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The integrals in Gradshteyn and Ryzhik. Part 16: Complete elliptic integrals

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ABSTRACT. The table of Gradshteyn and Ryzhik contains many entries that are related to elliptic integrals. We present a systematic derivation of some of them.

1. Introduction

Elliptic integrals were at the center of analysis at the end of 19^{th}-century. The **complete elliptic integral of the first kind** defined by

$$\mathbf{K}(k) := \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{1.1}$$

is a function of the so-called **modulus** k^2. The corresponding **complete elliptic integral of the second kind** defined by

$$\mathbf{E}(k) := \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}\, dx. \tag{1.2}$$

The total collection of complete elliptic integrals contains one more, the so-called **complete elliptic integral of the third kind** defined by

$$\mathbf{\Pi}(n,k) := \int_0^1 \frac{dx}{(1-n^2x^2)\sqrt{(1-x^2)(1-k^2x^2)}}. \tag{1.3}$$

The **complementary integrals** are defined by

$$\mathbf{K}'(k) := \mathbf{K}(k') \tag{1.4}$$

where $k' = \sqrt{1-k^2}$ is the so-called complementary modulus.

The change of variables $x = \sin t$ yields the trigonometric versions

$$\mathbf{K}(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}} \text{ and } \mathbf{E}(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2 t}\, dt, \tag{1.5}$$

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with a similar expression for $\mathbf{\Pi}(n,k)$.

In general, an elliptic integral is one of the form

$$I := \int_a^b \frac{P(x)\,dx}{y}, \tag{1.6}$$

where y^2 is a cubic or quartic polynomial in x. The integral is called **complete** if a and b are roots of $y=0$. It is clear that $\mathbf{K}(k)$ is elliptic. The same is true for $\mathbf{E}(k)$, written in the form

$$\mathbf{E}(k) := \int_0^1 \frac{(1-k^2x^2)\,dx}{\sqrt{(1-x^2)(1-k^2x^2)}}. \tag{1.7}$$

2. Some examples

In this section we offer some evaluations from [**4**] that follow directly from the definitions. Some special values are offered first. The evaluations of these integrals is facilitated by Legendre's relation

$$\mathbf{K}(k)\mathbf{E}'(k) + \mathbf{K}'(k)\mathbf{E}(k) - \mathbf{K}(k)\mathbf{K}'(k) = \frac{\pi}{2}. \tag{2.1}$$

The reader will find this identity as Exercise 4 in section 2.4 of [**7**].

EXAMPLE 2.1.

$$\mathbf{K}\left(\sqrt{-1}\right) = \frac{1}{4\sqrt{2\pi}}\Gamma^2\left(\frac{1}{4}\right). \tag{2.2}$$

The proof is direct. The integral is

$$\mathbf{K}(\sqrt{-1}) = \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4}\int_0^1 y^{-3/4}(1-y)^{-1/2}\,dy = \frac{\Gamma(1/4)\,\Gamma(1/2)}{4\Gamma(3/4)}. \tag{2.3}$$

The result now follows from the symmetry rule

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a} \tag{2.4}$$

for the gamma function and the special value $\Gamma(1/2) = \sqrt{\pi}$. This example appear as entry **3.166.16** in [**4**]. Entry 3.166.18 states that

$$\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} = \frac{1}{\sqrt{2\pi}}\Gamma^2\left(\frac{3}{4}\right). \tag{2.5}$$

The proof consists of a reduction to a special value of the beta function. The change of variables $t = x^4$ gives

$$\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} = \frac{1}{4}\int_0^1 t^{-1/4}(1-t)^{-1/2}\,dt. \tag{2.6}$$

This integral is $\frac{1}{4}B\left(\frac{3}{4},\frac{1}{2}\right)$. The simplified result is obtained as above.

Formula (1.7) with $k=\sqrt{-1}$ shows that

$$\mathbf{E}(\sqrt{-1}) = \int_0^1 \frac{1+x^2}{\sqrt{1-x^4}}\,dx. \tag{2.7}$$

The values given above show that

$$\mathbf{E}(\sqrt{-1}) = \frac{1}{4\sqrt{2\pi}}\left[\Gamma^2\left(\frac{1}{4}\right) + 4\Gamma^2\left(\frac{3}{4}\right)\right]. \tag{2.8}$$

EXAMPLE 2.2.

$$\mathbf{K}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right). \tag{2.9}$$

This appears as entry **8.129.1** in [**4**]. This value comes from the previous example and the identity

$$\mathbf{K}(\sqrt{-1}k) = \frac{1}{\sqrt{1+k^2}}K\left(\frac{k}{\sqrt{1+k^2}}\right), \tag{2.10}$$

with $k = 1$. The identity (2.10) follows by the change of variables $x \mapsto x/\sqrt{1+k^2(1-x^2)}$ in the left-hand side integral.

The values of the modulus k for which $\mathbf{K}'/\mathbf{K}$ is the square root of an integer are of considerable interest. These are called the **singular values**. The previous example shows that $1/\sqrt{2}$ is the simplest of them: in this case

$$\frac{\mathbf{K}'}{\mathbf{K}}\left(\frac{1}{\sqrt{2}}\right) = 1. \tag{2.11}$$

A list of the first few values k_r for which

$$\frac{\mathbf{K}'}{\mathbf{K}}(k_r) = \sqrt{r} \tag{2.12}$$

is given in [**3**] and it starts with

$$k_2 = \sqrt{2}-1,\ k_3 = \frac{\sqrt{2}(\sqrt{3}-1)}{4},\ k_4 = 3-2\sqrt{2},\ k_5 = \frac{1}{2}\left(\sqrt{\sqrt{5}-1} - \sqrt{3-\sqrt{5}}\right).$$

3. An elementary transformation

Elementary manipulations can be emploted to evaluate certain entries in [**4**]. For instance, direct integration by parts on the integrals defining the functions **K** and **E** produces

$$\int_0^1 \frac{x\ \arcsin x}{\sqrt{(1-k^2x^2)^3}}\,dx = \frac{1}{k^2}\left(\frac{\pi}{2k'} - \mathbf{K}(k)\right) \tag{3.1}$$

and

$$\int_0^1 \frac{x\ \arcsin x}{\sqrt{1-k^2x^2}}\,dx = \frac{1}{k^2}\left(\mathbf{E}(k) - \frac{\pi}{2}k'\right). \tag{3.2}$$

This last evaluation appears as entry **4.522.4** in [**4**].

On the other hand, several entries in [**4**] may be evaluated also by integration by parts choosing the inverse trigonometric term to be differentiated. Such procedure gives

$$\int_0^1 \frac{x \arccos x\, dx}{\sqrt{1-k^2x^2}} = \frac{1}{k^2}\left(\frac{\pi}{2} - \mathbf{E}(k)\right), \tag{3.3}$$

that appears as entry **4.522.5**,

$$\int_0^1 \frac{x \arcsin x\, dx}{\sqrt{k'^2+k^2x^2}} = \frac{1}{k^2}\left(\frac{\pi}{2} - \mathbf{E}(k)\right), \tag{3.4}$$

that appears as entry **4.522.6**, and finally **4.522.7**:

$$\int_0^1 \frac{x \arccos x\, dx}{\sqrt{k'^2+k^2x^2}} = \frac{1}{k^2}\left(-\frac{\pi}{2}k' + \mathbf{E}(k)\right). \tag{3.5}$$

In this section we derive a different type of elementary transformation for integrals and use it to obtain the value of some elliptic integrals appearing in [**4**].

LEMMA 3.1. *Let f be an odd periodic function of period a. Then*

$$\int_0^\infty \frac{f(x)}{x}\, dx = \frac{\pi}{a}\int_0^{a/2} \frac{f(x)}{\tan\frac{\pi x}{a}}\, dx. \tag{3.6}$$

PROOF. The result follows by splitting the integral as

$$\begin{aligned}\int_0^\infty \frac{f(x)}{x}\, dx &= \sum_{k=0}^\infty \int_0^a \frac{f(x)}{x+ka}\, dx \\ &= \sum_{k=0}^\infty \int_0^{a/2} f(x)\left[\frac{1}{x+ka} - \frac{1}{(k+1)a-x}\right] dx\end{aligned}$$

and using the partial fraction decomposition

$$\tan\frac{\pi b}{2} = \frac{4b}{\pi}\sum_{j=1}^\infty \frac{1}{(2j-1)^2-b^2}, \tag{3.7}$$

given as entry **1.421.1** in [**4**]. □

COROLLARY 3.1. *Let f be an even function with period a. Then*

$$\int_0^\infty \frac{f(x)}{x}\sin\frac{\pi x}{a}\, dx = \frac{\pi}{a}\int_0^{a/2} f(x)\, dx. \tag{3.8}$$

In particular, for $a=\pi$,

$$\int_0^\infty \frac{f(x)}{x}\sin x\, dx = \int_0^{\pi/2} f(x)\, dx. \tag{3.9}$$

PROOF. Apply the lemma to the function $f(x)\sin\frac{\pi x}{a}$ which is odd and it has period $2a$. The result follows from the half-angle formula

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x} \tag{3.10}$$

and the value

$$\int_0^a f(x)\cos\frac{\pi x}{a}\, dx = 0. \tag{3.11}$$

□

A similar results holds for odd functions. These appear as entry **3.033** in [**4**].

COROLLARY 3.2. *Let f be an odd function with period a. Then*

$$\int_0^\infty \frac{f(x)}{x} \sin \frac{\pi x}{a}\, dx = \frac{\pi}{a} \int_0^{a/2} f(x) \cos \frac{\pi x}{a}\, dx. \tag{3.12}$$

In particular, for $a = \pi$,

$$\int_0^\infty \frac{f(x)}{x} \sin x \, dx = \int_0^{\pi/2} f(x) \cos x \, dx. \tag{3.13}$$

EXAMPLE 3.1. The function $f(x) \equiv 1$ and $a = \pi$ in Corollary 3.1 gives the classical integral

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}. \tag{3.14}$$

This is entry **3.721.1** in [**4**]. The reader will find in [**5, 6**] a couple of articles by G. H. Hardy with an evaluation of the many proofs of this identity. These papers are available in volume 5 of his Complete Works.

EXAMPLE 3.2. Entry **3.842.3** of [**4**] consists of four evaluations, the first of which

$$\int_0^\infty \frac{\sin x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} = \mathbf{K}(k). \tag{3.15}$$

This follows from Corollary 3.1 by choosing $a = \pi$ and $f(x) = 1/\sqrt{1 - k^2 \sin^2 x}$. A different proof of this evaluation is offered in Section 6 below.

EXAMPLE 3.3. A second integral appearing in **3.842.3** is

$$\int_0^\infty \frac{\sin x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \mathbf{K}(k). \tag{3.16}$$

also follows from Corollary 3.1. This is also true for entry **3.841.1**

$$\int_0^\infty \sin x \sqrt{1 - k^2 \sin^2 x} \, \frac{dx}{x} = \mathbf{E}(k) \tag{3.17}$$

and its companion entry **3.841.2**

$$\int_0^\infty \sin x \sqrt{1 - k^2 \cos^2 x} \, \frac{dx}{x} = \mathbf{E}(k). \tag{3.18}$$

EXAMPLE 3.4. The elementary method introduced here may be used to evaluate all integrals of the type

$$I_{m,n}(k) := \int_0^\infty \frac{\sin^n x \, \cos^m x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} \tag{3.19}$$

and the companion family

$$J_{m,n}(k) := \int_0^\infty \frac{\sin^n x \, \cos^m x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x}. \tag{3.20}$$

All entries in the sections **3.844** and **3.846** match one of these forms.

Example 3.5. Many other evaluations can be produced by this method. For instance,

$$\int_0^\infty \frac{\sin x \, \log(1-k^2\sin^2 x)}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x} = \int_0^{\pi/2} \frac{\log(1-k^2\sin^2 x)}{\sqrt{1-k^2\sin^2 x}}\, dx. \tag{3.21}$$

The integral on the left appears as entry **4.432.1** and the one on the right is entry **4.414.1** in [**4**]. A proof of the identity

$$\int_0^{\pi/2} \frac{\log(1-k^2\sin^2 x)}{\sqrt{1-k^2\sin^2 x}}\, dx = \mathbf{K}(k)\, \ln k', \tag{3.22}$$

is given in Example 7.2.

4. Some principal value integrals

The method described above can be employed to evaluate some entries of [**4**] provided the integrals are interpreted as Cauchy principal values.

Example 4.1. The first example is

$$\int_0^\infty \frac{\tan x}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x} = \mathbf{K}(k), \tag{4.1}$$

that appears as one of the four entries in **3.842.3** of [**4**].

Let $I_1(k)$ denote the integral and introduce the notation

$$f(x) = \frac{\tan x}{\sqrt{1-k^2\sin^2 x}}. \tag{4.2}$$

Then $f(x)$ is odd and it has period π. The principal value of the integral is given by

$$I_1(k) = \lim_{\epsilon\to 0} \sum_{j=0}^{\infty} \left(\int_0^{\pi/2-\epsilon} \frac{f(x)}{x}\, dx + \int_{\pi/2+\epsilon}^{\pi} \frac{f(x)}{x+j\pi}\, dx \right). \tag{4.3}$$

The substitution $y = \pi - x$ in the second integral above produces

$$\begin{aligned} I_1(k) &= \lim_{\epsilon\to 0} \sum_{j=0}^{\infty} \int_0^{\pi/2-\epsilon} \left(\frac{1}{x} + \frac{1}{x-(j+1)\pi} \right) f(x)\, dx \\ &= \lim_{\epsilon\to 0} \int_0^{\pi/2-\epsilon} \left(\frac{1}{x} + \sum_{j=1}^{\infty} \frac{2x}{x^2-j^2\pi^2} \right) f(x)\, dx. \end{aligned}$$

The series corresponds to the partial fraction expansion of the cotangent function. This completes the evaluation of (4.1). The reader will note that this proof is very similar to that of Lemma 3.1.

The value

$$\int_0^\infty \frac{\tan x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x} = \mathbf{K}(k), \tag{4.4}$$

that also appears in **3.842.3** is established using the same type of argument. This completes the evaluation of the integrals in that entry of [**4**].

EXAMPLE 4.2. Entry **3.841.3** of [**4**]

$$\int_0^\infty \tan x\,\sqrt{1-k^2\sin^2 x}\,\frac{dx}{x} = \mathbf{E}(k) \tag{4.5}$$

and its companion **3.841.4**

$$\int_0^\infty \tan x\,\sqrt{1-k^2\cos^2 x}\,\frac{dx}{x} = \mathbf{E}(k) \tag{4.6}$$

can be established by the method described in the previous example.

5. The hypergeometric connection

The identites among elliptic integrals often make use of the series representations

$$\mathbf{K}(k) = \frac{\pi}{2}\,{}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2};\, k^2\\ & 1\end{matrix}\right] = \frac{\pi}{2}\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)_j\left(\frac{1}{2}\right)_j}{j!}\frac{k^{2j}}{j!}, \tag{5.1}$$

and

$$\mathbf{E}(k) = \frac{\pi}{2}\,{}_2F_1\left[\begin{matrix}-\frac{1}{2} & \frac{1}{2};\, k^2\\ & 1\end{matrix}\right] = \frac{\pi}{2}\sum_{j=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_j\left(\frac{1}{2}\right)_j}{j!}\frac{k^{2j}}{j!}, \tag{5.2}$$

where ${}_2F_1$ is the classical hypergeometric function

$${}_2F_1\left[\begin{matrix}a & b;\, x\\ & c\end{matrix}\right] = \sum_{j=0}^{\infty}\frac{(a)_j\,(b)_j}{(c)_j\,j!}x^j \tag{5.3}$$

and

$$(a)_j = a(a+1)(a+2)\cdots(a+j-1), \tag{5.4}$$

is the Pochhammer symbol. The value $(a)_0 = 1$ is adopted.

6. Evaluation by series expansions

In this section we describe a method to evaluate many of the elliptic integrals appearing in [**4**].

EXAMPLE 6.1. The first example is entry **3.842.3**

$$\int_0^\infty \frac{\sin x}{\sqrt{1-k^2\sin^2 x}}\,\frac{dx}{x} = \mathbf{K}(k) \tag{6.1}$$

that has been evaluated in Section 3.

Define

$$I_1(k^2) := \int_0^\infty \frac{\sin x}{\sqrt{1-k^2\sin^2 x}}\,\frac{dx}{x}. \tag{6.2}$$

To evaluate the integral, let $m = k^2$ and expand the integrand in power series using

$$\left(\frac{d}{dm}\right)^j \frac{\sin x}{x\,\sqrt{1-m\sin^2 x}} = \left(\frac{1}{2}\right)_j \frac{\sin^{2j+1} x}{x}(1-m\sin^2 x)^{-1/2-j}. \tag{6.3}$$

Therefore,

$$I_1(m) = \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)_j \frac{m^j}{j!} \int_0^{\infty} \frac{\sin^{2j+1} x}{x}\, dx. \tag{6.4}$$

The remaining integral is entry **3.821.7** in [**4**]:

$$\int_0^{\infty} \frac{\sin^{2j+1} x}{x}\, dx = \frac{(2j-1)!!}{(2j)!!} \frac{\pi}{2}. \tag{6.5}$$

The value of the integral (6.1) now follows from the series representation of $\mathbf{K}(k)$ given in (5.1).

Proof of (6.5). Start with

$$\sin^{2j+1} x = 2^{-2j} \sum_{\nu=0}^{j} (-1)^{j-\nu} \binom{2j+1}{\nu} \sin(2j-2\nu+1)x \tag{6.6}$$

and the integral in Example 3.1 in the form

$$\int_0^{\infty} \frac{\sin \alpha x}{x}\, dx = \frac{\pi}{2} \tag{6.7}$$

for $\alpha > 0$, to obtain

$$\int_0^{\infty} \frac{\sin^{2j+1} x}{x}\, dx = \frac{\pi}{2^{2j+1}} \sum_{\nu=0}^{j} (-1)^{j-\nu} \binom{2j+1}{\nu}. \tag{6.8}$$

It follows that

$$I_1(m) = \frac{\pi}{2} \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)_j \frac{(-1)^j}{2^{2j}} \frac{m^j}{j!} \times \sum_{\nu=0}^{j} (-1)^{\nu} \binom{2j+1}{\nu}. \tag{6.9}$$

The result now follows from the next lemma.

Lemma 6.1. Let $j,\, k \in \mathbb{N}$. Then

$$\sum_{\nu=0}^{k} (-1)^{j} \binom{2j+1}{\nu} = (-1)^k \binom{2j}{k}. \tag{6.10}$$

PROOF. The proof is by induction on k. The case $k=0$ is clear. The induction hypothesis is used to produce

$$\sum_{\nu=0}^{k} (-1)^{\nu} \binom{2j+1}{\nu} = (-1)^{k-1} \binom{2j}{k-1} + (-1)^k \binom{2j+1}{k}, \tag{6.11}$$

and an elementary calculation reduces this to $(-1)^k \binom{2j}{k}$. This completes the proof of (6.5). □

Second proof of (6.5): apply the identity (3.13) to the function $f(x) = \sin^{2j} x$ to obtain

$$\int_0^{\infty} \frac{\sin^{2j+1} x}{x}\, dx = \int_0^{\pi/2} \sin^{2j} x\, dx. \tag{6.12}$$

This last integral is the classical Wallis' formula given by

$$\int_0^{\pi/2} \sin^{2j} x \, dx = \frac{\pi}{2} \frac{\left(\frac{1}{2}\right)_j}{j!}. \tag{6.13}$$

The reader will find in [**1**] information about this formula.

Example 6.2. Entry **3.841.1** in [**4**]

$$\int_0^{\infty} \sin x \sqrt{1 - k^2 \sin^2 x} \, \frac{dx}{x} = \mathbf{E}(k) \tag{6.14}$$

is established by the same method employed above. The proof starts with the expansion of the integrand using

$$\left(\frac{d}{dm} \right)^j \frac{\sin x}{x} \sqrt{1 - m \sin^2 x} = \left(-\tfrac{1}{2}\right)_j \frac{\sin^{2j+1} x}{x} (1 - m \sin^2 x)^{1/2 - j} \tag{6.15}$$

and then identify the result with (5.2).

Example 6.3. Entry **3.842.4** in [**4**] states that

$$I_2(k) := \int_0^{\pi/2} \frac{x \, \sin x \, \cos x}{\sqrt{1 - k^2 \sin^2 x}} \, dx = -\frac{\pi k'}{2k^2} + \frac{E(k)}{k^2}. \tag{6.16}$$

The parameter k' is the complementary modulus $k' = \sqrt{1 - k^2}$.

Write $m = k^2$ and expand the integrand in series using

$$\left(\frac{d}{dm} \right)^j \frac{x \sin x \, \cos x}{\sqrt{1 - m \sin^2 x}} = \left(\frac{1}{2} \right)_j \frac{x \sin^{2j+1} x \, \cos x}{\sqrt{1 - m \sin^2 x}}. \tag{6.17}$$

Therefore

$$I_2(m) = \sum_{j=0}^{\infty} \left(\frac{1}{2} \right)_j \frac{m^j}{j!} \int_0^{\pi/2} x \sin^{2j+1} x \, \cos x \, dx. \tag{6.18}$$

Integration by parts gives

$$\int_0^{\pi/2} x \sin^{2j+1} x \, \cos x \, dx = \frac{\pi}{4(j+1)} - \frac{1}{4(j+1)} B\left(j + \tfrac{3}{2}, \tfrac{1}{2}\right), \tag{6.19}$$

where

$$B(u,v) = \int_0^1 t^{u-1} (1-t)^{v-1} \, dt = 2 \int_0^{\pi/2} \sin^{2u-1} \varphi \, \cos^{2v-1} \varphi \, d\varphi, \tag{6.20}$$

is the classical beta function. It follows that

$$I_2(m) = \frac{\pi}{4} \sum_{j=0}^{\infty} \left(\frac{1}{2} \right)_j \frac{m^j}{(j+1)!} - \frac{1}{4} \sum_{j=0}^{\infty} \left(\frac{1}{2} \right)_j B\left(j + \frac{3}{2}, \frac{1}{2} \right) \frac{m^j}{(j+1)!}. \tag{6.21}$$

The two series are now treated separately.

The first sum is computed by the binomial theorem

$$(1-x)^{-a} = \sum_{j=0}^{\infty} \frac{(a)_j}{j!} x^j \tag{6.22}$$

as

$$\frac{\pi}{4} \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)_j \frac{m^j}{(j+1)!} = \frac{\pi}{2(1+\sqrt{1-m})} = \frac{\pi}{2m}(1-\sqrt{1-m}). \tag{6.23}$$

The second sum is

$$\begin{aligned} -\frac{1}{4}\sum_{j=0}^{\infty}\left(\frac{1}{2}\right)_j B\left(j+\frac{3}{2},\frac{1}{2}\right)\frac{m^j}{(j+1)!} &= -\frac{\sqrt{\pi}}{4}\sum_{j=0}^{\infty}\left(\frac{1}{2}\right)_j \frac{\Gamma(j+\frac{3}{2})}{(j+1)!\Gamma(j+2)} m^j \\ &= -\frac{\pi}{8}\sum_{j=0}^{\infty}\left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_j \frac{m^j}{(j+1)!} \\ &= \frac{\pi}{2m}\sum_{j=0}^{\infty}\left(-\frac{1}{2}\right)_{j+1}\left(\frac{1}{2}\right)_{j+1}\frac{m^{j+1}}{(j+1)!} \\ &= \frac{\pi}{2m}\left[{}_2F_1\left(\begin{matrix}-\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}; m\right) - 1\right]. \end{aligned}$$

The hypergeometric representation (5.2) and (6.21) give

$$I_2(m) = -\frac{\pi\sqrt{1-m}}{2m} + \frac{\mathbf{E}(k)}{m} \tag{6.24}$$

as claimed.

7. A small correction to a formula in Gradshteyn and Ryzhik

In this section we present the evaluation of some elliptic integrals in [**4**]. In particular, a small error in formula **4.395.1** is corrected.

PROPOSITION 7.1. *Let* $k' = \sqrt{1-k^2}$ *be the complementary modulus. Then*

$$\int_0^{\infty} \frac{\ln x \, dx}{\sqrt{(1+x^2)(k'^2+x^2)}} = \frac{1}{2}\mathbf{K}(k) \ln k'. \tag{7.1}$$

PROOF. Let $m = k'^2$ and use

$$\left(\frac{d}{dm}\right)^j \frac{\ln x}{\sqrt{(1+x^2)(m+x^2)}} = (-1)^j \left(\frac{1}{2}\right)_j \frac{\ln x}{\sqrt{(1+x^2)(m+x^2)^{j+1/2}}} \tag{7.2}$$

to expand the integrand around $m = -1$. It follows that

$$\int_0^{\infty} \frac{\ln x \, dx}{\sqrt{(1+x^2)(k'^2+x^2)}} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left(\tfrac{1}{2}\right)_j \int_0^{\infty} \frac{\ln x \, dx}{(1+x^2)^{j+1}} (m-1)^j. \tag{7.3}$$

This last integral is given by

$$\begin{aligned}\int_0^\infty \frac{\ln x\,dx}{(1+x^2)^{j+1}} &= \frac{1}{4}\int_0^\infty \frac{\ln x\,dx}{\sqrt{x}\,(1+x)^{j+1}} \\ &= \frac{1}{4}\frac{d}{d\alpha}B(\alpha, j-\alpha+1)\Big|_{\alpha=1/2} \\ &= \frac{1}{4}B\left(\tfrac{1}{2}, j+\tfrac{1}{2}\right)\left[\psi\left(\tfrac{1}{2}\right)-\psi\left(j+\tfrac{1}{2}\right)\right] \\ &= \frac{\pi}{2j!}\left(\frac{1}{2}\right)_j \sum_{i=0}^{j-1}\frac{1}{2i+1}.\end{aligned}$$

Therefore, the left-hand side of (7.1) satisfies

$$\int_0^\infty \frac{\ln x\,dx}{\sqrt{(1+x^2)(k'^2+x^2)}} = \frac{\pi}{2}\sum_{j=0}^\infty \frac{\left(\frac{1}{2}\right)_j^2}{j!^2}\sum_{i=0}^{j-1}\frac{1}{2i+1}(1-m)^j. \tag{7.4}$$

The series expansion for the complete elliptic integral now shows that the right-hand side of (7.1) is given by

$$\begin{aligned}\frac{1}{4}\ln m\,\mathbf{K}(\sqrt{1-m}) &= \frac{\pi}{8}\left[\sum_{j=1}^\infty \frac{(1-m)^j}{j}\right]\times\left[\sum_{j=0}^\infty \frac{\left(\frac{1}{2}\right)_j^2}{j!^2}(1-m)^j\right] \\ &= \frac{\pi}{8}\sum_{j=0}^\infty\left[\sum_{i=0}^{j-1}\frac{1}{j-i}\frac{\left(\frac{1}{2}\right)_i^2}{i!^2}\right](1-m)^j.\end{aligned}$$

The result follows from the identity established in the next lemma. □

LEMMA 7.1. *Let $j \in \mathbb{N}$. Define*

$$a_r = \frac{\left(\frac{1}{2}\right)_r^2}{r!^2}. \tag{7.5}$$

Then

$$\sum_{i=0}^{j-1}\frac{a_i}{j-i} = 4a_j\sum_{i=0}^{j-1}\frac{1}{2i+1}. \tag{7.6}$$

PROOF. The relations

$$(-x)_k = (-1)^k(x-k+1)_k \text{ and } \left(\tfrac{1}{2}\right)_{n-k}\left(\tfrac{1}{2}-n\right)_k = (-1)^k\left(\tfrac{1}{2}\right)_n \tag{7.7}$$

can be used to rewrite the left-hand side as

$$\begin{aligned}\sum_{i=0}^{j-1}\frac{\left(\frac{1}{2}\right)_i^2}{i!^2}\frac{1}{j-i} &= \sum_{k=0}^{j-1}\frac{\left(\frac{1}{2}\right)_{j-k-1}^2}{(j-k-1!^2}\frac{1}{k+1} \\ &= \frac{\left(\frac{1}{2}\right)_j^2}{j!^2}\sum_{k=0}^{j-1}\frac{(-j)_{k+1}^2}{\left(\frac{1}{2}-j\right)_{k+1}^2}\frac{1}{k+1}.\end{aligned}$$

Thus the assertion of the lemma is equivalent to

$$\sum_{k=0}^{j-1} \frac{(-j)_{k+1}^2}{\left(\frac{1}{2}-j\right)_{k+1}^2} \frac{1}{k+1} = \sum_{k=0}^{j-1} \frac{4}{2k+1}. \tag{7.8}$$

Next apply the fact that $(x)_{k+1} = x(x+1)_k$ to obtain

$$\begin{aligned} \sum_{k=0}^{j-1} \frac{(-j)_{k+1}^2}{\left(\frac{1}{2}-j\right)_{k+1}^2} \frac{1}{k+1} &= \frac{j^2}{\left(\frac{1}{2}-j\right)^2} \sum_{k=0}^{j-1} \frac{(1-j)_k^2}{\left(\frac{3}{2}-j\right)_k^2} \frac{1}{k+1} \\ &= \frac{j^2}{\left(\frac{1}{2}-j\right)^2} \sum_{k=0}^{j-1} \frac{(1-j)_k^2 (1)_k^2}{\left(\frac{3}{2}-j\right)_k^2 (2)_k \, k!}. \end{aligned}$$

The right-hand side is a balanced ${}_4F_3$ series and it can be transformed using

$$\begin{aligned} &{}_4F_3\left[\begin{matrix} x & y & z & -m \\ u & v & w \end{matrix} ;1\right] = \\ &\frac{(v-z)_m\,(w-z)_m}{(v)_m\,(w)_m}\, {}_4F_3\left[\begin{matrix} u-x & u-y & z & -m \\ 1-v+z-m & 1-w+z-m & u \end{matrix} ;1\right]. \end{aligned}$$

See [**2**], page 56. Now let $y = z = 1$, $x = 1-j$, $m = j-1$, $u = v = \frac{3}{2}-j$ and $w = 2$. It follows that

$$\begin{aligned} &{}_4F_3\left[\begin{matrix} 1 & 1 & 1-j & 1-j \\ \frac{3}{2}-j & \frac{3}{2}-j & 2 \end{matrix} ;1\right] = \\ &\qquad \frac{(\frac{1}{2}-j)_{j-1}\,(1)_{j-1}}{(\frac{3}{2}-j)_{j-1}\,(2)_{j-1}}\, {}_4F_3\left[\begin{matrix} 1 & \frac{1}{2} & -j+\frac{1}{2} & 1-j \\ -j+\frac{3}{2} & \frac{3}{2} & 1-j \end{matrix} ;1\right]. \end{aligned}$$

The last hypergeometric terms is now simplified

$$\begin{aligned} \frac{2j-1}{j} \sum_{k=0}^{j-1} \frac{\left(\frac{1}{2}\right)_k \left(-j+\frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k \left(-j+\frac{3}{2}\right)_k} &= \frac{(2j-1)^2}{j} \sum_{k=0}^{j-1} \frac{1}{(2k+1)(2j-1-2k)} \\ &= \frac{(2j-1)^2}{2j^2} \sum_{k=0}^{j-1} \left(\frac{1}{2k+1} + \frac{1}{2j-1-2k} \right) \\ &= \frac{(2j-1)^2}{j^2} \sum_{k=0}^{j-1} \frac{1}{2k+1}, \end{aligned}$$

as claimed. □

An automatic proof. The result of Lemma 7.1 also admits an automatic proof as described in [**8**]. Define the functions $F(i,j)$ and $G(i,j)$, respectively, as

$$F(i,j) = \frac{\left(\frac{1}{2}\right)_i^2 j!^2}{\left(\frac{1}{2}\right)_j^2 i!^2} \frac{1}{j-i} \text{ and } G(i,j) = -\frac{\left(\frac{1}{2}\right)_i^2 j!^2}{\left(\frac{1}{2}\right)_{j+1}^2 i!^2} \frac{i^2}{j-i+1}. \tag{7.9}$$

The stated result is equivalent to the identity $a(j) = b(j)$, where

$$a(j) = \sum_{i=0}^{j-1} F(i,j) \text{ and } b(j) = \sum_{i=0}^{j-1} \frac{1}{2j+1}. \tag{7.10}$$

Zeilberger algorithm finds the non-homogeneous recurrence

$$F(i+1,j) - F(i,j) = G(i+1,j) - G(i,j). \tag{7.11}$$

Summing this for i from 0 to $j-1$ and using the telescoping of the right-hand side, produces

$$\begin{aligned} \sum_{i=0}^{j-1} F(i,j+1) - \sum_{i=0}^{j-1} F(i,j) &= \sum_{i=0}^{j-1} G(i+1,j) - \sum_{i=0}^{j-1} G(i,j) \\ &= G(j,j) - G(0,j) \\ &= -\frac{4j^2}{(2j+1)^2}. \end{aligned}$$

Now observe that

$$\begin{aligned} a(j+1) - a(j) &= \frac{4(j+1)^2}{(2j+1)^2} + \sum_{i=0}^{j-1} F(i,j+1) - \sum_{i=0}^{j-1} F(i,j) \\ &= \frac{4(j+1)^2}{(2j+1)^2} - \frac{4j^2}{(2j+1)^2} \\ &= \frac{4}{2j+1}. \end{aligned}$$

The sequence $b(j)$ satisfies the same recurrence. Therefore $a(j) - b(j)$ is a constant. Since $a(1) = b(1) = 4$ this constant vanishes. This establishes the result.

The next result corrects entry **4.395.1** in [**4**].

COROLLARY 7.1. *The value*

$$\int_0^\infty \frac{\ln \tan\theta \, d\theta}{\sqrt{1-k^2\sin^2\theta}} = -\frac{1}{2} \ln k' \mathbf{K}(k) \tag{7.12}$$

holds.

PROOF. Let $x \mapsto \tan\theta$ in (7.1). □

EXAMPLE 7.1. Entry **4.242.1** states

$$\int_0^\infty \frac{\ln x \, dx}{\sqrt{(a^2+x^2)(b^2+x^2)}} = \frac{1}{2a} \mathbf{K}\left(\frac{\sqrt{a^2-b^2}}{a} \right) \ln ab. \tag{7.13}$$

Formula (7.1) corresponds to the special case $a = 1$. The change of variables $x = at$ produces

$$\int_0^\infty \frac{\ln x \, dx}{\sqrt{(a^2+x^2)(b^2+x^2)}} = \frac{1}{b} \int_0^\infty \frac{\ln t \, dt}{\sqrt{(1+t^2)(c^2+t^2)}} + \frac{\ln a}{b} \int_0^\infty \frac{dt}{\sqrt{(1+t^2)(1+c^2t^2)}}$$

with $c = b/a$. The first integral is evaluated using (7.1) and let $t = \tan\varphi$ to see that the second integral is $\mathbf{K}(\sqrt{1-c^2})$. This establishes the result.

EXAMPLE 7.2. The techniques illustrated here are now employed to prove entry **4.414.1** in [**4**]:

$$\int_0^{\pi/2} \frac{\ln(1-k^2\sin^2 x)}{\sqrt{1-k^2\sin^2 x}}\,dx = \mathbf{K}(k)\ln k'. \tag{7.14}$$

Let $m = k^2$ and observe that

$$\frac{d}{dm}\frac{\alpha_j + \beta_j \ln(1-m\sin^2 x)}{(1-m\sin^2 x)^{j+1/2}}\sin^{2j} x = \frac{\alpha_{j+1} + \beta_{j+1}\ln(1-m\sin^2 x)}{(1-m\sin^2 x)^{j+3/2}}\sin^{2j+2} x \tag{7.15}$$

where the parameters α_j, β_j satisfy

$$\alpha_{j+1} = (j+\tfrac{1}{2})\alpha - j - \beta_j \text{ and } \beta_{j+1} = (j+\tfrac{1}{2})\beta_j. \tag{7.16}$$

Now choose $\alpha_0 = 0$ and $\beta_0 = 1$ to obtain

$$\left(\frac{d}{dm}\right)^j \frac{\ln(1-m\sin^2 x)}{\sqrt{1-m\sin^2 x}} = \frac{\alpha_j + \beta_j\ln(1-m\sin^2 x)}{(1-m\sin^2 x)^{j+1/2}}\sin^{2j} x. \tag{7.17}$$

Expand the integrand of (7.14) around $m = 0$ and use

$$\int_0^{\infty} \sin^{2j} x\,dx = \frac{\pi}{2}\frac{\left(\frac{1}{2}\right)_j}{j!^2} \tag{7.18}$$

and the expressions

$$\alpha_j = \left(\tfrac{1}{2}\right)_j \sum_{i=0}^{j-1} \frac{2}{2i+1} \text{ and } \beta_j = \left(\tfrac{1}{2}\right)_j \tag{7.19}$$

to see that

$$\int_0^{\pi/2} \frac{\ln(1-k^2\sin^2 x)}{\sqrt{1-k^2\sin^2 x}}\,dx = \pi\sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)_j^2}{j!^2}\left(\sum_{i=0}^{j-1}\frac{1}{2i+1}\right) m^j. \tag{7.20}$$

The result now follows from the evaluation given in the proof of Proposition 7.1.

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