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On the fractional calculus of generalized Mittag-Leffler function

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Abstract. The paper is devoted to the study of generalized fractional calculus of the generalized Mittag-Leffler function $E^{\delta}_{\nu,\rho}(z)$ which is an entire function of the form

$$E^{\delta}_{\nu,\rho}(z) = \sum_{s=0}^{\infty} \frac{(\delta)_s z^s}{\Gamma(\nu n+\rho)s!},$$

where $\nu > 0$ and $\rho > 0$. For δ =1, it reduces to Mittag-Leffler function $E_{\nu,\rho}(z)$. We have shown that the generalized functional calculus operators transform such functions with power multipliers in to generalized Wright function. Some elegant results obtained by Kilbas and Saigo [11], Saxena and Saigo [24] are the special cases of the results derived in this paper.

1. Introduction and Preliminaries

The function $E_{\nu}(z)$ defined by the series representation

$$E_{\nu}(z) = \sum_{s=0}^{\infty} \frac{z^s}{\Gamma(\nu s+1)}, \ (\nu > 0, z \in C). \tag{1.1}$$

Mittag-Leffler [19, 20], Wiman [26, 27], Agarwal [1], Humbert and Agarwal [11], investigated the generalizations of the above function $E_{\nu}(z)$ in the following manner; see [4, Section 18.1]

$$E_{\nu,\rho}(z) = \sum_{s=0}^{\infty} \frac{z^s}{\Gamma(\nu s+\rho)}, \ (\nu > 0, \rho > 0, z \in C), \tag{1.2}$$

where C be the setof complex numbers. For a detailed study of various properties, generalizations and applications of this function we can refer to papers of Dzherashyan

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[2], Kilbas and Saigo [9, 10, 11, and 12], Kilbas, Saigo and Saxena [15], Gorenflo and Mainardi [7] Gorenflo, Kilbas and Rogosin [5] and Gorenflo, Luchko and Rogosin [6].

A more generalized form of (1.2) is introduced by Prabhaker [21] as:

$$E^{\delta}_{\nu,\rho}(z) = \sum_{s=0}^{\infty} \frac{(\delta)_s z^s}{\Gamma(\nu s+\rho)s!}, \tag{1.3}$$

where $\nu, \rho, \delta \in \mathrm{C}$ $(R(\nu) > 0)$ and $E^{\delta}_{\nu,\rho}(z)$ is an entire function of order $[Re(\nu)],^{-1}$ $[21, p.7]$. For various properties and other details of (1.3), see [14].

The generalized Wright function ${}_p\Psi_q(z)$ defined for $z \in C, a_i, b_j \in C$ and $\alpha_i, \beta_j \in R(\alpha_i, \beta_j \neq 0; i = 1, 2, ..., p; j = 1, 2, ..., q)$ is given by the series

$$ {}_p\Psi_q(z) = {}_p\Psi_q \left[\begin{array}{c|c} (a_i, \alpha_i)_{(1,p)} & \\ & z \\ (b_j, \beta_j)_{(1,q)} & \end{array} \right] = \sum_{s=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i s) z^s}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j s) s!}, \tag{1.4}$$

where C is the set of complex numbers and $\Gamma(z)$ is the Euler gamma function [3, section1.1] and the function (1.4) was introduced by Wright [29] and known as generalized Wright function. Conditions for the existence of the generalized Wright function (1.4) together with its representation in terms of Melline-Barnes integral and in terms of the H-function were established in [13].
Some particular cases of generalized Wright function (1.4) were presented in [13, Section 6]. Wright in [28], [31] investigated, by "steepest descent"method, the asymptotic expansions of the function $\phi(\alpha, \beta; z)$ for large values of z in the cases $\alpha > 0$ and $-1 < \alpha < 0$, respectively. In [28] Wright indicated the application of the obtained results to the asymptotic theory of partitions. In [29], [30], [32] Wright extended the last result to the generalized Wright function (1.4) and proved several theorems on the asymptotic expansion of generalized Wright function ${}_p\Psi_q(z)$ for all values of the argument z under the condition.

$$\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1. \tag{1.5}$$

For a detailed study of various properties, generalizations and applications of Wright function and generalized Wright function, we refer to papers of Wright [28, 29, 30, 31, and 32], Luchko [16, 17] and Kilbas [13].

2. Fractional Calculus Operators and Generalized Fractional Calculus Operators

The left and right-sided Riemann-Liouville fractional calculus operators are defined by Samko, Kilbas and Marichev [23, Section 5.1]. For $\alpha \in \mathrm{C}$ $(Re(\alpha) > 0)$

$$(2.1) \qquad (I_{0+}^{\alpha}f)(x) = \frac{1}{\Gamma\alpha}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt; (x>0),$$

$$(2.2) \qquad (I_{-}^{\alpha}f)(x) = \frac{1}{\Gamma\alpha}\int_x^\infty \frac{f(t)}{(t-x)^{1-\alpha}}dt; (x>0),$$

$$(2.3) \quad (D_{0+}^{\alpha}f)(x) = \Big(\frac{d}{dx}\Big)^{[Re(\alpha)]+1}\Big[I_{0+}^{1-\alpha+[Re(\alpha)]}f\Big](x)$$
$$= \Big(\frac{d}{dx}\Big)^{[Re(\alpha)]+1}\frac{1}{\Gamma 1-\alpha+[Re(\alpha)]}\int_0^x \frac{f(t)}{(x-t)^{\alpha-[Re(\alpha)]}}dt; (x>0),$$

$$(2.4) \quad (D_{-}^{\alpha}f)(x) = \Big(-\frac{d}{dx}\Big)^{[Re(\alpha)]+1}\Big[I_{-}^{1-\alpha+[Re(\alpha)]}f\Big](x)$$
$$= \Big(-\frac{d}{dx}\Big)^{[Re(\alpha)]+1}\frac{1}{\Gamma 1-\alpha+[Re(\alpha)]}\int_x^\infty \frac{f(t)}{(t-x)^{\alpha-[Re(\alpha)]}}dt; (x>0),$$

where $[Re(\alpha)]$ is the integral of $Re(\alpha)$.

An interesting and useful generalization of the Riemann-Liouville and Erdlyi-Kober fractional integral operators has been introduced by Saigo [22] in terms of Gauss hypergeometric function as given below. Let $\alpha, \beta, \gamma \in C$ and $x \in R_+$, then the generalized fractional integration and fractional differentiation operators associated with Gauss hypergeometric function are defined as follows:

$$(2.5) \qquad (I_{0+}^{\alpha,\beta,\gamma}f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma\alpha}\int_0^x (x-t)^{\alpha-1}{}_2F_1\Big(\alpha+\beta,-\gamma;\alpha;1-\frac{t}{x}\Big)f(t)dt;$$
$$(Re(\alpha)>0),$$

$$(2.6) \qquad (I_{-}^{\alpha,\beta,\gamma}f)(x) = \frac{1}{\Gamma\alpha}\int_x^\infty (t-x)^{\alpha-1}t^{-\alpha-\beta}{}_2F_1\Big(\alpha+\beta,-\gamma;\alpha;1-\frac{x}{t}\Big)f(t)dt;$$
$$(Re(\alpha)>0),$$

$$(2.7) \qquad (D_{0+}^{\alpha,\beta,\gamma}f)(x) = \Big(I_{0+}^{-\alpha-\beta,\alpha+\gamma}f\Big)(x) = \Big(\frac{d}{dx}\Big)^k\Big(I_{0+}^{-\alpha+k,-\beta-k,\alpha+\gamma-k}f\Big)(x);$$
$$(Re(\alpha)>0); k=[Re(\alpha)]+1,$$

$$(2.8) \qquad (D_{-}^{\alpha,\beta,\gamma}f)(x) = \Big(I_{-}^{-\alpha-\beta,\alpha+\gamma}f\Big)(x) = \Big(-\frac{d}{dx}\Big)^k\Big(I_{-}^{-\alpha+k,-\beta-k,\alpha+\gamma}f\Big)(x);$$
$$(Re(\alpha)>0); k=[Re(\alpha)]+1.$$

Operators (2.5) - (2.8) reduce to that in (2.1) - (2.4) as the follows:

$$(2.9) \qquad (I_{0+}^{\alpha,-\alpha,\gamma}f)(x) = (I_{0+}^{\alpha}f)(x),$$

$$(I_{-}^{\alpha,-\alpha,\gamma}f)(x) = (I_{-}^{\alpha}f)(x), \tag{2.10}$$

$$(D_{0+}^{\alpha,-\alpha,\gamma}f)(x) = (D_{0+}^{\alpha}f)(x), \tag{2.11}$$

$$(D_{-}^{\alpha,-\alpha,\gamma}f)(x) = (D_{-}^{\alpha}f)(x). \tag{2.12}$$

lemma 1. Let $\alpha, \beta, \gamma \in C$ $(Re(\alpha) > 0)$ and $\rho \in C$

(a) If $Re(\rho) > \max[0, Re(\beta - \gamma)]$, then

$$\left(I_{0+}^{\alpha,\beta,\gamma} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho-\beta+\gamma)}{\Gamma(\rho-\beta)\Gamma(\rho+\alpha+\gamma)} x^{\rho-\beta-1}, \tag{2.13}$$

(b) If $Re(\rho) > \max[Re(-\beta), Re(-\gamma)]$, then

$$\left(I_{-}^{\alpha,\beta,\gamma} t^{-\rho}\right)(x) = \frac{\Gamma(\rho+\beta)\Gamma(\rho+\gamma)}{\Gamma(\rho)\Gamma(\rho+\alpha+\beta+\gamma)} x^{-\rho-\beta}. \tag{2.14}$$

3. Left-Sided Generalized Fractional Integration of the Generalized Mittag-Leffler Function

In this section we consider the left-sided generalized fractional integration formula of the generalized Mittage-Leffler function.

Theorem 1. Let $\alpha, \beta, \gamma, \rho, \delta, \in C$ be complex numbers such that $Re(\alpha) > 0, Re(\rho + \gamma - \beta) > 0, \nu > 0$ and $a \in R$. If the condition (1.5) is satisfied and $I_{0+}^{\alpha,\beta,\gamma}$ be the left-sided operator of the generalized fractional integration associated with Gauss hypergeometric function, then there holds the following relationship

$$\left(I_{0+}^{\alpha,\beta,\gamma}(t^{\rho-1}E_{\nu,\rho}^{\delta}[at^{\nu}])\right)(x) = \frac{x^{\rho-\beta-1}}{\Gamma\delta}\, {}_2\Psi_2\left[\begin{array}{c|c} (\rho+\gamma-\beta,\nu),(\delta,1) & \\ & ax^{\nu} \\ (\rho-\beta,\nu),(\rho+\alpha+\gamma,\nu) & \end{array}\right], \tag{3.1}$$

provided each member of the equation (3.1) exists.

proof. By using the definition of generalized Mittag-Leffler function (1.3) and fractional integral formula (2.5), we have

$$\begin{aligned}\Omega &= \left(I_{0+}^{\alpha,\beta,\gamma}(t^{\rho-1}E_{\nu,\rho}^{\delta}[at^{\nu}])\right)(x)\\ &= \frac{x^{-\alpha-\beta}}{\Gamma\alpha}\int_0^x (x-t)^{\alpha-1}{}_2F_1\Big(\alpha+\beta,-\gamma;\alpha;1-\frac{t}{x}\Big)(t^{\rho-1}E_{\nu,\rho}^{\delta}[at^{\nu}])dt.\end{aligned}$$

By the use of Gaussian hypergeometric series [25,p.18,equation 17], series form of generalized Mittag-Leffler function (1.3), interchanging the order of integration and summations and evaluating the inner integral by the use of the known formula of Beta integral. Finally by the virtue of Gauss summation theorem, we have

$$\Omega = \frac{x^{\rho-\beta-1}}{\Gamma\delta}\sum_{s=0}^{\infty}\frac{\Gamma(\delta+s)\Gamma(\rho+\gamma-\beta+\nu s)}{\Gamma(\rho-\beta+\nu s)\Gamma(\rho+\alpha+\gamma+\nu s)}\frac{(ax^{\nu})^{s}}{s!}$$

or

$$\Omega = \frac{x^{\rho-\beta-1}}{\Gamma\delta}{}_2\Psi_2\left[\begin{array}{c|c} (\rho+\gamma-\beta,\nu),(\delta,1) & \\ & ax^{\nu} \\ (\rho-\beta,\nu),(\rho+\alpha+\gamma,\nu) & \end{array}\right].$$

Interchanging the order of integration and summations, which is permissible under the conditions, stated with the theorem due to convergence of the integrals involved in the process. This completes the proof of the theorem.

Corollary 1. For $Re(\alpha) > 0, Re(\rho+\gamma-\beta) > 0, \nu > 0$ and $a \in R$. If the condition (1.5) is satisfied, then there holds the formula

$$\text{(3.2)}\quad \left(I_{0+}^{\alpha,\beta,\gamma}(t^{\rho-1}E_{\nu,\rho}[at^{\nu}])\right)(x) = x^{\rho-\beta-1}{}_2\Psi_2\left[\begin{array}{c|c} (\rho+\gamma-\beta,\nu),(1,1) & \\ & ax^{\nu} \\ (\rho-\beta,\nu),(\rho+\alpha+\gamma,\nu) & \end{array}\right],$$

provided each member of the equation (3.2) makes sense.

Remark 1. If we put $\beta = -\alpha$ in our result (3.1), we arrive at the result $[24, p.145, Eq.14]$ given by Saxena and Saigo.

Remark 2. If we set $\beta = -\alpha$ in our formula (3.2), it reduces in to the well known result [23,table 9.1, formula (23)].

4. Right-Sided Generalized Fractional Integration of the Generalized Mittag-Leffler Function

In this section we have discussed the right-sided generalized fractional integration formula of the generalized Mittag-Leffler function.

Theorem 2. Let $\alpha, \beta, \gamma, \rho, \delta, \in C$ be complex numbers such that $Re(\alpha) > 0, Re(\alpha+\rho) > \max[-Re(\beta), -Re(\gamma)]$ with the conditions $Re(\beta) \neq Re(\gamma), \nu > 0$ and $a \in R$. If the condition (1.5) is satisfied and $I_{-}^{\alpha,\beta,\gamma}$ be the right-sided operator of the generalized fractional integration associated with Gauss hypergeometric function, then there holds the formula

$$\text{(4.1)}\quad \left(I_{-}^{\alpha,\beta,\gamma}(t^{-\alpha-\rho}E_{\nu,\rho}^{\delta}[at^{-\nu}])\right)(x)$$

$$= \frac{x^{-\rho-\alpha-\beta}}{\Gamma\delta}{}_3\Psi_3\left[\begin{array}{c|c} (\alpha+\beta+\rho,\nu),(\alpha+\gamma+\rho,\nu),(\delta,1) & \\ & ax^{-\nu} \\ (\rho,\nu),(\alpha+\rho,\nu),(2\alpha+\beta+\gamma+\rho,\nu) & \end{array}\right],$$

provided both the sides of (4.1) exist.

proof. By using the definition of generalized Mittag-Leffler function (1.3), generalized fractional integral formula (2.6) and proceeding similarly to the proof of theorem 1, we obtain

$$\Lambda = \Big(I_{-}^{\alpha,\beta,\gamma}(t^{-\alpha-\rho}E_{\nu,\rho}^{\delta}[at^{-\nu}])\Big)(x)$$

$$= \frac{1}{\Gamma\alpha}\int_x^\infty (t-x)^{\alpha-1}t^{-\alpha-\beta}{}_2F_1\Big(\alpha+\beta,-\gamma;\alpha;1-\frac{x}{t}\Big)(t^{-\alpha-\rho}E_{\nu,\rho}^{\delta}[at^{-\nu}])dt$$

$$= \frac{x^{-\rho-\alpha-\beta}}{\Gamma\delta}\sum_{s=0}^{\infty}\frac{\Gamma(\alpha+\beta+\rho+\nu s)\Gamma(\alpha+\gamma+\rho+\nu s)\Gamma(\delta+s)}{\Gamma(\rho+\nu s)\Gamma(\alpha+\rho+\nu s)\Gamma(2\alpha+\beta+\gamma+\rho+\nu s)}\frac{(ax^{-\nu})^s}{s!}$$

or

$$\Lambda = \frac{x^{-\rho-\alpha-\beta}}{\Gamma\delta}{}_3\Psi_3\left[\begin{array}{c|c} (\alpha+\beta+\rho,\nu),(\alpha+\gamma+\rho,\nu),(\delta,1) & \\ & ax^{-\nu} \\ (\rho,\nu),(\alpha+\rho,\nu),(2\alpha+\beta+\gamma+\rho,\nu) & \end{array}\right].$$

Corollary 2. For $Re(\alpha) > 0, Re(\alpha+\rho) > \max[-Re(\beta), -Re(\gamma)]$ with the conditions $Re(\beta) \neq Re(\gamma), \nu > 0$ and $a \in R$. If the condition (1.5) is satisfied, then there holds the following formula

$$\text{(4.2)}\quad \Big(I_{-}^{\alpha,\beta,\gamma}(t^{-\alpha-\rho}E_{\nu,\rho}[at^{-\nu}])\Big)(x)$$

$$= x^{-\rho-\alpha-\beta}{}_3\Psi_3\left[\begin{array}{c|c} (\alpha+\beta+\rho,\nu),(\alpha+\gamma+\rho,\nu),(1,1) & \\ & ax^{\nu} \\ (\rho,\nu),(\alpha+\rho,\nu),(2\alpha+\beta+\gamma+\rho,\nu) & \end{array}\right],$$

provided each member of the equation(4.2) makes sense.

Remark 3. If we set $\beta = -\alpha$ in (4.1), we get the result $[24, p.147, Eq.(23)]$ given by Saxena and Saigo.

Remark 4. If we set $\beta = -\alpha$ in (4.2), we arrive at the known formula $[24, p.148, Eq.(24)]$.

5. Left-Sided Generalized Fractional Differentiation of the Generalized Mittag-Leffler Function

In this section we study the left-sided generalized fractional differentiation formula of the generalized Mittag-Leffler function.

Theorem 3. Let $\alpha, \beta, \gamma, \rho, \delta, \in C$ be complex numbers such that $Re(\alpha) > 0, Re(\rho+\beta+\gamma) > 0, \nu > 0$ and $a \in R$. If the condition (1.5) is satisfied and $D_{0+}^{\alpha,\beta,\gamma}$ be the left-sided operator of the generalized fractional differentiation associated with Gauss hypergeometric function, then there holds the following elegant relationship

$$(5.1)\quad \Big(D_{0+}^{\alpha,\beta,\gamma}(t^{\rho-1}E_{\nu,\rho}^{\delta}[at^{\nu}])\Big)(x)=\frac{x^{\rho+\beta-1}}{\Gamma\delta}\,{}_2\Psi_2\left[\begin{array}{c|c}(\alpha+\beta+\gamma+\rho,\nu),(\delta,1) & \\ & ax^{\nu}\\ (\rho+\gamma,\nu),(\rho+\beta,\nu) & \end{array}\right],$$

provided each member of the equation (5.1) exists.

proof. By using the definition of generalization Mittag-Leffler function (1.3) and fractional derivative formula (2.7), we have

$$\Theta=\Big(D_{0+}^{\alpha,\beta,\gamma}(t^{\rho-1}E_{\nu,\rho}^{\delta}[at^{\nu}])\Big)(x)$$
$$=\Big(\frac{d}{dx}\Big)^{k}\Big(I_{0+}^{-\alpha+k,-\beta-k,\alpha+\gamma-k}(t^{\rho-1}E_{\nu,\rho}^{\delta}[at^{\nu}])\Big)(x)$$
$$=\Big(\frac{d}{dx}\Big)^{k}x^{\alpha+\beta}\int_0^x(x-t)^{-\alpha+k-1}{}_2F_1(-\alpha-\beta,-\gamma-\alpha+k;-\alpha+k;1-\frac{t}{x})(t^{\rho-1}E_{\nu,\rho}^{\delta}[at^{\nu}])dt$$
$$=\sum_{s=0}^{\infty}\frac{(\delta)_s}{\Gamma(\rho+\gamma+\nu s)}\frac{\Gamma(\alpha+\beta+\gamma+\rho+\nu s)}{\Gamma(\rho+\beta+\nu s)}\frac{a^s x^{\rho+\beta+\nu s-1}}{s!}$$

or

$$\Theta=\frac{x^{\rho+\beta-1}}{\Gamma\delta}\,{}_2\Psi_2\left[\begin{array}{c|c}(\alpha+\beta+\gamma+\rho,\nu),(\delta,1) & \\ & ax^{\nu}\\ (\rho+\gamma,\nu),(\rho+\beta,\nu) & \end{array}\right].$$

This completes the proof.

Corollary 3. For$Re(\alpha)>0, Re(\rho+\beta+\gamma)>0, \nu>0$ and $a\in R$. If the condition (1.5) is satisfied, then there holds the formula

$$(5.2)\quad \Big(D_{0+}^{\alpha,\beta,\gamma}(t^{\rho-1}E_{\nu,\rho}[at^{\nu}])\Big)(x)=x^{\rho+\beta-1}\,{}_2\Psi_2\left[\begin{array}{c|c}(\alpha+\beta+\gamma+\rho,\nu),(1,1) & \\ & ax^{\nu}\\ (\rho+\gamma,\nu),(\rho+\beta,\nu) & \end{array}\right],$$

provided both the sides of (5.2) make sense.

Remark 5. If we put $\beta=-\alpha$ in result (5.1), we arrive at the known result $[24, p.149, Eq.(29)]$.

Remark 6. If we set $\beta=-\alpha$ in result (5.2), it reduces to the known relation $[24, p.149, Eq.(30)]$.

6. Right-Sided Generalized Fractional Differentiation of the Generalized Mittag-Leffler Function

In this section we discuss the right-sided generalized fractional derivative of generalized Mittag-Leffler functions .

Theorem 4. Let $\alpha,\beta,\gamma,\rho,\delta,\in C$ be complex numbers such that $Re(\alpha)>0, Re(\rho)>\max[Re(\alpha+\beta)+k,-Re(\gamma)], \nu>0$ and $a\in R$ with $Re(\alpha+\beta+\gamma)+k\neq 0$ (where $k=[Re(\alpha)]+1$). If the condition (1.5) is satisfied and $D_{-}^{\alpha,\beta,\gamma}$ be the right-sided operator of the generalized fractional differentiation associated with Gauss hypergeometric function, then there holds the following elegant relationship

$$(6.1)\quad \Big(D_{-}^{\alpha,\beta,\gamma}(t^{\alpha-\rho}E_{\nu,\rho}^{\delta}[at^{-\nu}])\Big)(x) = \frac{x^{\alpha+\beta-\rho}}{\Gamma\delta}\,{}_3\Psi_3\left[\begin{array}{c} (\rho+\gamma,\nu),(\rho-\alpha-\beta,\nu),(\delta,1) \\ \\ (\rho,\nu),(\rho-\alpha,\nu),(\rho+\gamma-\alpha-\beta,\nu) \end{array}\middle|\, ax^{-\nu}\right],$$

provided each member of (6.1) is in existence.

proof. By virtue of (1.3) and (2.8), we have

$$\Delta = \Big(D_{-}^{\alpha,\beta,\gamma}(t^{\alpha-\rho}E_{\nu,\rho}^{\delta}[at^{-\nu}])\Big)(x)$$

$$= \Big(-\frac{d}{dx}\Big)^{k}\Big(I_{-}^{-\alpha+k,-\beta-k,\alpha+\gamma}(t^{\alpha-\rho}E_{\nu,\rho}^{\delta}[at^{-\nu}])\Big)(x)$$

$$= \Big(-\frac{d}{dx}\Big)^{k}\frac{1}{\Gamma(-\alpha+k)}\int_{x}^{\infty}(t-x)^{-\alpha+k-1}t^{\alpha+\beta}{}_2F_1(-\alpha-\beta,-\alpha-\gamma;-\alpha+k;1-\frac{x}{t})$$

$$.(t^{\alpha-\rho}E_{\nu,\rho}^{\delta}[at^{-\nu}])dt$$

$$= \frac{x^{\alpha+\beta-\rho}}{\Gamma\delta}\,{}_3\Psi_3\left[\begin{array}{c} (\rho+\gamma,\nu),(\rho-\alpha-\beta,\nu),(\delta,1) \\ \\ (\rho,\nu),(\rho-\alpha,\nu),(\rho+\gamma-\alpha-\beta,\nu) \end{array}\middle|\, ax^{-\nu}\right].$$

This completes the proof.

Corollary 4. For$Re(\alpha) > 0, Re(\rho) > \max[Re(\alpha+\beta)+k, -Re(\gamma)], \nu > 0$ and $a \in R$ with $Re(\alpha+\beta+\gamma)+k \neq 0$ (where $k = [Re(\alpha)]+1$). If the condition (1.5) is satisfied, then there holds the formula

$$(6.2)\quad \Big(D_{-}^{\alpha,\beta,\gamma}(t^{\alpha-\rho}E_{\nu,\rho}[at^{-\nu}])\Big)(x) = x^{\alpha+\beta-\rho}{}_3\Psi_3\left[\begin{array}{c} (\rho+\gamma,\nu),(\rho-\alpha-\beta,\nu),(1,1) \\ \\ (\rho,\nu),(\rho-\alpha,\nu)(\rho+\gamma-\alpha-\beta,\nu) \end{array}\middle|\, ax^{-\nu}\right],$$

provided each member of (6.2) exists.

Remark 7. On setting $\beta = -\alpha$ in the result (6.1) we can produce the known result $[24, p.150, Eq.(35)]$.

Remark 8. On taking $\beta = -\alpha$ in the result (6.2) we can obtained the known relation $[24, p.151, Eq.(36)]$.

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