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The Inverse Tangent Integral and its Association with Euler Sums

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Abstract. In the existing literature, integrals that involve the inverse tangent integral in their integrands are not commonly found. However, in this paper, we delve into specific families of integrals that do contain the inverse tangent integral and reveal that their expressions are closely linked to certain variants of Euler harmonic sums. Furthermore, we provide explicit demonstrations of certain particular cases of our primary findings.

1. Introduction, Preliminaries and Notations

Many researchers have dedicated their efforts to studying integrals that involve logarithmic, polylogarithmic, and various other elementary functions combinations. Specifically, there has been significant attention on integrals that pertain to logarithms [5], [6], [7], [8], [9], [12], [14], [17], [18], [19], [24], [25], [26], [27], [29],[30], [33], [34], [35], [36], [37], [38], [40], [46], [56], [59], [61], [62], [63], [64]; integrals associated with polylogarithms [1], [9], [18], [22], [23], [29], [30], [41], [56], [61], [62], [63]; integrals connected to generalized harmonic sums and polylogarithms [2]; log-trigonometric integrals log-trigonometric integrals [6], [13], [15], [17], [24], [25], [27], [28], [29], [34], [35], [56], [59], [60]; integrals associated with logarithms and polylogarithms [22], [30], [51]; integrals incorporating arctangent and arcsine and their powers [31], [42], and so on.

The objective of this study is to enhance the comprehension and usage of integrals in (1.1) by investigating those that involve integrands composed of logarithmic, inverse tangent, and other elementary function combinations:

$$U_I^{\pm}(a,m,p,q,t) := \int_I \frac{x^a\,(\log^p x)\,\mathrm{Ti}_t(x^q)}{(1+\delta x)^{m+1}}\,\mathrm{d}x \tag{1.1}$$

$$\left(a \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant -2};\ m \in \mathbb{Z}_{\geqslant -1};\ t \in \mathbb{Z}_{\geqslant 0};\ p,\,q \in \mathbb{N};\ \delta = \pm 1\right),$$

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where I is the unit interval $(0,1)$ or the positive real half line $(0,\infty)$, and $\mathrm{Ti}_n(x)$ is the inverse tangent integral of order n defined by

$$\mathrm{Ti}_n(x) := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^n}\, x^{2k-1} \quad (|x| \leqslant 1,\, n \in \mathbb{N}). \tag{1.2}$$

Note that it satisfies the recurrence

$$\mathrm{Ti}_n(x) = \int_0^x \frac{\mathrm{Ti}_{n-1}(t)}{t}\,\mathrm{d}t, \tag{1.3}$$

the domain of which can be extended to ∞ (see, e.g., [60, p. 180 and p. 186]).

The polylogarithm function $\mathrm{Li}_p(z)$ of order p is defined by

$$\mathrm{Li}_p(z) := \sum_{m=1}^{\infty} \frac{z^m}{m^p} \quad (|z| \leqslant 1;\ p \in \mathbb{Z}_{\geqslant 2}). \tag{1.4}$$

It satisfies the following integral representation:

$$\mathrm{Li}_p(z) = \int_0^z \frac{\mathrm{Li}_{p-1}(t)}{t}\,\mathrm{d}t \quad (p \in \mathbb{Z}_{\geqslant 3}). \tag{1.5}$$

In particular, the dilogarithm function $\mathrm{Li}_2(z)$ is given by

$$\begin{aligned} \mathrm{Li}_2(z) &= \sum_{m=1}^{\infty} \frac{z^m}{m^2} \quad (|z| \leqslant 1) \\ &= -\int_0^z \frac{\log(1-t)}{t}\,\mathrm{d}t. \end{aligned} \tag{1.6}$$

Likewise, the polylogarithm $\mathrm{Li}_p(z)$ for $p \in \mathbb{Z}_{\leqslant 1}$ can be defined as follows:

$$\mathrm{Li}_1(z) := -\log(1-z), \quad \mathrm{Li}_0(z) := \frac{z}{1-z}, \tag{1.7}$$

and

$$\mathrm{Li}_{-n}(z) = \left(z\frac{d}{dz}\right)^n \frac{z}{1-z} = \sum_{j=0}^{n} j!\,S(n+1,j+1)\left(\frac{z}{1-z}\right)^{j+1} \quad (n \in \mathbb{Z}_{\geqslant 0}), \tag{1.8}$$

where $S(n+1,j+1)$ are Stirling numbers of the second kind. It can be equivalently written as

$$\mathrm{Li}_{-n}(z) = \frac{1}{(1-z)^{n+1}} \sum_{j=0}^{n-1} \left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle z^{n-j} \quad (n \in \mathbb{N}), \tag{1.9}$$

where $\left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle$ are the Eulerian numbers which are explicitly given as follows:

$$\left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle = \sum_{r=0}^{j+1} (-1)^r \binom{n+1}{r} (j+1-r)^n.$$

The polylogarithm function $\mathrm{Li}_p(z)$ in (1.4) with an integer order p is extended to $\mathrm{Li}_s(z)$ with a complex-valued order s as follows (see, e.g., [60, p. 198]):

$$\mathrm{Li}_s(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^s} \tag{1.10}$$

$$(s \in \mathbb{C} \text{ and } |z| < 1;\ \Re(s) > 1 \text{ and } |z| = 1)\,.$$

Jonquiére's formula is recalled (see, e.g., [20, pp. 30–31], [60, pp. 197–198]; see also [16]):

$$\mathrm{Li}_s(z) + e^{is\pi}\,\mathrm{Li}_s\left(\frac{1}{z}\right) = \frac{(2\pi)^s}{\Gamma(s)} e^{\frac{1}{2} i\pi s}\,\zeta\left(1-s, \frac{\log z}{2\pi i}\right). \tag{1.11}$$

Setting $s = m$ in (1.11) and using the following relation (see, e.g., [20, p. 27], [60, p. 151]):

$$\zeta(-m, a) = -\frac{B_{m+1}(a)}{m+1} \quad (m \in \mathbb{Z}_{\geqslant 0})\,, \tag{1.12}$$

one gets

$$\mathrm{Li}_m(z) + (-1)^m\,\mathrm{Li}_m\left(\frac{1}{z}\right) = -\frac{(2\pi i)^m}{m!}\,B_m\left(\frac{\log z}{2\pi i}\right) \quad (m \in \mathbb{Z}_{\geqslant 2}). \tag{1.13}$$

Here $B_n(x)$ are the Bernoulli polynomials in (1.52). Both (1.11) and (1.13) provide the analytic continuation of $\mathrm{Li}_s(z)$ in (1.10) outside its disk of convergence $|z| \leqslant 1$.

Setting $z = iy$ $(y \in \mathbb{R})$ in (1.4), we have

$$\mathrm{Li}_n(iy) = \frac{1}{2^n}\,\mathrm{Li}_n\left(-y^2\right) + i\,\mathrm{Ti}_n(y) \quad (y \in \mathbb{R}), \tag{1.14}$$

The Dirichlet beta function $\beta(z)$ is defined by

$$\beta(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^z} \quad (\Re(z) > 0). \tag{1.15}$$

Among various properties and formulas for $\beta(z)$, we recall the followings:

$$\begin{aligned} \beta(z) &= 4^{-z}\left\{\zeta\left(z, \tfrac{1}{4}\right) - \zeta\left(z, \tfrac{3}{4}\right)\right\} \\ &= \frac{i}{2}\left\{\mathrm{Li}_z(-i) - \mathrm{Li}_z(i)\right\}, \end{aligned} \tag{1.16}$$

and

$$\beta(k+1) = \frac{(-1)^{k+1}}{k!\,4^{k+1}}\left\{\psi^{(k)}\left(\tfrac{1}{4}\right) - \psi^{(k)}\left(\tfrac{3}{4}\right)\right\} \quad (k \in \mathbb{N}), \tag{1.17}$$

where $\psi^{(k)}(z)$ is the polygamma function in (1.26) and $\zeta(s, z)$ is the generalized zeta function in (1.28). From (1.2) and (1.15), we find

$$\mathrm{Ti}_n(1) = \beta(n) \quad (n \in \mathbb{N})\,. \tag{1.18}$$

Recall

$$\mathrm{Ti}_{2n+1}(1) = \frac{(-1)^n\,\pi^{2n+1}}{(2n)!\,2^{2n+2}}\,E_{2n} \quad (n \in \mathbb{Z}_{\geqslant 0})\,, \tag{1.19}$$

where E_n are Euler numbers (see, e.g., [60, pp. 86–90]). The first few values of $\mathrm{Ti}_{2n+1}(1)$ are

$$\mathrm{Ti}_1(1) = \frac{\pi}{4}, \quad \mathrm{Ti}_3(1) = \frac{\pi^3}{32}, \quad \mathrm{Ti}_5(1) = \frac{5\,\pi^5}{1536}, \quad \mathrm{Ti}_7(1) = \frac{61\,\pi^7}{184320}, \ldots .$$

Recall the Spence's formula (see, e.g., [60, p. 189, Eq. (97)]):

$$\begin{aligned} &\mathrm{Ti}_n(x) + (-1)^{n-1}\,\mathrm{Ti}_n\left(\frac{1}{x}\right) \\ &= \frac{\pi}{2}\,\frac{\log^{n-1} x}{(n-1)!} + 2\sum_{k=1}^{\left[\frac{1}{2}(n-1)\right]} \frac{\log^{n-2k-1} x}{(n-2k-1)!}\,\mathrm{Ti}_{2k+1}(1) \\ &\qquad (x \in \mathbb{R}_{>0},\ n \in \mathbb{Z}_{\geqslant 2}). \end{aligned} \tag{1.20}$$

The $\beta_n^{(z)}(\alpha)$ are defined by

$$\beta_n^{(z)}(\alpha) := \sum_{k=1}^{n} \frac{(-1)^{k+1}}{(2k-1+\alpha)^z} \tag{1.21}$$

$$(n \in \mathbb{N},\ \alpha \in \mathbb{C} \text{ with } \alpha \neq -(2k-1),\ k \in \mathbb{N})$$

and $\beta_0^{(z)} := 0$, and are called the nth beta numbers with parameter α of order z and $\beta_n^{(z)} := \beta_n^{(z)}(0)$.

The Catalan constant G is given as

$$G = \beta\,(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.91597. \tag{1.22}$$

There are also numerous identities for G. For example,

$$G = -\int_0^1 \frac{\log x}{1+x^2}\,\mathrm{d}x = \int_0^1 \frac{\arctan x}{x}\,\mathrm{d}x = \Im\left(\mathrm{Li}_2\,(i)\right). \tag{1.23}$$

Here and elsewhere, let $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{Z}$ denote the sets of complex numbers, real numbers, and integers, respectively. Also let $A_{\geqslant \ell}$, $A_{>\ell}$, $A_{\leqslant \ell}$, and $A_{<\ell}$ be the subsets of the set A ($\mathbb{R}$ or $\mathbb{Z}$) whose elements are greater than or equal to, greater than, less than or equal to, and less than some $\ell \in \mathbb{R}$, respectively. In particular, let $\mathbb{N} := \mathbb{Z}_{\geqslant 1}$.

There are very useful formulae, identities with the inverse tangent integral in [29, Chapter 2] and the generalized inverse tangent integral in [29, Chapter 3], which are involved in $\mathrm{Ti}_2(x)$. Vălean [62, Sections 3.25 and 3.26] treated some surprising integrals involving $\mathrm{Ti}_2(x)$ and the other functions such as logarithm and arctangent. There are relatively abundant integral formulas associated with $\mathrm{Ti}_1(x) = \arctan x$ (see, e.g., [31], [42]). Except for two monographs [29] and [62], we were unable to find any published research articles about integrals involving the inverse tangent integrals of higher order. Our goal is to obtain a closed-form expression of the integral (1.1) using special functions, numbers, and constants, such as the Riemann zeta function and harmonic numbers. To achieve this, we will utilize particular values of these special functions and mathematical constants to convert the integrals into closed-form representations such as

$$\int_0^1 \frac{x(\log x)\,\mathrm{Ti}_2\,(x)}{1+x}\,\mathrm{d}x = -\tfrac{\log 2}{4} - \tfrac{\pi^2}{12} + \tfrac{\pi}{2} + \tfrac{\pi^2 G}{16} - \tfrac{21\,\pi\,\zeta(3)}{64} - G + 2\,\beta(4), \tag{1.24}$$

which is shown in (5.17).

For our purpose, certain polynomials, numbers, mathematical constants, and special functions are recalled. The harmonic numbers H_n are given by

$$H_n = \sum_{j=1}^{n} \frac{1}{j} = \gamma + \psi\,(n+1) \quad (n \in \mathbb{N}) \quad \text{and} \quad H_0 := 0. \tag{1.25}$$

Here γ is the familiar Euler-Mascheroni constant (see, e.g., [**60**, Section 1.2]) and $\psi\,(z)$ denotes the digamma (or psi) function defined by

$$\psi\,(z) := \frac{d}{dz}\left(\log \Gamma\,(z)\right) = \frac{\Gamma'\,(z)}{\Gamma\,(z)} \quad \left(\mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}\right),$$

where $\Gamma\,(z)$ is the Gamma function (see, e.g., [**60**, Section 1.1]). Here and elsewhere, an empty sum is assumed to be nil. The polygamma function $\psi^{(k)}(z)$ defined by

$$\psi^{(k)}(z) := \frac{d^k}{dz^k}\{\psi(z)\} = (-1)^{k+1}\,k!\,\sum_{r=0}^{\infty} \frac{1}{(r+z)^{k+1}} = (-1)^{k+1}\,k!\,\zeta(k+1,z) \tag{1.26}$$

$$(k \in \mathbb{N};\ z \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0})$$

has the recurrence

$$\psi^{(k)}(z+1) = \psi^{(k)}(z) + \frac{(-1)^k\,k!}{z^{k+1}} \quad (k \in \mathbb{Z}_{\geqslant 0};\ z \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}) \tag{1.27}$$

and $\psi^{(0)}(z) = \psi(z)$. Here $\zeta(s,z)$ is the generalized (or Hurwitz) zeta function defined by

$$\zeta(s,z) = \sum_{m=0}^{\infty} \frac{1}{(m+z)^s} \quad \left(\Re(s) > 1,\ z \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}\right). \tag{1.28}$$

The following identities are used:

$$\zeta(s,1) = \zeta(s) \quad \text{and} \quad \zeta(s,z) = \zeta(s,n+z) + \sum_{m=0}^{n-1} \frac{1}{(m+z)^s} \quad (n \in \mathbb{N}) \tag{1.29}$$

and

$$\zeta(s,1) = \zeta(s) = (2^s - 1)^{-1}\,\zeta\left(s, \tfrac{1}{2}\right), \tag{1.30}$$

where $\zeta(s)$ is the Riemann zeta function defined by

$$\zeta(s) := \sum_{j=1}^{\infty} \frac{1}{j^s} \quad (\Re(s) > 1). \tag{1.31}$$

The generalized harmonic numbers $H_n^{(s)}(u)$ of order s are defined by

$$H_n^{(s)}(u) := \sum_{j=1}^{n} \frac{1}{(j+u)^s} \quad \left(s \in \mathbb{C},\ u \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant -1},\ n \in \mathbb{N}\right), \tag{1.32}$$

and $H_n^{(s)} := H_n^{(s)}(0)$ are the harmonic numbers of order s. Also $H_\alpha^{(m)}$ are extended harmonic numbers of order $m \in \mathbb{N}$ with index $\alpha \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant -1}$ defined by (see [**57**])

$$H_\alpha^{(m)} := \begin{cases} \gamma + \psi(\alpha+1) & (m=1), \\ \zeta(m) + \frac{(-1)^{m-1}}{(m-1)!}\,\psi^{(m-1)}(\alpha+1) & (m \in \mathbb{Z}_{\geqslant 2}). \end{cases} \tag{1.33}$$

The case $m=1$ in (1.33) is given in (1.25). Employing (1.33) in (1.27) gives

$$H_\alpha^{(m)} = H_{\alpha-1}^{(m)} + \frac{1}{\alpha^m} \quad \left(m \in \mathbb{N},\ \alpha \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}\right). \tag{1.34}$$

Applying (1.33) to the multiplication formula for polygamma functions (see, e.g., [**32**, p. 14]):

$$\psi^{(n)}(mz) = \delta_{n,0}\log m + \frac{1}{m^{n+1}}\sum_{j=1}^{m}\psi^{(n)}\left(z + \tfrac{j-1}{m}\right) \quad \left(m \in \mathbb{N},\ n \in \mathbb{Z}_{\geqslant 0}\right), \tag{1.35}$$

$\delta_{n,j}$ being the Kronecker delta, provide the following multiplication formula for the extended harmonic numbers:

$$\begin{gathered} H_{m\alpha}^{(p)} = \frac{1}{m^p}\sum_{j=1}^{m} H_{\alpha+\frac{j}{m}-1}^{(p)} + \left(1 - m^{1-p}\right)\zeta(p) \\ \left(m \in \mathbb{N},\ p \in \mathbb{Z}_{\geqslant 2};\ m\alpha+1,\ \alpha + \frac{j}{m} \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}\right). \end{gathered} \tag{1.36}$$

The odd harmonic numbers $O_n^{(s)}$ of order s are defined by

$$O_n^{(s)} := \sum_{j=1}^{n} \frac{1}{(2j-1)^s} \quad \left(s \in \mathbb{C},\ n \in \mathbb{N}\right) \tag{1.37}$$

and $O_n := O_n^{(1)}$ $(n \in \mathbb{N})$. Also let us denote the generalized odd harmonic numbers $O_n^{(s)}(\alpha)$ of order s by

$$\begin{gathered} O_n^{(s)}(\alpha) := \sum_{j=1}^{n} \frac{1}{(2j-1+\alpha)^s} \\ \left(s \in \mathbb{C},\ n \in \mathbb{N};\ \alpha \in \mathbb{C},\ \alpha \neq -(2j-1),\ j \in \mathbb{N}\right) \end{gathered} \tag{1.38}$$

and $O_n^{(s)}(0) = O_n^{(s)}$.

From (1.37), one obtains

$$O_n^{(s)} = H_{2n}^{(s)} - 2^{-s}\,H_n^{(s)} \quad \left(s \in \mathbb{C},\ n \in \mathbb{N}\right). \tag{1.39}$$

The alternating harmonic numbers $A_n^{(s)}$ of order s are defined by

$$A_n^{(s)} := \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j^s} \quad \left(s \in \mathbb{C},\ n \in \mathbb{N}\right) \tag{1.40}$$

and $A_n := A_n^{(1)}$. The alternating harmonic numbers and the harmonic numbers have the following relation

$$A_n^{(s)} = H_n^{(s)} - 2^{1-s} H_{[n/2]}^{(s)}. \tag{1.41}$$

Here and elsewhere, $[x]$ indicates the greatest integer less than or equal to $x \in \mathbb{R}$. The generalized alternating harmonic numbers $A_n^{(s)}(u)$ are defined by

$$A_n^{(s)}(u) := \sum_{j=1}^{n} \frac{(-1)^{j+1}}{(j+u)^s} \quad (s \in \mathbb{C},\, u \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1},\, n \in \mathbb{N}). \tag{1.42}$$

Flajolet and Salvy [21] presented and explored a total of four distinct types of linear Euler sums which are denoted by

$$\mathtt{S}_{\mu,z}^{++} = \sum_{\tau=1}^{\infty} \frac{H_\tau^{(\mu)}}{\tau^z}, \qquad \mathtt{S}_{\mu,z}^{+-} = \sum_{\tau=1}^{\infty} (-1)^{\tau+1} \frac{H_\tau^{(\mu)}}{\tau^z},$$
$$\mathtt{S}_{\mu,z}^{-+} = \sum_{\tau=1}^{\infty} \frac{A_\tau^{(\mu)}}{\tau^z}, \qquad \mathtt{S}_{\mu,z}^{--} = \sum_{\tau=1}^{\infty} (-1)^{\tau+1} \frac{A_\tau^{(\mu)}}{\tau^z}.$$

Likewise, Alzer and Choi [3] introduced and investigated four distinct kinds of parametric linear Euler sums:

$$\begin{aligned} \mathtt{S}_{\mu,z}^{++}(\alpha,\beta) &= \sum_{\tau=1}^{\infty} \frac{H_\tau^{(\mu)}(\alpha)}{(\tau+\beta)^z}, \qquad & \mathtt{S}_{\mu,z}^{+-}(\alpha,\beta) &= \sum_{\tau=1}^{\infty} (-1)^{\tau+1} \frac{H_\tau^{(\mu)}(\alpha)}{(\tau+\beta)^z}, \\ \mathtt{S}_{\mu,z}^{-+}(\alpha,\beta) &= \sum_{\tau=1}^{\infty} \frac{A_\tau^{(\mu)}(\alpha)}{(\tau+\beta)^z}, \qquad & \mathtt{S}_{\mu,z}^{--}(\alpha,\beta) &= \sum_{\tau=1}^{\infty} (-1)^{\tau+1} \frac{A_\tau^{(\mu)}(\alpha)}{(\tau+\beta)^z}. \end{aligned} \tag{1.43}$$

Clearly

$$\mathtt{S}_{\mu,z}^{++}(0,0) = \mathtt{S}_{\mu,z}^{++}, \;\; \mathtt{S}_{\mu,z}^{+-}(0,0) = \mathtt{S}_{\mu,z}^{+-}, \;\; \mathtt{S}_{\mu,z}^{-+}(0,0) = \mathtt{S}_{\mu,z}^{-+}, \;\; \mathtt{S}_{\mu,z}^{--}(0,0) = \mathtt{S}_{\mu,z}^{--}.$$

The Dirichlet eta function $\eta(s)$ is given by

$$\eta(s) := \lim_{n \to \infty} A_n^{(s)} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^s} \quad (\Re(s) > 0). \tag{1.44}$$

Particularly,

$$\eta(1) = \log 2 \quad \text{and} \quad \eta(0) = \frac{1}{2}. \tag{1.45}$$

The generalized Dirichlet eta function $\eta(s,z)$ is defined by

$$\eta(s,z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+z)^s} \quad \left(\Re(s) > 0,\; z \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}\right). \tag{1.46}$$

The following identities are noteworthy:

$$\eta(s,1) = \eta(s) \quad \text{and} \quad \eta(s,z) = (-1)^n \, \eta(s, n+z) + \sum_{m=0}^{n-1} \frac{(-1)^m}{(m+z)^s} \quad (n \in \mathbb{N}). \tag{1.47}$$

The Dirichlet lambda function $\lambda(s)$ is defined as the termwise arithmetic mean of the Dirichlet eta function and the Riemann zeta function:

$$\lambda(s) = \frac{\eta(s) + \zeta(s)}{2} = \lim_{n\to\infty} O_n^{(s)} = \sum_{j=1}^{\infty} \frac{1}{(2j-1)^s} \quad (\Re(s) > 1). \tag{1.48}$$

The Stirling numbers of the first kind $s(n, j)$ are defined by the generating function

$$z(z-1)\cdots(z-n+1) = \sum_{j=0}^{n} s(n,j)\, z^j. \tag{1.49}$$

Or, equivalently,

$$n!\binom{z}{n} = \sum_{j=0}^{n} s(n,j)\, z^j.$$

We recall the following properties for $s(n, j)$ (see, e.g., [60, p. 76]):

$$s(n+1, j) = s(n, j-1) - n\, s(n, j) \quad (n \geqslant j \geqslant 1); \tag{1.50}$$

$$s(n, 0) = 0 \quad (n \in \mathbb{N}); \qquad s(n, n) = 1 \quad (n \in \mathbb{Z}_{\geqslant 0}). \tag{1.51}$$

The Bernoulli polynomials $B_n(x)$ are defined by the generating function:

$$\frac{z\, e^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\, \frac{z^n}{n!} \quad (|z| < 2\pi) \tag{1.52}$$

The numbers $B_n := B_n(0)$ are called the Bernoulli numbers generated by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n\, \frac{z^n}{n!} \quad (|z| < 2\pi). \tag{1.53}$$

The Euler numbers E_n are defined by means of the following generating function:

$$\frac{2\, e^z}{e^{2z} + 1} = \operatorname{sech} z = \sum_{n=0}^{\infty} E_n\, \frac{z^n}{n!} \quad \left(|z| < \frac{\pi}{2}\right). \tag{1.54}$$

It is noted that

$$B_{2n+1} = 0 \;\; (n \in \mathbb{N}) \quad \text{and} \quad E_{2n+1} = 0 \;\; (n \in \mathbb{Z}_{\geqslant 0}). \tag{1.55}$$

The first few of the Bernoulli and Euler numbers are given below:

$$B_0 = 1,\; B_1 = -\frac{1}{2},\; B_2 = \frac{1}{6},\; B_4 = -\frac{1}{30},\; B_6 = \frac{1}{42},\; B_8 = -\frac{1}{30},\; \ldots$$

and

$$E_0 = 1,\; E_2 = -1,\; E_4 = 5,\; E_6 = -61,\; E_8 = 1385,\; \ldots.$$

As mentioned above, except for two monographs [29] and [62] and integrals associated with $\mathrm{Ti}_1(x) = \arctan(x)$, as far as it is searched, there is no published research articles about integrals involving the inverse tangent integral of higher order. The integral in (1.24) is a certain particular instance of our main identities, which is also demonstrated in (5.17). It is found that these integrals cannot be evaluated directly by means of a current CAS software package.

2. Required results

In this section, we will review some known variant Euler harmonic sums, as well as a mathematical constant that will be necessary for the upcoming sections.

LEMMA 2.1. *([43]) Let $p \in \mathbb{Z}_{\geqslant 0}$. Then*

$$\begin{aligned}\sum_{n\geqslant 1}\frac{(-1)^{n+1}H_n}{(2n+1)^{2p+1}} &= \frac{1}{2^{2p+1}}\mathrm{S}^{+-}_{1,2p+1}\left(0,\tfrac{1}{2}\right)\\ &= \frac{1}{(2p)!}\left(\frac{\pi}{2}\right)^{2p+1}|E_{2p}|\log 2-(2p+1)\,\beta\,(2p+2)\\ &\quad+\left(\frac{\pi}{2}\right)^{2p+1}\sum_{j=1}^{p}\frac{1}{(2p-2j)!}\left(\frac{2}{\pi}\right)^{2j}|E_{2p-2j}|\,\lambda\,(2j+1)\,,\end{aligned}\tag{2.1}$$

where E_n are the Euler numbers (1.54), $\lambda(s)$ is the Dirichlet lambda function (1.48) and $\beta(s)$ is the Dirichlet beta function (1.15).

Note that (2.1) is a simplified version by using the following well-known relation between the Bernoulli numbers B_{2j} (1.53) and the Riemann zeta function (1.31) (see, e.g., [60, p. 166, Eq. (18)]):

$$\frac{(2\pi)^{2j}}{2\,(2j)!}\,|B_{2j}|=\zeta(2j)\qquad(j\in\mathbb{Z}_{\geqslant 1})\,.$$

LEMMA 2.2. *Let $p \in \mathbb{N}$. Then*

$$\begin{aligned}&\frac{1}{2^2}\chi^{+-}_{2p,2}\left(0,-\tfrac{1}{2}\right)=\sum_{n\geqslant 1}\frac{(-1)^{n+1}\left(H_n^{(2p)}-H_{n-\frac{1}{2}}^{(2p)}\right)}{(2n-1)^2}\\ &=2^{2p}\left(\beta\,(2)-2p\beta\,(1)\right)+\sum_{j=1}^{2p}j2^{j+1}\eta\,(2p+1-j)\\ &\quad+\frac{2^{2p}}{(2p-1)!}\left((2p)!\beta\,(1)\,\eta\,(2p+1)-\frac{(2p-1)!}{2}\beta\,(2p+2)\right)\\ &\quad-\frac{2^{2(p-3)}\pi}{(2p-1)!}\lim_{a\to 0}\frac{d^{2p-1}}{da^{2p-1}}\left(\begin{array}{c}\csc\left(\frac{\pi a}{2}\right)\left(\psi'\left(\frac{1-a}{4}\right)-\psi'\left(\frac{3-a}{4}\right)\right)\\ +\sec\left(\frac{\pi a}{2}\right)\left(\psi'\left(\frac{2-a}{4}\right)-\psi'\left(\frac{4-a}{4}\right)\right)-32\,G\,\csc\left(\pi a\right)\end{array}\right),\end{aligned}\tag{2.2}$$

where $\eta(z)$ is the Dirichlet eta function (1.44) and G is the Catalan constant (1.22).

PROOF. One manipulates a result in [62, p. 148] to obtain (2.2). □

LEMMA 2.3. *The following identities hold.*

$$\sum_{n\geqslant 0}\frac{(-1)^n\left(2\,H^{(p+1)}_{\frac{an}{2}}-2^{p+1}\,H^{(p+1)}_{an}\right)}{(2n+1)^{t+1}} \tag{2.3}$$
$$=-2^{p+1}\,\eta(p+1)\,\beta(t+1)+\frac{(-1)^{p+t}\,2^{p+1}}{p!\,t!}\int_0^1\int_0^1\frac{(\log^p x)\,(\log^t y)}{(1+x)\,(1+x^a y^2)}\,\mathrm{d}x\mathrm{d}y$$

$$\left(a\in\mathbb{R}_{>0},\ (p,t)\in\mathbb{Z}_{\geqslant 0}\times\mathbb{Z}_{\geqslant 0}\right).$$

In particular,

$$\sum_{n\geqslant 0}\frac{(-1)^n\,\left(H_n-H_{2n}\right)}{(2n+1)^{t+1}} \tag{2.4}$$
$$=\frac{2(-1)^t}{t!}\int_0^1\frac{\log^t y\,\left\{2y\,\arctan y-\log\left(1+y^2\right)\right\}}{(1+y^2)}\,\mathrm{d}y$$

and

$$\sum_{n\geqslant 0}\frac{(-1)^n\left(2\,H^{(p+1)}_{n}-2^{p+1}\,H^{(p+1)}_{2n}\right)}{2n+1} \tag{2.5}$$
$$=-2^{p+1}\,\pi\eta(p+1)\,+\frac{(-1)^p\,2^{p+1}}{p!}\int_0^1\frac{(\log^p x)\,\arctan x}{x\,(1+x)}\,\mathrm{d}x.$$

For $t\in\mathbb{N}$,

$$\sum_{n\geqslant 1}\frac{(-1)^{n+1}\,H^{(t)}_n}{(2n+1)^t}=\frac{(-1)^t}{2\,(t-1)!}\int_0^\infty\frac{(\log^{t-1}x)\,\mathrm{Li}_t\left(-x^2\right)}{1+x^2}\,\mathrm{d}x$$
$$+(-1)^t\sum_{j=0}^{\left[\frac{t}{2}\right]}2^{t-2j}\binom{2t-2j-1}{t-2j}\eta\,(2j)\,\beta\,(2t-2j) \tag{2.6}$$
$$=\frac{(-1)^t}{2\,(t-1)!}\left(\left(\frac{\pi}{2}\right)^t|E_{t-1}|\,\zeta\,(t)-\sum_{j=0}^{t-1}\frac{2^j\pi^{t-j}}{j!}\,|E_{t-1-j}|\,\lambda\,(t+j)\right)$$
$$+(-1)^t\sum_{j=0}^{\left[\frac{t}{2}\right]}2^{t-2j}\binom{2t-2j-1}{t-2j}\eta\,(2j)\,\beta\,(2t-2j)\,.$$

PROOF. The formula (2.3) is recalled from [**53**]. Setting $a=2$, $p=0$ and $a=2$, $t=0$ in (2.3) yields, respectively, (2.4) and (2.5). The identity (2.6) comes from [**50**, Corollary 3]. ☐

Recall an intriguing and useful mathematical constant $\mathscr{G}$ defined by

$$\mathscr{G} := \Im\left(\operatorname{Li}_3\left(\frac{1+i}{2}\right)\right) \approx .570077, \tag{2.7}$$

which was considered among useful mathematical constants and investigated in [**10**] (see also [**55**]). This constant $\mathscr{G}$ is a natural companion to the Catalan's constant G in (1.22) in many ways and has appeared in various literature (for example, see the references in [**10**]). One finds from (2.7) that (see, e.g., [**55**]; see also [**10**])

$$\mathscr{G} = \sum_{n\geqslant 1} \frac{\sin\left(\frac{\pi n}{4}\right)}{2^{\frac{n}{2}} n^3} = \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{2^{2n}} \left(\frac{2}{(4n-3)^3} + \frac{2}{(4n-2)^3} + \frac{1}{(4n-1)^3}\right). \tag{2.8}$$

We also recall the real part expression of $\operatorname{Li}_3\left(\frac{1+i}{2}\right)$

$$\Re\left(\operatorname{Li}_3\left(\frac{1+i}{2}\right)\right) = \frac{35\zeta(3)}{64} - \frac{5\pi^2 \log 2}{192} + \frac{\log^3 2}{48}, \tag{2.9}$$

which is derived by setting $\theta = \frac{\pi}{2}$ in the formula [**29**, Eq. (6.54)] and using [**29**, Eq. (6.6)] (or [**29**, p. 296, Entry A.2.6-(5)]). An integral expression for $\mathscr{G}$ is recalled (see, e.g., [**10**], [**55**]):

$$\mathscr{G} = \frac{1}{2} \int_0^1 \frac{\log^2(1-x)}{1+x^2}\, \mathrm{d}x. \tag{2.10}$$

The constant $\mathscr{G}$ occurs in many Euler type sums such as (see, e.g., see [**44**])

$$\sum_{n\geqslant 1} \frac{(-1)^{n+1} H_n}{(2n-1)^2} = -\frac{\pi}{2} - \frac{\pi^3}{64} + \log 2 - \frac{\pi \log^2 2}{16} + 2G - G\log 2 + 2\mathscr{G}; \tag{2.11}$$

$$\sum_{n\geqslant 0} \frac{(-1)^{n+1} H_n^{(2)}}{2n+1} = \frac{11\pi^3}{6} + \frac{\pi \log^2 2}{8} - 2G\log 2 - 4\mathscr{G}; \tag{2.12}$$

$$\sum_{n\geqslant 1} \frac{(-1)^{n+1} H_{2n}}{(2n-1)^2} = \pi - \frac{\pi^2}{12} - \frac{11\,\pi^3}{96} - \frac{\pi \log^2 2}{8} + 2G\log 2 - 2\log 2 + 4\mathscr{G}; \tag{2.13}$$

$$\sum_{n\geqslant 1} \frac{(-1)^{n+1} H_n^{(2)}}{2n-1} = \pi - \frac{\pi^2}{12} - \frac{11\,\pi^3}{96} - \frac{\pi \log^2 2}{8} + 2G\log 2 - 2\log 2 + 4\mathscr{G}; \tag{2.14}$$

$$\sum_{n\geqslant 1} \frac{(-1)^{n+1} H_{2n}^{(2)}}{2n-1} = \frac{\pi}{4} - \frac{\pi^2}{48} - \frac{\pi^3}{48} - \frac{\pi \log^2 2}{16} + \frac{G\log 2}{2} - \frac{\log 2}{2} + 2\mathscr{G}. \tag{2.15}$$

Setting $m = 2$ and $\alpha = n$ in (1.36) and using the second equality of (3.7) gives

$$H_{2n}^{(p)} = \eta(p) + \frac{1}{2^p} H_n^{(p)} + \frac{1}{2^p} H_{n-\frac{1}{2}}^{(p)} \quad (p,\, n \in \mathbb{N}). \tag{2.16}$$

Using (2.16) in (2.13) and (2.15), with the aid of (2.11) and (2.14), offers, respectively,

$$\sum_{n\geqslant 1}\frac{(-1)^{n+1}\,H_{n-\frac{1}{2}}}{(2n-1)^2}=\frac{5\pi}{2}-\frac{\pi^2}{6}-\frac{\pi^3}{192}-\frac{3\pi\,\log^2 2}{16}+3G\log 2-2G-5\log 2+6\mathscr{G}, \tag{2.17}$$

and

$$\sum_{n\geqslant 1}\frac{(-1)^{n+1}\,H^{(2)}_{n-\frac{1}{2}}}{2n-1}=-\frac{5\,\pi^3}{96}-\frac{\pi\,\log^2 2}{8}+4\,\mathscr{G}. \tag{2.18}$$

Remark 2.1. *Euler launched a sequence of inquiry for the linear harmonic sums* (2.19) *during his contact with Goldbach beginning in 1742 and was the first to explore the following sums (see, e.g.,* [**14, 21**]*)*

$$\mathtt{S}^{++}_{p,q}=\mathtt{S}_{p,q}:=\sum_{n=1}^{\infty}\frac{H^{(p)}_n}{n^q}. \tag{2.19}$$

Euler, whose research was finished by Nielsen in 1906 (see [**39**]*), demonstrated that the linear harmonic sums in* (2.19) *can be evaluated in the following instances: $p=1$; $p=q$; $p+q$ odd; the pairs (p,q) with $p+q$ even are only the set $\{(2,4),\,(4,2)\}$. Of these particular cases, in the ones with $p\neq q$, if $\mathtt{S}_{p,q}$ is evaluated, then $\mathtt{S}_{q,p}$ is determined by virtue of the symmetry relation*

$$\mathtt{S}_{p,q}+\mathtt{S}_{q,p}=\zeta(p)\,\zeta(q)+\zeta(p+q) \tag{2.20}$$

and vice versa.

3. Integrals involving the inverse tangent integral

This section explores the integral (1.1). Let us denote

$$\mathtt{R}^{++}_{t,u}(a,b):=\sum_{n=1}^{\infty}\frac{O^{(t)}_n(a)}{(n+b)^u}\quad\text{and}\quad \mathtt{R}^{-+}_{t,u}(a,b):=\sum_{n=1}^{\infty}\frac{\beta^{(t)}_n(a)}{(n+b)^u}, \tag{3.1}$$

where

$$\mathtt{R}^{++}_{t,u}(0,0):=\mathtt{R}^{++}_{t,u}\quad\text{and}\quad \mathtt{R}^{-+}_{t,u}(0,0):=\mathtt{R}^{-+}_{t,u}. \tag{3.2}$$

Theorem 3.1. *Let $t\in\mathbb{Z}_{\geqslant 0}$, $u\in\mathbb{Z}_{\geqslant 2}$, $b\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant -1}$ and $a\in\mathbb{C}$ with $a\neq -(2k-1)$ $(k\in\mathbb{Z}_{\geqslant 1})$. Then*

$$\begin{aligned}\mathtt{R}^{++}_{t,u}(a,b)&=\sum_{k=1}^{\infty}\frac{\zeta(u,k+b)}{(2k-1+a)^t}\\&=\frac{1}{2^t}\,\mathtt{S}^{++}_{t,u}\Big(\frac{a-1}{2},b\Big);\end{aligned} \tag{3.3}$$

$$\begin{aligned}\mathtt{R}^{-+}_{t,u}(a,b)&=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\,\zeta(u,k+b)}{(2k-1+a)^t}\\&=\frac{1}{2^t}\,\mathtt{S}^{-+}_{t,u}\Big(\frac{a-1}{2},b\Big);\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&\mathtt{R}_{t,u}^{++}(a,b)+\mathtt{R}_{t,u}^{-+}(a,b)\\
&\quad=\frac{1}{2^{u+2t-1}}\left\{\mathtt{S}_{t,u}^{++}\left(\frac{a-3}{4},\frac{b}{2}\right)+\mathtt{S}_{t,u}^{++}\left(\frac{a-3}{4},\frac{b-1}{2}\right)\right\};
\end{aligned}\tag{3.5}
$$

$$
\begin{aligned}
&\mathtt{R}_{t,u}^{++}(a,b)-\mathtt{R}_{t,u}^{-+}(a,b)\\
&\quad=\frac{1}{2^{u+2t-1}}\left\{\mathtt{S}_{t,u}^{++}\left(\frac{a-1}{4},\frac{b}{2}\right)+\mathtt{S}_{t,u}^{++}\left(\frac{a-1}{4},\frac{b-1}{2}\right)\right.\\
&\qquad\qquad\left.-\sum_{n\geqslant 1}\frac{1}{\left(n+\frac{b-1}{2}\right)^u\left(n+\frac{a-1}{4}\right)^t}\right\}.
\end{aligned}\tag{3.6}
$$

Proof. We prove only (3.5). Let $\mathcal{L}$ be the left member of (3.5). We have

$$
\begin{aligned}
\mathcal{L}&=\frac{1}{2^t}\sum_{n=1}^{\infty}\frac{1}{(n+b)^u}\sum_{k=1}^{n}\frac{1+(-1)^{k+1}}{\left(k+\frac{a-1}{2}\right)^t}\\
&=\frac{1}{2^{t-1}}\sum_{n=1}^{\infty}\frac{1}{(n+b)^u}\sum_{k=1}^{[\frac{n+1}{2}]}\frac{1}{\left(2k-1+\frac{a-1}{2}\right)^t}\\
&=\frac{1}{2^{2t-1}}\sum_{n=1}^{\infty}\frac{1}{(n+b)^u}\sum_{k=1}^{[\frac{n+1}{2}]}\frac{1}{\left(k+\frac{a-3}{4}\right)^t},
\end{aligned}
$$

which, upon decomposing even and odd indices of n, yields

$$
\begin{aligned}
\mathcal{L}&=\frac{1}{2^{2t-1}}\left\{\sum_{n=1}^{\infty}\frac{1}{(2n+b)^u}\sum_{k=1}^{n}\frac{1}{\left(k+\frac{a-3}{4}\right)^t}\right.\\
&\qquad\qquad\left.+\sum_{n=1}^{\infty}\frac{1}{(2n-1+b)^u}\sum_{k=1}^{n}\frac{1}{\left(k+\frac{a-3}{4}\right)^t}\right\}\\
&=\frac{1}{2^{u+2t-1}}\left\{\sum_{n=1}^{\infty}\frac{1}{(n+\frac{b}{2})^u}\sum_{k=1}^{n}\frac{1}{\left(k+\frac{a-3}{4}\right)^t}\right.\\
&\qquad\qquad\left.+\sum_{n=1}^{\infty}\frac{1}{(n+\frac{b-1}{2})^u}\sum_{k=1}^{n}\frac{1}{\left(k+\frac{a-3}{4}\right)^t}\right\}.
\end{aligned}
$$

Finally, use of the notations for the parametric linear sums in (1.43) in the last two summations is found to show (3.5).

The other ones are left to the interested reader. □

Remark 3.1. *(i) Recall the known identity (see, e.g., [**60**, p. 164, Eq. (1)]):*

$$
\zeta(s)=\begin{cases}\sum\limits_{n\geqslant 1}\frac{1}{n^s}=\frac{\lambda(s)}{1-2^{-s}} & (\Re(s)>1)\\ \frac{\eta(s)}{1-2^{1-s}} & (\Re(s)>0;\ s\neq 1).\end{cases}\tag{3.7}
$$

(ii) The linear Euler sums $\mathtt{S}^{++}_{p,q}$, $\mathtt{S}^{+-}_{p,q}$, $\mathtt{S}^{-+}_{p,q}$, *and* $\mathtt{S}^{--}_{p,q}$ *are evaluated in terms of Riemann zeta functions* $\zeta(s)$ *when* $p+q$ *is odd (see, e.g.,* [**5**] *and* [**21**, p. 33]*).*

(iii) Note that the parametric linear Euler sum $\mathtt{S}^{++}_{p,q}\left(0,\frac{1}{2}\right)$ *reduces to linear Euler sums as follows (see* [**51**, p. 405]*):*

$$\mathtt{S}^{++}_{p,q}\left(0,\frac{1}{2}\right)=(2^p+1)\,\zeta(p)\,\zeta(q)+\left(1-2^{p+q-1}\right)\mathtt{S}^{++}_{p,q}+2^{p+q-1}\mathtt{S}^{+-}_{p,q}. \tag{3.8}$$

Remark 3.2. *(i) As noted in Remark* (3.1)*, (ii), together with* (3.8)*, the following particular instances of the identities in Theorem 3.1, when* $t+u$ *is odd, are expressed in terms of Riemann zeta functions:*

$$\mathtt{R}^{++}_{t,u}(1,0)=\frac{1}{2^t}\sum_{k=1}^{\infty}\frac{\zeta(u,k)}{k^t}=\frac{1}{2^t}\,\mathtt{S}^{++}_{t,u}; \tag{3.9}$$

$$\mathtt{R}^{-+}_{t,u}(1,0)=\frac{1}{2^t}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^t}\,\zeta(u,k)=\frac{1}{2^t}\,\mathtt{S}^{-+}_{t,u}; \tag{3.10}$$

$$\mathtt{R}^{++}_{t,u}(3,1)+\mathtt{R}^{-+}_{t,u}(3,1)=\frac{1}{2^{u+2t-1}}\left\{\mathtt{S}^{++}_{t,u}\left(0,\frac{1}{2}\right)+\mathtt{S}^{++}_{t,u}\right\}; \tag{3.11}$$

$$\mathtt{R}^{++}_{t,u}(1,1)-\mathtt{R}^{-+}_{t,u}(1,1)=\frac{1}{2^{u+2t-1}}\left\{\mathtt{S}^{++}_{t,u}\left(0,\frac{1}{2}\right)+\mathtt{S}^{++}_{t,u}-\zeta(u+t)\right\}. \tag{3.12}$$

(ii) As in [**3**, Eq. (3.3)]*, the summation in* (3.6)

$$\sum_{n\geqslant 1}\frac{1}{\left(n+\frac{b-1}{2}\right)^u\left(n+\frac{a-1}{4}\right)^t}$$

are expressed in terms of finite sums of psi and polygamma functions.

Lemma 3.1. *Let* $p,\,k\in\mathbb{Z}_{\geqslant 0}$, $q,\,\mu\in\mathbb{N}$, *and* $\Re(a)>-1$. *Then*

$$\frac{x^q\,\mathrm{Ti}_t(x^q)}{1-x}=\sum_{n=1}^{\infty}\beta^{(t)}_{[n/2q]}\,x^n; \tag{3.13}$$

$$\frac{x^q\,\mathrm{Ti}_t(x^q)}{1+x}=\sum_{n=1}^{\infty}(-1)^n\,\beta^{(t)}_{[n/2q]}\,x^n; \tag{3.14}$$

$$\left(\mathrm{Ti}_t\left(x^q\right)\right)^{(\mu)}=\frac{1}{x^\mu}\sum_{\lambda=1}^{\mu}q^{\mu+1-\lambda}\,s(\mu,\mu+1-\lambda)\,\mathrm{Ti}_{t+\lambda-1-\mu}(x^q); \tag{3.15}$$

In particular,

$$\frac{d}{dx}\mathrm{Ti}_t\left(x^q\right)=\frac{q}{x}\,\mathrm{Ti}_{t-1}\left(x^q\right);$$

$$\left(\frac{x^q}{1-x}\right)^{(k)} = k! \sum_{j=0}^{k} \binom{q}{j} \frac{x^{q-j}}{(1-x)^{k-j+1}}; \tag{3.16}$$

$$\left(\frac{x^q}{1+x}\right)^{(k)} = (-1)^k \, k! \sum_{j=0}^{k} (-1)^j \binom{q}{j} \frac{x^{q-j}}{(1+x)^{k-j+1}}; \tag{3.17}$$

$$\int_0^1 x^a \, \log^p x \, \mathrm{d}x = \frac{(-1)^p \, p!}{(a+1)^{p+1}}; \tag{3.18}$$

$$\int_0^1 \frac{x^{p-1} \log^n x}{1-x^q} \, \mathrm{d}x = -\frac{1}{q^{n+1}} \, \psi^{(n)}\left(\tfrac{p}{q}\right) \quad (p,\, q \in \mathbb{R}_{>0},\, n \in \mathbb{N})\,; \tag{3.19}$$

$$\int_0^1 \frac{x^{p-1} \log^n x}{1+x^q} \, \mathrm{d}x = \frac{1}{q^{n+1}} \, \mathbf{b}^{(n)}\left(\tfrac{p}{q}\right) \quad (p,\, q \in \mathbb{R}_{>0},\, n \in \mathbb{N})\,, \tag{3.20}$$

where the function $\mathbf{b}(z)$ *is defined by*

$$\begin{aligned} \mathbf{b}(z) :&= \frac{1}{2}\left\{\psi\left(\frac{z+1}{2}\right) - \psi\left(\frac{z}{2}\right)\right\} \\ &= \int_0^1 \frac{x^{z-1}}{1+x} \, \mathrm{d}x \quad (\Re(z) > 0). \end{aligned} \tag{3.21}$$

Proof. With the aid of (1.50) and (1.51), the identity (3.15) may be proved by mathematical induction on μ. The other ones are easily derivable or a known result. For example, the formula (3.18) is recorded in [**58**, Entry 18.90]), and the definition and its integral formula in (3.21) are found in [**20**, p. 20]. □

THEOREM 3.2. *Let $t,\, q \in \mathbb{N}$, $a \in \mathbb{C}$ with $\Re(a) > q-1$, $m \in \mathbb{Z}_{\geqslant 0}$, and $p \in \mathbb{N}$ with $p \geqslant m+1$. Then the following formulas hold:*

$$\begin{aligned}
&\sum_{j=0}^{m} \binom{q}{j} \int_0^1 \frac{x^{a+q-j} \left(\log^p x\right) \mathrm{Ti}_t\left(x^q\right)}{(1-x)^{m-j+1}}\, \mathrm{d}x \\
&\quad = (-1)^p\, p! \sum_{n=1}^{\infty} \binom{n}{m} \frac{\beta^{(t)}_{[n/2q]}}{(a+n-m+1)^{p+1}} \\
&\qquad - \sum_{\mu=1}^{m} \sum_{\lambda=1}^{\mu} \sum_{j=0}^{m-\mu} \binom{q}{j} \frac{q^{\mu+1-\lambda}}{\mu!}\, s(\mu, \mu+1-\lambda) \\
&\qquad \times \int_0^1 \frac{x^{a+q-\mu-j} \left(\log^p x\right) \mathrm{Ti}_{t+\lambda-1-\mu}(x^q)}{(1-x)^{m-\mu-j+1}}\, \mathrm{d}x;
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
&\sum_{j=0}^{m} \binom{q}{j} \int_0^1 \frac{x^{a+q-j} \left(\log^p x\right) \mathrm{Ti}_t\left(x^q\right)}{(1-x)^{m-j+1}}\, \mathrm{d}x \\
&\quad = \frac{(-1)^p\, p!}{(2q)^{p+1}} \sum_{j=0}^{2q-1} \sum_{n=0}^{\infty} \binom{2qn+j}{m} \frac{\beta^{(t)}_{n}}{(n+\frac{a+j-m+1}{2q})^{p+1}} \\
&\qquad - \sum_{\mu=1}^{m} \sum_{\lambda=1}^{\mu} \sum_{j=0}^{m-\mu} \binom{q}{j} \frac{q^{\mu+1-\lambda}}{\mu!}\, s(\mu, \mu+1-\lambda) \\
&\qquad \times \int_0^1 \frac{x^{a+q-\mu-j} \left(\log^p x\right) \mathrm{Ti}_{t+\lambda-1-\mu}(x^q)}{(1-x)^{m-\mu-j+1}}\, \mathrm{d}x;
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
U^{-}_{(0,1)}\left(a+q,0,p,q,t\right) &= \int_0^1 \frac{x^{a+q} \left(\log^p x\right) \mathrm{Ti}_t\left(x^q\right)}{1-x}\, \mathrm{d}x \\
&= \frac{(-1)^p p!}{(2q)^{p+1}} \sum_{j=1}^{2q} \sum_{n=1}^{\infty} \frac{\beta^{(t)}_{n}}{\left(n+\frac{j+a}{2q}\right)^{p+1}} \\
&= \frac{(-1)^p p!}{(2q)^{p+1}} \sum_{j=1}^{2q} \mathtt{R}^{-+}_{t,p+1}\left(0, \frac{j+a}{2q}\right).
\end{aligned} \tag{3.24}$$

PROOF. Using Leibnitz's rule for higher-order differentiation of product of two functions, with the aid of (3.15), we obtain

$$\begin{aligned}\left(\frac{x^q\,\mathrm{Ti}_t(x^q)}{1-x}\right)^{(m)} &= \mathrm{Ti}_t\left(x^q\right)\left(\frac{x^q}{1-x}\right)^{(m)}\\ &+\sum_{\mu=1}^{m}\binom{m}{\mu}\frac{1}{x^\mu}\sum_{\lambda=1}^{\mu} q^{\mu+1-\lambda}\, s(\mu,\mu+1-\lambda)\,\mathrm{Ti}_{t+\lambda-1-\mu}(x^q)\left(\frac{x^q}{1-x}\right)^{(m-\mu)}.\end{aligned}$$

Employing (3.16), we get

$$\begin{aligned}\left(\frac{x^q\,\mathrm{Ti}_t(x^q)}{1-x}\right)^{(m)} &= m!\,\mathrm{Ti}_t\left(x^q\right)\sum_{j=0}^{m}\binom{q}{j}\frac{x^{q-j}}{(1-x)^{m-j+1}}\\ &+m!\sum_{\mu=1}^{m}\sum_{\lambda=1}^{\mu}\sum_{j=0}^{m-\mu}\binom{q}{j}\frac{q^{\mu+1-\lambda}}{\mu!}\, s(\mu,\mu+1-\lambda)\,\frac{x^{q-\mu-j}\,\mathrm{Ti}_{t+\lambda-1-\mu}(x^q)}{(1-x)^{m-\mu-j+1}}.\end{aligned}$$

Use of (3.13) offers

$$\begin{aligned}\sum_{n=1}^{\infty}\binom{n}{m}\beta^{(t)}_{[n/2q]}\,x^{n-m} &= \sum_{j=0}^{m}\binom{q}{j}\frac{x^{q-j}\,\mathrm{Ti}_t\left(x^q\right)}{(1-x)^{m-j+1}}\\ &+\sum_{\mu=1}^{m}\sum_{\lambda=1}^{\mu}\sum_{j=0}^{m-\mu}\binom{q}{j}\frac{q^{\mu+1-\lambda}}{\mu!}\, s(\mu,\mu+1-\lambda)\,\frac{x^{q-\mu-j}\,\mathrm{Ti}_{t+\lambda-1-\mu}(x^q)}{(1-x)^{m-\mu-j+1}}.\end{aligned}\tag{3.25}$$

Multiplying both sides of the identity (3.25) by $x^a\,(\log^p x)$ and integrating both sides of the resultant identity from $x=0$ to 1, and using (3.18), we have (3.22).

Employing the following ubiquitous identity which is a decomposition of the set of nonnegative integers into equivalent classes modulo $2q$:

$$\sum_{n=0}^{\infty}\Psi(n)=\sum_{j=0}^{2q-1}\sum_{n=0}^{\infty}\Psi(2qn+j)\qquad(q\in\mathbb{N}),$$

$\Psi:\mathbb{Z}_{\geqslant 0}\to\mathbb{C}$ being a function such that the involved series converges absolutely, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\binom{n}{m}\frac{\beta^{(t)}_{[n/2q]}}{(a+n-m+1)^{p+1}} &= \sum_{n=0}^{\infty}\binom{n}{m}\frac{\beta^{(t)}_{[n/2q]}}{(a+n-m+1)^{p+1}}\\ &=\frac{1}{(2q)^{p+1}}\sum_{j=0}^{2q-1}\sum_{n=0}^{\infty}\binom{2qn+j}{m}\frac{\beta^{(t)}_n}{(n+\frac{a+j-m+1}{2q})^{p+1}},\end{aligned}$$

which, upon putting (3.22), yields (3.23).

Putting $m=0$ in (3.23) and using the notation in (3.1) proves (3.24). $\square$

THEOREM 3.3. *Let t, $q \in \mathbb{N}$, $a \in \mathbb{C}$ with $\Re(a) > q-1$, $m \in \mathbb{Z}_{\geqslant 0}$, and $p \in \mathbb{N}$ with $p \geqslant m+1$. Then the following formulae hold:*

$$\begin{aligned}
&\sum_{j=0}^{m} (-1)^j \binom{q}{j} \int_0^1 \frac{x^{a+q-j} \left(\log^p x\right) \operatorname{Ti}_t\left(x^q\right)}{(1+x)^{m-j+1}} \, \mathrm{d}x \\
&\quad = (-1)^p \, p! \sum_{n=1}^{\infty} (-1)^{n+m} \binom{n}{m} \frac{\beta_{[n/2q]}^{(t)}}{(a+n-m+1)^{p+1}} \\
&\qquad - \sum_{\mu=1}^{m} \sum_{\lambda=1}^{\mu} \sum_{j=0}^{m-\mu} (-1)^{\mu+j} \frac{(m-\mu)!}{\mu!} \binom{q}{j} q^{\mu+1-\lambda} \, s(\mu, \mu+1-\lambda) \\
&\qquad \times \int_0^1 \frac{x^{a+q-\mu-j} \left(\log^p x\right) \operatorname{Ti}_{t+\lambda-1-\mu}(x^q)}{(1+x)^{m-\mu-j+1}} \, \mathrm{d}x;
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
&\sum_{j=0}^{m} (-1)^j \binom{q}{j} \int_0^1 \frac{x^{a+q-j} \left(\log^p x\right) \operatorname{Ti}_t\left(x^q\right)}{(1+x)^{m-j+1}} \, \mathrm{d}x \\
&\quad = \frac{(-1)^p \, p!}{(2q)^{p+1}} \sum_{j=0}^{2q-1} \sum_{n=0}^{\infty} (-1)^{j+m} \binom{2qn+j}{m} \frac{\beta_n^{(t)}}{(n+\frac{a+j-m+1}{2q})^{p+1}} \\
&\qquad - \sum_{\mu=1}^{m} \sum_{\lambda=1}^{\mu} \sum_{j=0}^{m-\mu} (-1)^{\mu+j} \frac{(m-\mu)!}{\mu!} \binom{q}{j} q^{\mu+1-\lambda} \, s(\mu, \mu+1-\lambda) \\
&\qquad \times \int_0^1 \frac{x^{a+q-\mu-j} \left(\log^p x\right) \operatorname{Ti}_{t+\lambda-1-\mu}(x^q)}{(1+x)^{m-\mu-j+1}} \, \mathrm{d}x;
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
U_{(0,1)}^{+}(a,0,p,q,t) &= \int_0^1 \frac{x^a \left(\log^p x\right) \operatorname{Ti}_t\left(x^q\right)}{1+x} \, \mathrm{d}x \\
&= \frac{(-1)^p p!}{(2q)^{p+1}} \sum_{j=1}^{2q} \sum_{n=1}^{\infty} \frac{(-1)^{j+1} \, \beta_n^{(t)}}{\left(n+\frac{j+a-q}{2q}\right)^{p+1}} \\
&= \frac{(-1)^p p!}{(2q)^{p+1}} \sum_{j=1}^{2q} (-1)^{j+1} \, \mathtt{R}_{t,p+1}^{-+}\left(0, \frac{j+a-q}{2q}\right).
\end{aligned} \tag{3.28}$$

Proof. Using Leibnitz's rule for higher-order differentiation of product of two functions, with the aid of (3.15), we obtain

$$\left(\frac{x^q\,\mathrm{Ti}_t(x^q)}{1+x}\right)^{(m)} = \mathrm{Ti}_t\,(x^q)\,\left(\frac{x^q}{1+x}\right)^{(m)} + \sum_{\mu=1}^{m}\binom{m}{\mu}\frac{1}{x^\mu}\sum_{\lambda=1}^{\mu} q^{\mu+1-\lambda}\,s(\mu,\mu+1-\lambda)\,\mathrm{Ti}_{t+\lambda-1-\mu}(x^q)\,\left(\frac{x^q}{1+x}\right)^{(m-\mu)}.$$

Employing (3.17), we get

$$\begin{aligned}\left(\frac{x^q\,\mathrm{Ti}_t(x^q)}{1+x}\right)^{(m)} =&(-1)^m\,m!\sum_{j=0}^{m}(-1)^j\binom{q}{j}\frac{x^{q-j}\,\mathrm{Ti}_t\,(x^q)}{(1+x)^{m-j+1}}\\ &+m!\sum_{\mu=1}^{m}\sum_{\lambda=1}^{\mu}\sum_{j=0}^{m-\mu}(-1)^{m-\mu+j}\,\frac{(m-\mu)!}{\mu!}\\ &\times\binom{q}{j}\,q^{\mu+1-\lambda}\,s(\mu,\mu+1-\lambda)\,\frac{x^{q-\mu-j}\,\mathrm{Ti}_{t+\lambda-1-\mu}(x^q)}{(1+x)^{m-\mu-j+1}}.\end{aligned}$$

Use of (3.14) offers

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^n\binom{n}{m}\beta^{(t)}_{[n/2q]}\,x^{n-m} =& \sum_{j=0}^{m}(-1)^{m+j}\binom{q}{j}\frac{x^{q-j}\,\mathrm{Ti}_t\,(x^q)}{(1+x)^{m-j+1}}\\ &+\sum_{\mu=1}^{m}\sum_{\lambda=1}^{\mu}\sum_{j=0}^{m-\mu}(-1)^{m-\mu+j}\,\frac{(m-\mu)!}{\mu!}\\ &\times\binom{q}{j}\,q^{\mu+1-\lambda}\,s(\mu,\mu+1-\lambda)\,\frac{x^{q-\mu-j}\,\mathrm{Ti}_{t+\lambda-1-\mu}(x^q)}{(1+x)^{m-\mu-j+1}}.\end{aligned}\tag{3.29}$$

Multiplying both sides of the identity (3.29) by $x^a\log^p(x)$ and integrating both sides of the resultant identity from $x=0$ to 1, and using (3.18), we obtain (3.26).

Similar process as in the proof of (3.23) and (3.24) can prove (3.27) and (3.28). The details are omitted. □

4. Integrals on the half real line $x > 0$

This section reveals that certain integrals on the half real line $x > 0$, whose integrands are of the types in (1.1), can be transformed into those on the $(0, 1)$ as in Section 3.

Two integral formulas are provided in the following lemma, before stating theorems in this section.

Lemma 4.1. *The following integral formulae hold true:*

$$\int_0^1 \frac{x^{m-1}\log^{\alpha-1}x}{(1-x)^{m+1}}\,\mathrm{d}x = (-1)^{\alpha-1}(\alpha-1)!\sum_{j=0}^{\infty}\frac{(m)_j}{j!}\,\zeta(\alpha,m+j) \quad (\alpha\in\mathbb{Z}_{\geqslant 3},\, m\in\mathbb{N}), \tag{4.1}$$

and

$$\int_0^1 \frac{x^{m-1}\log^{\alpha-1}x}{(1+x)^{m+1}}\,\mathrm{d}x = (-1)^{\alpha-1}(\alpha-1)!\sum_{j=0}^{\infty}\frac{(-1)^j\,(m)_j}{j!}\,\eta(\alpha,m+j) \quad (\alpha\in\mathbb{Z}_{\geqslant 2},\, m\in\mathbb{N}). \tag{4.2}$$

In particular,

$$\sum_{j=0}^{\infty}\zeta(\alpha,j+1)=\zeta(\alpha-1) \quad (\alpha\in\mathbb{Z}_{\geqslant 3}), \tag{4.3}$$

and

$$\sum_{j=0}^{\infty}(-1)^j\,\eta(z,j+1)=\eta(z-1) \quad (\alpha\in\mathbb{Z}_{\geqslant 2}), \tag{4.4}$$

where

$$\int_0^1 \frac{\log^{\alpha-1}x}{(1-x)^2}\,\mathrm{d}x = (-1)^{\alpha-1}(\alpha-1)!\,\zeta(\alpha-1) \quad (\alpha\in\mathbb{Z}_{\geqslant 3}), \tag{4.5}$$

and

$$\int_0^1 \frac{\log^{\alpha-1}x}{(1+x)^2}\,\mathrm{d}x = (-1)^{\alpha-1}(\alpha-1)!\,\eta(\alpha-1) \quad (\alpha\in\mathbb{Z}_{\geqslant 2}). \tag{4.6}$$

It is noted in passing that, in view of (4.5) and (4.6), the following known integral formulas are recalled (cf., [**24**, p. 547, Entries 4.272-9 and 4.272-8]):

$$\int_0^1 \frac{\log^{\alpha-1}x}{1-x}\,\mathrm{d}x = (-1)^{\alpha-1}(\alpha-1)!\,\zeta(\alpha) \quad (\alpha\in\mathbb{Z}_{\geqslant 2}), \tag{4.7}$$

and

$$\int_0^1 \frac{\log^{\alpha-1}x}{1+x}\,\mathrm{d}x = (-1)^{\alpha-1}(\alpha-1)!\,\eta(\alpha) \quad (\alpha\in\mathbb{Z}_{\geqslant 1}). \tag{4.8}$$

PROOF. Let I_l be the left-sided integral of (4.1). Then we have

$$I_l = \sum_{k=0}^{\infty} \int_0^1 \frac{x^{m+k-1} \log^{\alpha-1} x}{(1-x)^m} \, \mathrm{d}x.$$

Using

$$(1-x)^{-m} = \sum_{j=0}^{\infty} \frac{(m)_j}{j!} x^j \quad (|x| < 1),$$

we obtain

$$I_l = \sum_{j=0}^{\infty} \frac{(m)_j}{j!} \sum_{k=0}^{\infty} \int_0^1 x^{m+k+j-1} \log^{\alpha-1} x \, \mathrm{d}x.$$

By using the integral formula (3.18), we get

$$\begin{aligned} I_l &= (-1)^{\alpha-1} (\alpha-1)! \sum_{j=0}^{\infty} \frac{(m)_j}{j!} \sum_{k=0}^{\infty} \frac{1}{(k+m+j)^\alpha} \\ &= (-1)^{\alpha-1} (\alpha-1)! \, \zeta(\alpha, m+j), \end{aligned}$$

which is just the right member of (4.1). Setting $m = 1$ in (4.1) gives (4.3).
Similarly, the formulas (4.2) and (4.4) can be proved. The details are omitted. □

THEOREM 4.1. *Let m, p, $q \in \mathbb{N}$, $t \in \mathbb{Z}_{\geqslant 2}$, and $p \geqslant m+1$. Then*

$$\begin{aligned} V^-(m,p,q,t) &:= \int_0^{\infty} \frac{(\log^p x) \mathrm{Ti}_t(x^q)}{(1-x)^{m+1}} \, \mathrm{d}x \\ &= U^-_{(0,1)}(0,m,p,q,t) + (-1)^{m+p+t+1} U^-_{(0,1)}(m-1,m,p,q,t) \\ &\quad + \frac{\pi}{2} (-1)^{m+1} p! \binom{p+t-1}{p} q^{t-1} \sum_{j=0}^{\infty} \frac{(m)_j}{j!} \zeta(p+t, m+j) \\ &\quad + 2(-1)^{m+1} p! \sum_{k=1}^{\left[\frac{1}{2}(t-1)\right]} \binom{p+t-2k-1}{p} q^{t-2k-1} \mathrm{Ti}_{2k+1}(1) \\ &\quad \times \sum_{j=0}^{\infty} \frac{(m)_j}{j!} \zeta(p+t-2k, m+j). \end{aligned} \tag{4.9}$$

PROOF. Put

$$\Omega(m,p,q,t;x) := \frac{(\log^p x) \, \mathrm{Ti}_t(x^q)}{(1-x)^{m+1}}.$$

Noticing that $\lim_{x \downarrow 0} \Omega(m,p,q,t;x)$, $\lim_{x \uparrow \infty} \Omega(m,p,q,t;x)$ and $\lim_{x \to 1} \Omega(m,p,q,t;x)$ exit, we write

$$V^-(m,p,q,t) = \int_0^1 \Omega(m,p,q,t;x) \, \mathrm{d}x + \int_1^{\infty} \Omega(m,p,q,t;x) \, \mathrm{d}x. \tag{4.10}$$

Using the transformation $y = \frac{1}{x}$ in the last integral in (4.10) and recovering the variable x instead of y in the resultant integral, we obtain

(4.11)
$$V^{-}(m,p,q,t) = \int_0^1 \Omega(m,p,q,t;x)\,\mathrm{d}x + (-1)^{m+p+1}\int_0^1 \frac{x^{m-1}(\log^p x)\operatorname{Ti}_t\left(\frac{1}{x^q}\right)}{(1-x)^{m+1}}\,\mathrm{d}x.$$

Employing (1.20) and using (4.1), we get

$$\begin{aligned}\int_0^1 \frac{x^{m-1}(\log^p x)\operatorname{Ti}_t\left(\frac{1}{x^q}\right)}{(1-x)^{m+1}}\,\mathrm{d}x &= (-1)^t\int_0^1 \frac{x^{m-1}(\log^p x)\operatorname{Ti}_t(x^q)}{(1-x)^{m+1}}\,\mathrm{d}x \\ &+\frac{\pi}{2}(-1)^p\,p!\binom{p+t-1}{p}q^{t-1}\sum_{j=0}^{\infty}\frac{(m)_j}{j!}\,\zeta(p+t,m+j) \\ &+2\,(-1)^p\,p!\sum_{k=1}^{\left[\frac{1}{2}(t-1)\right]}\binom{p+t-2k-1}{p}q^{t-2k-1}\operatorname{Ti}_{2k+1}(1) \\ &\times\sum_{j=0}^{\infty}\frac{(m)_j}{j!}\,\zeta(p+t-2k,m+j).\end{aligned} \tag{4.12}$$

Finally, setting (4.12) in the last integral in (4.11) yields the desired result (4.9). □

Corollary 4.1. *Let $q \in \mathbb{N}$ and $p,\, t \in \mathbb{Z}_{\geqslant 2}$. Then if $p+t$ is even,*

$$\begin{aligned}&\int_0^\infty \frac{(\log^p x)\operatorname{Ti}_t(x^q)}{(1-x)^2}\,\mathrm{d}x = 2\int_0^1 \frac{(\log^p x)\operatorname{Ti}_t(x^q)}{(1-x)^2}\,\mathrm{d}x \\ &+2\,p!\sum_{k=0}^{\left[\frac{1}{2}(t-1)\right]}(-1)^k\binom{p+t-2k-1}{p}\frac{q^{t-2k-1}\,\pi^{2k+1}E_{2k}\,\zeta(p+t-2k-1)}{(2k)!2^{2k+2}}.\end{aligned} \tag{4.13}$$

if $p+t$ is odd,

$$\begin{aligned}&\int_0^\infty \frac{(\log^p x)\operatorname{Ti}_t(x^q)}{(1-x)^2}\,\mathrm{d}x \\ &=2\,p!\sum_{k=0}^{\left[\frac{1}{2}(t-1)\right]}(-1)^k\binom{p+t-2k-1}{p}\frac{q^{t-2k-1}\,\pi^{2k+1}E_{2k}\,\zeta(p+t-2k-1)}{(2k)!2^{2k+2}}.\end{aligned} \tag{4.14}$$

Proof. Setting $m=1$ in the result in Theorem 4.1 provides the identities here. Here we used (1.19) and the identity (4.3). □

THEOREM 4.2. *Let* $p,\ q \in \mathbb{N},\ t \in \mathbb{Z}_{\geqslant 2},$ *and* $p \geqslant m+1$. *Then*

$$\begin{aligned} V^{+}(m,p,q,t) &:= \int\limits_0^\infty \frac{(\log^p x)\,\mathrm{Ti}_t(x^q)}{(1+x)^{m+1}}\,\mathrm{d}x \\ &= U^{+}_{(0,1)}(0,m,p,q,t) + (-1)^{p+t}\,U^{+}_{(0,1)}(m-1,m,p,q,t) \\ &\quad + \frac{\pi\,p!}{2}\binom{p+t-1}{p} q^{t-1} \sum_{j=0}^{\infty} \frac{(-1)^j\,(m)_j}{j!}\,\eta(p+t,m+j) \\ &\quad + 2\,p! \sum_{k=1}^{\left[\frac{1}{2}(t-1)\right]} q^{t-2k-1}\binom{p+t-2k-1}{p}\mathrm{Ti}_{2k+1}(1) \\ &\quad \times \sum_{j=0}^{\infty} \frac{(-1)^j\,(m)_j}{j!}\,\eta(p+t-2k,m+j). \end{aligned} \tag{4.15}$$

PROOF. The proof would run parallel that of Theorem 4.1. Here, instead of (4.1), we use (4.2). □

COROLLARY 4.2. *Let* $q \in \mathbb{N}$ *and* $p,\ t \in \mathbb{Z}_{\geqslant 2}$. *Then if* $p+t$ *is even,*

$$\begin{aligned} &\int\limits_0^\infty \frac{(\log^p x)\,\mathrm{Ti}_t(x^q)}{(1+x)^2}\,\mathrm{d}x = 2\int\limits_0^1 \frac{(\log^p x)\,\mathrm{Ti}_t(x^q)}{(1+x)^2}\,\mathrm{d}x \\ &\quad + 2\,p! \sum_{k=0}^{\left[\frac{1}{2}(t-1)\right]} (-1)^k \binom{p+t-2k-1}{p} \frac{q^{t-2k-1}\,\pi^{2k+1}E_{2k}\ \eta(p+t-2k-1)}{(2k)!2^{2k+2}}. \end{aligned} \tag{4.16}$$

if $p+t$ *is odd,*

$$\begin{aligned} &\int\limits_0^\infty \frac{(\log^p x)\,\mathrm{Ti}_t(x^q)}{(1+x)^2}\,\mathrm{d}x \\ &= 2\,p! \sum_{k=0}^{\left[\frac{1}{2}(t-1)\right]} (-1)^k \binom{p+t-2k-1}{p} \frac{q^{t-2k-1}\,\pi^{2k+1}E_{2k}\ \eta(p+t-2k-1)}{(2k)!2^{2k+2}}. \end{aligned} \tag{4.17}$$

PROOF. Setting $m=1$ in the result in Theorem 4.2 yields the identities here. Here we employed (1.19) and the identity (4.4). □

5. Particular cases

This section explores ceratin particular instances of (1.1). The following theorem considers the case $m = -1$ in (1.1).

THEOREM 5.1. *Let $t,\, p \in \mathbb{Z}_{\geqslant 0}$, $q \in \mathbb{R}_{>0}$, and $a \in \mathbb{C}$ with $|a+1| < q$. Then*

$$\int_0^1 x^a \left(\log^p x\right) \mathrm{Ti}_t\left(x^q\right) \mathrm{d}x = \frac{(-1)^p\, p!}{q^{p+1}} \sum_{j=0}^{\infty} \frac{(-1)^j\, (p+1)_j}{j!} \left(\tfrac{a+1}{q}\right)^j \beta(t+p+j+1). \tag{5.1}$$

In particular,

$$\int_0^1 x^{-1} \left(\log^p x\right) \mathrm{Ti}_t\left(x^q\right)\, \mathrm{d}x = \frac{(-1)^p\, p!}{q^{p+1}} \beta(p+t+1). \tag{5.2}$$

PROOF. Let $\mathcal{L}_1$ be the left member of (5.1). Using (1.2) and (3.18) gives

$$\begin{aligned} \mathcal{L}_1 &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^t} \int_0^1 x^{a+q(2k-1)} \left(\log^p x\right) \mathrm{d}x \\ &= \frac{(-1)^p\, p!}{q^{p+1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^t \left(2k-1+\frac{a+1}{q}\right)^{p+1}} \\ &= \frac{(-1)^p\, p!}{q^{p+1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^{t+p+1} \left(1+\frac{a+1}{q(2k-1)}\right)^{p+1}}. \end{aligned} \tag{5.3}$$

Employing binomial theorem provides

$$\frac{1}{\left(1+\frac{a+1}{q(2k-1)}\right)^{p+1}} = \sum_{j=0}^{\infty} \frac{(-1)^j (p+1)_j}{j!} \left(\frac{a+1}{q(2k-1)}\right)^j \quad \left(\left|\frac{a+1}{q(2k-1)}\right| < 1\right). \tag{5.4}$$

Setting (5.4) in (5.3) and using (1.2) proves the identity (5.1).

Obviously (5.2) is a particular case of (5.1) when $a = -1$. □

COROLLARY 5.1. *Let $t,\, p \in \mathbb{Z}_{\geqslant 0}$, $q \in \mathbb{R}_{>0}$, and $a - q \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -2}$. Then*

$$\begin{aligned} \int_0^1 \frac{x^a \left(\log^p x\right) \mathrm{Ti}_t\left(x^q\right)}{1-x}\, \mathrm{d}x &= (-1)^p\, p! \sum_{n=1}^{\infty} \frac{\beta_{[n/2q]}^{(t)}}{(n+a-q+1)^{p+1}} \\ &= (-1)^p\, p!\, \zeta\left(p+1\right) \beta(t) - (-1)^p\, p! \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, H_{2qn-q+a}^{(p+1)}}{(2n-1)^t}. \end{aligned} \tag{5.5}$$

PROOF. Setting $m = 0$ and replacing a by $a - q$ in (3.22) yields the first equality in (5.5). Expanding $\mathrm{Ti}_t\left(x^q\right)$ and using (3.19), via (1.33), shows the second equality in (5.5). □

COROLLARY 5.2. *Let $t,\, p \in \mathbb{Z}_{\geqslant 0}$, $q \in \mathbb{R}_{>0}$, and $a - q \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -2}$. Then*

$$\begin{aligned} \int_0^1 \frac{x^a\left(\log^p x\right)\operatorname{Ti}_t\left(x^q\right)}{1+x}\,\mathrm{d}x &= (-1)^p\, p! \sum_{n=1}^{\infty} (-1)^n \frac{\beta_{[n/2q]}^{(t)}}{(n+a-q+1)^{p+1}} \\ &= \frac{(-1)^p\, p!}{2^{p+1}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^t}\left(H_{qn+\left(\frac{a-q}{2}\right)}^{(p+1)} - H_{qn+\left(\frac{a-q-1}{2}\right)}^{(p+1)}\right). \end{aligned} \tag{5.6}$$

PROOF. Setting $m = 0$ and replacing a by $a - q$ in (3.26) proves the first equality in (5.6). The second equality in (5.6) can be shown by using (1.33) and (3.20). □

COROLLARY 5.3. *Let $t,\, p \in \mathbb{Z}_{\geqslant 0}$, $q \in \mathbb{R}_{>0}$, and $a - q \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -2}$. Then*

$$\int_0^1 \frac{x^a\left(\log^p x\right)\operatorname{Ti}_t\left(x^q\right)}{1-x^2}\,\mathrm{d}x = \frac{(-1)^p\, p!}{2^{p+1}} \sum_{n=1}^{\infty} \frac{\beta_{[n/q]}^{(t)}}{\left(n+\frac{a-q+1}{2}\right)^{p+1}}. \tag{5.7}$$

PROOF. Adding (5.5) and (5.6), side by side, offers (5.7). □

Examples. Some examples are demonstrated as follows:

(i) Setting $a = p = q = t = 1$ in (5.5), with the aid of (2.15) and

$$\beta(1) = \arctan(1) = \frac{\pi}{4},$$

gives

$$\begin{aligned} \int_0^1 \frac{x\left(\log x\right)\operatorname{Ti}_1\left(x\right)}{1-x}\,\mathrm{d}x &= -\sum_{n\geqslant 1} \frac{\beta_{[n/2]}^{(t)}}{(n+1)^2} = \sum_{n\geqslant 1} \frac{(-1)^{n+1} H_{2n}^{(2)}}{2n-1} - \zeta\left(2\right)\beta\left(1\right) \\ &= \frac{\pi}{4} - \frac{\pi^2}{48} - \frac{\pi^3}{16} - \frac{\pi\log^2 2}{16} + \frac{G\log 2}{2} - \frac{\log 2}{2} + 2\mathscr{G}. \end{aligned} \tag{5.8}$$

(ii) Setting $a = q = \frac{1}{2}$ and $p + 1 = t$ in (5.5) yields

$$\begin{aligned} \int_0^1 \frac{\sqrt{x}\,\left(\log^{t-1} x\right)\operatorname{Ti}_t(\sqrt{x})}{1-x}\,\mathrm{d}x &= (-1)^{t-1}\,(t-1)! \sum_{n\geqslant 1} \frac{\beta_n^{(t)}}{(n+1)^t} \\ &= (-1)^{t-1}\,(t-1)!\zeta(t)\,\beta(t) - (-1)^{t-1}\,(t-1)! \sum_{n\geqslant 1} \frac{(-1)^{n+1}\, H_n^{(t)}}{(2n-1)^t}, \end{aligned} \tag{5.9}$$

which can admit a complete closed form representation upon a shift in the index n and using (2.6). For $t=5$, we evaluate

$$\sum_{n\geqslant 1}\frac{(-1)^{n+1}\,H_n^{(5)}}{(2n+1)^5}=\frac{25}{256}\pi^5\zeta(5)+\frac{1905}{128}\pi^3\zeta(7)+\frac{17885}{32}\pi\zeta(9) -\frac{7}{72}\pi^4\beta(6)-\frac{70}{3}\pi^2\beta(8)-2016\,\beta(10). \tag{5.10}$$

(iii) Setting $a=-\frac{1}{2}$, $p=1$, $q=\frac{1}{2}$, and $t=2$ in (5.5) provides

$$\begin{aligned}\int_0^1\frac{(\log x)\,\mathrm{Ti}_2\left(x^{\frac{1}{2}}\right)}{x^{\frac{1}{2}}(1-x)}\,\mathrm{d}x &= -\sum_{n\geqslant 1}\frac{\beta_n^{(2)}}{n^2}=\sum_{n\geqslant 0}\frac{(-1)^n\,H_n^{(2)}}{(2n+1)^2}-\zeta\,(2)\,G\\ &=\frac{\pi^2\,G}{4}+\frac{7\pi}{4}\,\zeta(3)-6\,\beta(4),\end{aligned} \tag{5.11}$$

where

$$\sum_{n\geqslant 0}\frac{(-1)^n\,H_n^{(2)}}{(2n+1)^2}=\frac{7\pi}{4}\,\zeta(3)-\frac{\pi^2\,G}{12}-6\,\beta(4). \tag{5.12}$$

(iv) Setting $a=p=q=t=1$ in (5.6) offers

$$\begin{aligned}\int_0^1\frac{x\,(\log x)\,\mathrm{Ti}_1\,(x)}{1+x}\,\mathrm{d}x &= -\sum_{n\geqslant 1}^{\infty}\frac{(-1)^n\,\beta_{[n/2]}^{(1)}}{(n+1)^2}\\ &=-\frac{1}{4}\sum_{n\geqslant 1}\frac{(-1)^{n+1}\left(H_n^{(2)}-H_{n-\frac{1}{2}}^{(2)}\right)}{2n-1}.\end{aligned} \tag{5.13}$$

Applying (2.14) and (2.18) in (5.13), one obtains

$$\int_0^1\frac{x\,(\log x)\,\mathrm{Ti}_1\,(x)}{1+x}\,\mathrm{d}x=\frac{\pi^3}{64}+\frac{\pi^2}{48}-\frac{\pi}{4}-\frac{G\log 2}{2}+\frac{\log 2}{2}. \tag{5.14}$$

(v) Setting $a=1$, $q=1$, $t=2$ and replacing p by $2p-1$ in (5.6), with the aid of (2.2), one finds

$$\begin{aligned}\int_0^1\frac{x\,(\log^{2p-1}x)\,\mathrm{Ti}_2\,(x)}{1+x}\,\mathrm{d}x &= -(2p-1)!\sum_{n\geqslant 1}\frac{(-1)^n\,\beta_{[n/2]}^{(2)}}{(n+1)^{2p}}\\ &=-\frac{(2p-1)!}{2^{2p}}\sum_{n\geqslant 1}\frac{(-1)^{n+1}\left(H_n^{(2p)}-H_{n-\frac{1}{2}}^{(2p)}\right)}{(2n-1)^2}\\ &=-\frac{(2p-1)!}{2^{2p+2}}\,\chi_{2p,2}^{+-}\left(0,-\tfrac{1}{2}\right)\quad (p\in\mathbb{N}),\end{aligned} \tag{5.15}$$

which can be explicitly evaluated by the identity (2.2). In particular, setting $p=1$ in (2.2) offers

$$\chi_{2,2}^{+-}\left(0,-\tfrac{1}{2}\right)=64\,\log 2+\tfrac{4\,\pi^2}{3}-8\,\pi-\pi^2\,G+16\,G+\tfrac{21\,\pi\,\zeta(3)}{4}-32\,\beta(4). \tag{5.16}$$

Putting $p=1$ in (5.15), with the aid of (5.16), yields

$$\begin{aligned}\int_0^1 \frac{x\,(\log x)\,\mathrm{Ti}_2\,(x)}{1+x}\,\mathrm{d}x &= -\sum_{n\geqslant 1}\tfrac{(-1)^n\,\beta^{(2)}_{[n/2]}}{(n+1)^2}\\ &= -\tfrac{1}{4}\sum_{n\geqslant 1}\tfrac{(-1)^{n+1}\left(H_n^{(2)}-H^{(2)}_{n-\frac{1}{2}}\right)}{(2n-1)^2}=-\tfrac{1}{16}\,\chi_{2,2}^{+-}\left(0,-\tfrac{1}{2}\right)\\ &= -\tfrac{\log 2}{4}-\tfrac{\pi^2}{12}+\tfrac{\pi}{2}+\tfrac{\pi^2\,G}{16}-G-\tfrac{21\,\pi\,\zeta(3)}{64}+2\,\beta(4).\end{aligned}\tag{5.17}$$

Setting $p=2$ in (2.2) offers

$$\begin{aligned}\tfrac{1}{4}\,\chi_{4,2}^{+-}\left(0,-\tfrac{1}{2}\right)=&128\,\log 2+16\,G+12\,\zeta(3)+\tfrac{\pi^4}{180}+4\,\pi^2-16\,\pi\\ &+\tfrac{465\,\pi\,\zeta(5)}{32}-\pi^2\,\beta(4)-\tfrac{3\,\pi^3\,\zeta(3)}{32}+\tfrac{7\,G\,\pi^4}{48}-48\,\beta(6).\end{aligned}\tag{5.18}$$

Putting $p=2$ in (5.15), with the aid of (5.18), one obtains

$$\begin{aligned}\int_0^1 \frac{x\,(\log^3 x)\,\mathrm{Ti}_2\,(x)}{1+x}\,\mathrm{d}x &= -6\sum_{n\geqslant 1}\tfrac{(-1)^n\,\beta^{(2)}_{[n/2]}}{(n+1)^4}\\ &= -\tfrac{3}{8}\sum_{n\geqslant 1}\tfrac{(-1)^{n+1}\left(H_n^{(4)}-H^{(4)}_{n-\frac{1}{2}}\right)}{(2n-1)^2}=-\tfrac{3}{32}\,\chi_{4,2}^{+-}\left(0,-\tfrac{1}{2}\right)\\ &= -48\,\log 2-6\,G-\tfrac{9\,\zeta(3)}{2}-\tfrac{\pi^4}{480}-\tfrac{3\,\pi^2}{2}+6\,\pi\\ &\quad -\tfrac{1395\,\pi\,\zeta(5)}{256}+\tfrac{3\,\pi^2\,\beta(4)}{8}+\tfrac{9\,\pi^3\,\zeta(3)}{256}-\tfrac{21\,G\,\pi^4}{384}+18\,\beta(6).\end{aligned}\tag{5.19}$$

(vi) Combining the identities in Theorem 3.2 and Corollary 5.1, one gets

$$\begin{aligned}U^{-}_{(0,1)}\,(a,0,p,q,t) &= \int_0^1 \frac{x^a\,(\log^p x)\,\mathrm{Ti}_t\,(x^q)}{1-x}\,\mathrm{d}x\\ &= (-1)^p\,p!\zeta\,(p+1)\,\beta(t)-(-1)^p\,p!\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,H^{(p+1)}_{2qn-q+a}}{(2n-1)^t}\\ &= (-1)^p p!\sum_{j=1}^{2q}\sum_{n=1}^{\infty}\frac{\beta_n^{(t)}}{(2qn+j+a-q)^{p+1}},\end{aligned}$$

which, upon equating the 2nd and 3rd equalities, gives

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,H^{(p+1)}_{2qn-q+a}}{(2n-1)^t}=\zeta\,(p+1)\,\beta(t)-\sum_{j=1}^{2q}\sum_{n=1}^{\infty}\frac{\beta_n^{(t)}}{(2qn+j+a-q)^{p+1}}.\tag{5.20}$$

Setting $a = q = \frac{1}{2}$ and $a = 0,\ q = \frac{1}{2}$ in (5.20), respectively, produces

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(p+1)}}{(2n-1)^t} = \zeta(p+1)\,\beta(t) - \sum_{n=1}^{\infty} \frac{\beta_n^{(t)}}{(n+1)^{p+1}}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}^{(p+1)}}{(2n-1)^t} = \zeta(p+1)\,\beta(t) - \sum_{n=1}^{\infty} \frac{\beta_n^{(t)}}{\left(n+\frac{1}{2}\right)^{p+1}}.$$

Putting $t = 2$ and replacing p by $2p-1$ in the last two identities, one finds

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(2p)}}{(2n-1)^2} = \zeta(2p)\,\beta(2) - \sum_{n=1}^{\infty} \frac{\beta_n^{(2)}}{(n+1)^{2p}}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}^{(2p)}}{(2n-1)^2} = \zeta(2p)\,\beta(2) - \sum_{n=1}^{\infty} \frac{\beta_n^{(2)}}{\left(n+\frac{1}{2}\right)^{2p}}.$$

Subtracting the last identity from the penultimate one, side by side, in view of (2.2), one derives the following identity: For $p \in \mathbb{N}$,

$$\begin{aligned} \chi_{2p,2}^{+-}\left(0, -\frac{1}{2}\right) &= \sum_{n\geqslant 1} \frac{(-1)^{n+1}\left(H_n^{(2p)} - H_{n-\frac{1}{2}}^{(2p)}\right)}{(2n-1)^2} \\ &= 2^{2p} \sum_{n=1}^{\infty} \frac{\beta_n^{(2)}}{(2n+1)^{2p}} - \sum_{n=1}^{\infty} \frac{\beta_n^{(2)}}{(n+1)^{2p}}. \end{aligned} \tag{5.21}$$

(vii) For $t \in \mathbb{N}$,

$$\begin{aligned} U_{(0,1)}^{+}\left(-\frac{1}{2}, 0, 0, \frac{1}{2}, t\right) &= \int_0^1 \frac{\mathrm{Ti}_t\left(\sqrt{x}\right)}{(1+x)\sqrt{x}}\, \mathrm{d}x = \frac{1}{2} \sum_{n\geqslant 0} \frac{(-1)^n \left(H_{\frac{n}{2}} - H_{\frac{n}{2}-\frac{1}{2}}\right)}{(2n+1)^t} \\ &= t\lambda(t+1) - \sum_{j=0}^{t-1} \beta(t-j)\,\beta(j+1). \end{aligned} \tag{5.22}$$

(viii) Considering the case $m = 1$ in Theorem 3.3, after considerable algebraic simplification, one obtains

$$\begin{aligned} \int_0^1 \frac{(\log^p x)\,\mathrm{Ti}_t(x^q)}{(1+x)^2}\, \mathrm{d}x &= \frac{(-1)^p\, p!}{2^p} \sum_{n\geqslant 1} \frac{(-1)^{n+1}\left(H_{qn-\frac{q}{2}}^{(p)} - H_{qn-\frac{q+1}{2}}^{(p)}\right)}{(2n-1)^t} \\ &\quad + \frac{(-1)^p\, p!\,q}{2^{p+1}} \sum_{n\geqslant 1} \frac{(-1)^{n+1}\left(H_{qn-\frac{q}{2}}^{(p+1)} - H_{qn-\frac{q+1}{2}}^{(p+1)}\right)}{(2n-1)^{t-1}}. \end{aligned} \tag{5.23}$$

Here we are interested in the case $p = q = 1$ in (5.23), which may be useful in evaluating a similar integral on the half line $x \geqslant 0$ (see (4.16)). We also consider the following relation

$$\mathrm{Ti}_2\left(x^q\right) = x^q\, {}_3F_2\left[\begin{array}{c} \frac{1}{2}, \frac{1}{2}, 1 \\ \frac{3}{2}, \frac{3}{2} \end{array} \middle| -x^{2q}\right]$$

where ${}_pF_q(\cdot)$ is the generalized hypergeometric function (see, e.g., [**60**, Section 1.5]). Then we get

$$\begin{aligned} U^+_{(0,1)}\left(0,1,1,1,2\right) &= \int_0^1 \frac{(\log x)\,\mathrm{Ti}_2\left(x\right)}{\left(1+x\right)^2}\,\mathrm{d}x \\ &= \int_0^1 \frac{x\left(\log x\right)}{\left(1+x\right)^2}\, {}_3F_2\left[\begin{array}{c} \frac{1}{2}, \frac{1}{2}, 1 \\ \frac{3}{2}, \frac{3}{2} \end{array} \middle| -x^2\right]\mathrm{d}x \\ &= 3\mathscr{G} - \frac{7\pi^3}{128} - \frac{3\pi\log^2 2}{32}. \end{aligned} \tag{5.24}$$

Indeed, setting $p = q = 1$ in (5.23), with the aid of

$$H^{(p)}_{n-1} = H^{(p)}_n - \frac{1}{n^p} \quad (p \in \mathbb{N}),$$

gives

$$\begin{aligned} U^+_{(0,1)}\left(0,1,1,1,2\right) = \frac{1}{2}\sum_{n\geqslant 1}\frac{(-1)^{n+1}\,H_n}{\left(2n-1\right)^2} - \frac{1}{2}\sum_{n\geqslant 1}\frac{(-1)^{n+1}\,H_{n-\frac{1}{2}}}{\left(2n-1\right)^2} \\ -\frac{1}{4}\sum_{n\geqslant 1}\frac{(-1)^{n+1}\,H^{(2)}_n}{2n-1} + \frac{1}{4}\sum_{n\geqslant 1}\frac{(-1)^{n+1}\,H^{(2)}_{n-\frac{1}{2}}}{2n-1} \\ +\frac{1}{2}\sum_{n\geqslant 1}\frac{(-1)^n}{n(2n-1)^2} + \frac{1}{4}\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n^2(2n-1)}, \end{aligned}$$

each term of which is evaluated by, respectively, using (2.11), (2.17), (2.14), (2.18),

$$\sum_{n\geqslant 1}\frac{(-1)^n}{n\,(2n-1)^2} = \frac{\pi}{2} - \log 2 - 2G \tag{5.25}$$

and

$$\sum_{n\geqslant 1}\frac{(-1)^n}{n^2\,(2n-1)} = -\pi + \frac{\pi^2}{12} + 2\log 2 \tag{5.26}$$

to produce the desired result in (5.24).

(ix) By using [**64**, Eq. (3.2)], with the aid of (1.26) and (1.27), one finds from (3.3) that, for $a \in \mathbb{R}$ with $|a| < 1$,

$$\begin{aligned}\mathtt{R}^{++}_{1,2}(1,a) &= \frac{1}{2}\,\mathtt{S}^{++}_{1,2}(0,a)\\ &= \frac{1}{2}\left\{-\frac{\psi^{(1)}(a+1)}{a} + \psi^{(1)}(a)\Big(\psi(a+1)+\gamma\Big) - \frac{1}{2}\psi^{(2)}(a+1)\right\}.\end{aligned} \tag{5.27}$$

Note that, by the principle of analytic continuation, (5.27) holds true for $a \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant -1}$. In this regard, particular cases of (5.27) when $a = 0$ and $a = 1$ give well-known identities:

$$\mathtt{S}^{++}_{1,2}(0,0) = \mathtt{S}^{++}_{1,2} = 2\,\zeta(3) \quad \text{and} \quad \mathtt{S}^{++}_{1,2}(0,1) = \zeta(3). \tag{5.28}$$

Setting $a = \frac{1}{2}$ in (5.27) provides

$$\mathtt{S}^{++}_{1,2}(0,\tfrac{1}{2}) = 7\,\zeta(3) - \pi^2\,\log 2. \tag{5.29}$$

Differentiating both sides of (5.27) with respect to a, ℓ times, one obtains that, for $a \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant -1}$,

$$\begin{aligned}(-1)^{\ell}\,(\ell+1)!\,\mathtt{S}^{++}_{1,\ell+2}(0,a) &= \psi^{(\ell+1)}(a)\,\{\psi(a+1)+\gamma\} - \frac{1}{2}\,\psi^{(\ell+2)}(a+1)\\ &- \sum_{j=0}^{\ell}(-1)^j\,j!\binom{\ell}{j}\frac{\psi^{(\ell-j+1)}(a+1)}{a^{j+1}} + \sum_{j=0}^{\ell-1}\binom{\ell}{j}\,\psi^{(j+1)}(a)\,\psi^{(\ell-j)}(a+1).\end{aligned} \tag{5.30}$$

Setting $\ell = 1$ in (5.30) and then taking the limit as $a \to 0$ and putting $a = \frac{1}{2}$, one gets

$$\mathtt{S}^{++}_{1,3} = \frac{\pi^4}{72} = \frac{5}{4}\,\zeta(4) \tag{5.31}$$

and

$$\sum_{n\geqslant 1}\frac{H_n}{(2n+1)^3} = \frac{1}{8}\,\mathtt{S}^{++}_{1,3}\left(0,\tfrac{1}{2}\right) = \frac{\pi^4}{64} - \frac{7}{4}\,\zeta(3)\,\log 2. \tag{5.32}$$

Setting $\ell = 2$ in (5.30) and then taking the limit as $a \to 0$ and putting $a = \frac{1}{2}$, one obtains

$$\mathtt{S}^{++}_{1,4} = 3\,\zeta(5) - \zeta(2)\,\zeta(3) \tag{5.33}$$

and

$$\sum_{n\geqslant 1}\frac{H_n}{(2n+1)^4} = \frac{1}{16}\,\mathtt{S}^{++}_{1,4}\left(0,\tfrac{1}{2}\right) = \frac{31}{8}\zeta(5) - \frac{21}{16}\zeta(2)\,\zeta(3) - \frac{15}{8}\,\zeta(4)\,\log 2. \tag{5.34}$$

Chen [**11**, p. 8] recalled the following identity:

$$\begin{aligned}\sum_{n\geqslant 1}\frac{H_n}{(2n-1)^2} &= \frac{\pi^2}{4} - \frac{\pi^2\,\log 2}{4} - 2\,\log 2 + \frac{7}{4}\,\zeta(3)\\ &= \sum_{n\geqslant 0}\frac{H_n}{(2n+1)^2} + \sum_{n\geqslant 0}\frac{1}{(n+1)(2n+1)^2}.\end{aligned} \tag{5.35}$$

Since

$$\sum_{n\geqslant 0}\frac{1}{(n+1)(2n+1)^2}=\frac{\pi^2}{4}-2\,\log 2,$$

one gets

$$\sum_{n\geqslant 1}\frac{H_n}{(2n+1)^2}=\frac{1}{4}\,\mathtt{S}^{++}_{1,2}(0,\tfrac{1}{2})=\frac{7}{4}\,\zeta(3)-\frac{\pi^2\,\log 2}{4},\tag{5.36}$$

which is equivalent to the identity in (5.29). Combining (5.27) and (5.36) yields

$$\mathtt{R}^{++}_{1,2}(1,\tfrac{1}{2})=\frac{7}{2}\,\zeta(3)-\frac{\pi^2\,\log 2}{2}.\tag{5.37}$$

(x) Setting $p=3$ and $q=2$ in (3.19) and (3.20), with the aid of (1.26), (1.27), (1.30), and (1.17), one obtains

$$\int_0^1\frac{x^2\,(\log^n x)}{1-x^2}\,\mathrm{d}x=(-1)^n\,n!\,\left\{\left(1-2^{-n-1}\right)\,\zeta(n+1)-1\right\}\quad(n\in\mathbb{N})\tag{5.38}$$

and

$$\int_0^1\frac{x^2\,(\log^n x)}{1+x^2}\,\mathrm{d}x=(-1)^n\,n!\,\{1-\beta(n+1)\}\quad(n\in\mathbb{N})\,.\tag{5.39}$$

(xi) Setting $m=1$, $p=q=t=2$ in the result of Theorem 4.2 and noting that (see (1.20))

$$\mathrm{Ti}_2\left(x^2\right)-\mathrm{Ti}_2\left(\frac{1}{x^2}\right)=\pi\log x,$$

one derives

$$\int_0^\infty\frac{(\log^2 x)\,\mathrm{Ti}_2\left(x^2\right)}{(1+x)^2}\,\mathrm{d}x=2\,U^{+}_{(0,1)}\,(0,1,2,2,2)+\frac{9}{2}\pi\zeta\,(3)\,.\tag{5.40}$$

Similarly, for $m=2$,

$$\int_0^\infty\frac{(\log^2 x)\,\mathrm{Ti}_2\left(x^2\right)}{(1+x)^3}\,\mathrm{d}x=U^{+}_{(0,1)}\,(0,1,2,2,2)+\frac{21}{64}\pi\zeta\,(4)\,.\tag{5.41}$$

Here $U^{+}_{(0,1)}\,(0,1,2,2,2)$ can be evaluated as the expression in (4.16).

6. Concluding remarks and question

Other than the monograph [62] and integrals associated with $\mathrm{Ti}_1(x) = \arctan(x)$, there may be no published research papers about integrals involving the inverse tangent integrals. As has been the case with the many studies of integrals involving polylogarithms, it is anticipated that the current exploration of integrals associated with the inverse tangent integrals will stimulate future research on integrals involving the inverse tangent integrals. Most of those integrals in this article cannot be evaluated directly employing a current CAS software package.

Likewise, the issue of series involving zeta functions has caught the curiosity of numerous academics. The interested reader can consult, for instance, the monograph [60] for information on the subject's history and an astoundingly large number of identities (also check a recent study [4]). The series involving zeta functions presented in Theorem 3.1 are different from those of the type provided, for example, in [60, Chapter 3], but are of the same type in [52].

Like those in **Examples**, the interested researcher can give more closed form evaluations of certain particular cases of the main identities.

Comment. Recall (see [52, Eqs. (4.25) and (4.30)])

$$\sum_{j=1}^{\infty} \frac{\zeta(2,j)}{j^3} = \frac{11}{2}\zeta(5) - 2\,\zeta(2)\,\zeta(3) \tag{6.1}$$

and

$$\sum_{j=1}^{\infty} \frac{(-1)^{j+1}\eta(3,j)}{j^2} = \frac{5}{8}\,\zeta(2)\,\zeta(3) - \frac{11}{32}\zeta(5). \tag{6.2}$$

Also, by using (4.3) and (4.4), the following series associated zeta and eta functions are evaluated:

$$\sum_{j=1}^{\infty} \zeta(3,j) = \zeta(2) = \frac{\pi^2}{6} \tag{6.3}$$

and

$$\sum_{j=1}^{\infty} (-1)^{j+1}\,\eta(3,j) = \eta(2) = \frac{\pi^2}{12}. \tag{6.4}$$

Here one can check that the two series are convergent. For example, employing the asymptotic formula for $\zeta(s,a)$ (see [20, p. 48, Eq. (9)]):

$$\zeta(s,a) = O\left(a^{1-s}\right) \quad (\Re(s) > 1,\ |\arg a| < \pi,\ a \to \infty), \tag{6.5}$$

which is used to yield

$$\zeta(3,j) = O\left(j^{-2}\right) \tag{6.6}$$

for sufficiently large j. That is, there exist $J \in \mathbb{N}$ and $M \in \mathbb{R}_{>0}$ such that

$$\sum_{j=J}^{\infty} \zeta(3,j) \leqslant M \sum_{j=J}^{\infty} \frac{1}{j^2} \leqslant M\,\zeta(2).$$

Thus the series $\sum_{j=1}^{\infty} \zeta(3,j)$ converges.

The identities (6.3) and (6.4) look intriguing when they are compared with the following known results (see, e.g., [60, p. 285, Eq. (214) and Eq. (213)]):

$$\sum_{j=2}^{\infty} \{\zeta(j)-1\} = \sum_{j=2}^{\infty} \zeta(j,2) = 1 \tag{6.7}$$

and

$$\sum_{j=2}^{\infty} (-1)^j \,\{\zeta(j)-1\} = \sum_{j=2}^{\infty} (-1)^j \,\zeta(j,2) = \frac{1}{2}. \tag{6.8}$$

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