2k!\binom{k+m}{m}} \frac{d^k}{da^k}\left(\frac{\ln^2 a}{(a+1)^{m+1}}\right) \\ &\quad + \sum_{j=0}^{m-1} \frac{\binom{k+j}{j}}{\binom{k+m}{m}} \frac{H_{m-j-1}}{m-j} \frac{1}{(a+1)^{j+k+1}} + \frac{(-1)^k}{k!\binom{k+m}{m}} \sum_{j=0}^{m-1} \frac{1}{m-j} \frac{d^k}{da^k}\left(\frac{\ln a}{(a+1)^{j+1}}\right). \end{aligned}$$

THEOREM 14. *For $k \in \mathbb{N}_0$ and $a > 0$, we have*

$$\begin{aligned} &\int_0^\infty \frac{(x^k-1)\ln x\, dx}{(x-1)(x+a)^{k+1}} \\ &\qquad = \frac{1}{(a+1)^{k+1}} \sum_{j=0}^{k-1} \frac{(1+1/a)^{j+1}}{j+1} \left((a^{j+1}-1)H_j + (a^{j+1}+1)\ln a\right), \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\int_0^\infty \frac{(x^k+1)\ln x\, dx}{(x-1)(x+a)^{k+1}} \\ &\qquad = \frac{\pi^2 + \ln^2 a}{(a+1)^{k+1}} + \frac{1}{(a+1)^{k+1}} \sum_{j=0}^{k-1} \frac{(1+1/a)^{j+1}}{j+1} \left((a^{j+1}+1)H_j + (a^{j+1}-1)\ln a\right). \end{aligned}$$

PROOF. Evaluate $F(k,0,a) \pm F(0,k,a)$. □

Note that (3.9) can also be written as

$$\int_0^\infty \frac{\ln x \sum\limits_{j=0}^{k-1} x^j}{(x+a)^{k+1}} dx = \frac{1}{(a+1)^{k+1}} \sum_{j=0}^{k-1} \frac{(1+1/a)^{j+1}}{j+1} \left((a^{j+1}-1)H_j + (a^{j+1}+1)\ln a\right).$$

References

[1] T. Amdeberhan, A. Dixit, X. Guan, L. Jiu, A. Kuznetsov, V. H. Moll and Ch. Vignat, *The integrals in Gradshteyn and Ryzhik. Part 30: Trigonometric functions,* Sci. Ser. A Math. Sci. (N.S.) 27 (2016), pp. 47–74.

[2] G. Boros and V. H. Moll, *Irresistible Integrals Symbolics, Analysis and Experiments in the Evaluation of Integrals,* Cambridge University Press, 2004.

[3] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products,* Elsevier Academic Press, 2007.

[4] V. H. Moll, *The integrals in Gradshteyn and Ryzhik. Part 1: A family of logarithmic integrals,* Sci. Ser. A Math. Sci. (N.S.), 14 (2007), pp. 1–6.

[5] V. H. Moll, *Special Integrals of Gradshteyn and Ryzhik: The Proofs. Vol. 1,* CRC Press, 2015.

[6] V. H. Moll, *Special Integrals of Gradshteyn and Ryzhik: The Proofs. Vol. 2,* CRC Press, 2016.

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