SCIENTIA
Series A: *Mathematical Sciences*, Vol. 20 (2010), 161–169
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

Asymptotic formulas for unified elliptic type integrals having H-function

V. B. L. Chaurasia* and Anil Sharma**

Abstract. The object of the present paper is to introduce H-function in elliptic integral theory and to discuss about its asymptotic formulas. It is a unification and generalization of certain families of elliptic-type integrals which were studied in a number of earlier works on the subject due to their importance for possible applications in certain problems arising in radiation physics and nuclear technology. The obtained results are of general character and include the investigations carried out by several authors.

1.Introduction

Elliptic integrals occur in a number of physical problems [1,13], and frequently in the form of multiple integrals. For example, the problems dealing with the computation of the radiations field off axis from certain uniform circular disc radiating according to an arbitrary angular distribution law [4], when treated with Legendre polynomials expansion method, give rise to Epstein and Hubbell [2,5] family of elliptic-type integrals:

$$\Omega_{\rm j}({\rm k}) = \int (1-{\rm k}^2\cos\theta)^{-{\rm j}-\frac{1}{2}}{\rm d}\theta \ ; \ \ {\rm j}=0,1,2,... \tag{1.1}$$

and $0 < k < 1$.

Elliptic integrals (1.1) have been studied and generalized by many authors notably by Kalla [6,14] and Kalla et al. [7], Kalla and Al-Saqabi [8] and Saxena et al. [20], Kalla et al. [9], Srivastava and Bromberg [15] and others. We now describe the various generalizations of (1.1) as follows.

Kalla [6,14] introduced the generalization of the form:

$$\mathrm{R}_\mu(\mathrm{k},\alpha,\gamma) \ = \ \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\gamma-2\alpha-1}(\theta/2)}{(1-\mathrm{k}^2\cos\theta)^{\mu+1/2}} \ \mathrm{d}\theta \ , \tag{1.2}$$

where $0 \leqslant k < 1, Re(\gamma) > Re(\alpha) > 0, Re(\mu) > -1/2$. Results for this generalization are also derived by Glasser and Kalla [11].

2000 *Mathematics Subject Classification.* 26A33, 33E12, 33C45.
Key words and phrases. Elliptic-type integrals, Hypergeometric functions, Asymptotic formula.

Al-Saqabi [10] defined and studied the generalization given by the integral

$$B_\mu(\mathrm{k},\mathrm{m},\nu) \;=\; \int_0^\pi \frac{\cos^{2\mathrm{m}}(\theta)\sin^{2\nu}(\theta)}{(1-\mathrm{k}^2\cos\theta)^{\mu+1/2}}\,\mathrm{d}\theta, \tag{1.3}$$

asymptotic expansion of (1.3) has recently been discussed by Matera et al. [21].

The integral

$$\Lambda_\nu(\alpha,\mathrm{k}) = \int_0^\pi \frac{\exp[\alpha\sin^2(\theta/2)]}{(1-\mathrm{k}^2\cos\theta)^{\nu+1/2}}\,\mathrm{d}\theta, \tag{1.4}$$

where $0 \leqslant k < 1, \alpha, \nu \in R$; presents another generalization of (1.1), given by Siddiqi [12].

Srivastava and Siddiqi [16] have given an interesting unification and extension of the families of elliptic-type integrals in the following form:

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\mu)}(\rho,\mathrm{k}) = \int_0^1 \frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2)}{(1-\mathrm{k}^2\cos\theta)^{\mu+\frac{1}{2}}}\left[1-\rho\sin^2\left(\frac{\theta}{2}\right)\right]^{-\lambda}\mathrm{d}\theta, \tag{1.5}$$

where $0 \leqslant k < 1, Re(\alpha) > 0, Re(\beta) > 0, \lambda, \mu \in C, |\rho| < 1$.

Kalla and Tuan [17] generalized equation (1.5) by means of the following integral and also obtained its asymptotic expansion

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;\mathrm{k}) = \int_0^1 \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)(1-\mathrm{k}^2\cos\theta)^{-\mu-1/2}$$
$$\times\left(1-\rho\sin^2\left(\frac{\theta}{2}\right)\right)^{-\lambda}\left(1+\delta\cos^2\left(\frac{\theta}{2}\right)\right)\mathrm{d}\theta, \tag{1.6}$$

where $0 \leqslant k < 1, Re(\alpha) > 0, Re(\beta) > 0, \lambda, \mu, \gamma \in C$ and either $|\rho|, |\delta| < 1$ or $\rho(or\delta) \in C$ whenever $\lambda = m$ or $\gamma = -m, m \in N_0$, respectively.

Al-Zamel et al. [18] discussed a generalized family of elliptic-type integrals in the form:

$$Z^{(\alpha,\beta)}_{(\gamma)}(\mathrm{k}) = Z^{(\alpha,\beta)}_{(\gamma_1,\ldots,\gamma_\mathrm{n})}(\mathrm{k}_1,\ldots,\mathrm{k}_\mathrm{n})$$
$$= \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{\mathrm{j}=1}^{\mathrm{n}}(1-\mathrm{k}_\mathrm{j}^2\cos\theta)^{-\gamma_\mathrm{j}}\,\mathrm{d}\theta$$
$$= B(\alpha,\beta)\prod_{\mathrm{j}=1}^{\mathrm{n}}(1-\mathrm{k}_\mathrm{j}^2)^{-\gamma_\mathrm{j}}F_D^{(\mathrm{n})}\left(\beta;\,\gamma_1,\ldots,\gamma_\mathrm{n};\,\alpha+\beta\,;\,\frac{2\mathrm{k}_1^2}{\mathrm{k}_1^2-1},\ldots,\frac{2\mathrm{k}_\mathrm{n}^2}{\mathrm{k}_\mathrm{n}^2-1}\right), \tag{1.7}$$

where $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $|\,\mathrm{k_j}| < 1$, $\gamma_\mathrm{j} \in \mathrm{C}, \mathrm{j} = 1,\ldots,\mathrm{n}, F_\mathrm{d}^{(\mathrm{n})}(.)$ is the Lauricella hypergeometric function of n variables [24, p.163].

Saxena and Kalla [19] have studied a family of elliptic-type integrals of the form:

$$\Omega^{(\alpha,\beta)}_{(\sigma_1,\ldots,\sigma_{\mathrm{n}-2}:\delta,\mu)}(\rho_1,\ldots,\rho_{\mathrm{n}-2},\delta;\,\mathrm{k})$$
$$= \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{\mathrm{j}=1}^{\mathrm{n}-2}\left[1-\rho_\mathrm{j}\sin^2\left(\frac{\theta}{2}\right)\right]^{-\sigma_\mathrm{j}}\left[1+\delta\cos^2\left(\frac{\theta}{2}\right)\right]^{-\gamma}$$
$$\times(1-\mathrm{k}^2\cos\theta)^{-\mu-1/2}\,\mathrm{d}\theta, \tag{1.8}$$

where $0 \leqslant k < 1, \mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$; $\sigma_j\,(j = 1,...,n-2)$, γ, $\mu \in C$; $\max\left\{ |\rho_j|, \left|\frac{\delta}{1+\delta}\right|, \left|\frac{2k^2}{k^2-1}\right|\right\} < 1$.

In a recent paper, Saxena and Pathan [23] investigated an extension of Equation (1.8) in the form:

$$\Omega^{(\alpha,\beta)}_{(\sigma_1,...,\sigma_m,\gamma;\tau_1,...,\tau_n)}(\rho_1,...,\rho_m,\delta;\lambda_1,...,\lambda_n)$$
$$= \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{i=1}^m\left[1-\rho_i\sin^2\left(\frac{\theta}{2}\right)\right]^{-\sigma_i}$$
$$\times\left[1+\delta\cos^2\left(\frac{\theta}{2}\right)\right]^{-\gamma}\prod_{j=1}^n(1-\lambda_j^2\cos\theta)^{-\tau_j}d\theta, \tag{1.9}$$

where $\min[\mathrm{Re}(\alpha), \mathrm{Re}(\beta)] > 0$; $|\lambda_j| < 1$; $\sigma_i, \gamma, \tau_j \in C$; $\max\left\{|\rho_j|, \left|\frac{2\lambda_j^2}{\lambda_j^2-1}\right|, \left|\frac{\delta}{1+\delta}\right|\right\} < 1 \quad (i = 1,...,m; j = 1,...,n)$.

Garg et al. [22] gave a new concept of elliptic-type integrals in the form

$$A^{(\alpha,\beta)}_{(a,b,c)}(k) = A^{(\alpha,\beta)}_{(a_1,...,a_n,b_1,...,b_n,c_1,...,c_n)}(k)$$
$$= \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{j=1}^n\left\{{}_2F_1\left(a_j,b_j;c_j;\left(\frac{k_j^2}{k_j^2-1}\right)(1-\cos\theta)\right)\right\}d\theta, \tag{1.10}$$

where $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $|k_j| < 1$, $j = 1,...,n$.

Here we consider a new unified and generalized form of a family of elliptic type integrals:

$$H^{(\alpha,\beta)}_{(\sigma_i,k_j)}[(\rho_i),(\delta_i);\lambda_j]$$
$$= \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{i=1}^m\left[\rho_i\sin^2\left(\frac{\theta}{2}\right)+\delta_i\cos^2\left(\frac{\theta}{2}\right)\right]^{-\sigma_i}$$
$$\times\prod_{j=1}^n\frac{\Gamma(c_j)}{\Gamma(a_j)\Gamma(b_j)}\left\{H^{m',n'}_{p,q}\left[\frac{k_j^2}{(1-k_j^2)}(1-\cos\theta)^{-\tau_j}\left|\begin{matrix}(a^{(j)}_{j'},\alpha^{(j)}_{j'})_{1,p}\\(b^{(j)}_{j'},\beta^{(j)}_{j'})_{1,q}\end{matrix}\right.\right]\right\}d\theta, \tag{1.11}$$

where $\min[\mathrm{Re}(\alpha), \mathrm{Re}(\beta-\tau_j s)] > 0, |k_j| < 1$; $\sigma_i, \tau_j \in C$; $\max\left\{|\rho_j|, |\delta_j|, \left|\frac{\delta_i-\rho_i}{\delta_i}\right|\right\} < 1$, $(i = 1,...,m; j = 1,...,n)$, where $H^{m',n'}_{p,q}(z)$ is *H*-function introduced by Charles Fox [3] in 1961.

2.Special Cases

Taking $m' = 1, n' = p = q = 2, \tau_j = -1, \sigma_i = 0$; $a^{(j)}_1 = (1-a_j)$, $a^{(j)}_2 = (1-b_j)$, $\alpha^{(j)}_1 = \alpha^{(j)}_2 = \beta^{(j)}_1 = \beta^{(j)}_2 = 1$, $b^{(j)}_1 = 0$, $b^{(j)}_2 = (1-c_j)$ in (1.11) and simplifying, we get elliptic type integral defined by Garg et al. [22]. Further several more well known special cases can be defined as given in Garg et al. [22].

3. Explicit representation and asymptotic expansion

From (1.11), it follows that

$$H^{(\alpha,\beta)}_{(\sigma_i,k_j)}[(\rho_i),(\delta_i);\lambda_j]$$

$$= \prod_{i=1}^{m} (\delta_i)^{-\sigma_i} \prod_{j=1}^{n} \left(\frac{2k_j^2}{1-k_j^2} \right)^{-\tau_j s} \frac{\Gamma(c_j)}{\Gamma(a_j)\,\Gamma(b_j)} \sum_{h=1}^{m'} \sum_{r=0}^{\infty} \frac{\prod_{j'=1,j'\neq h}^{m'} \Gamma(b_{j'}^{(j)} - \beta_{j'}^{(j)} s)}{\prod_{j'=m'+1}^{q} \Gamma(1 - b_{j'}^{(j)} + \beta_{j'}^{(j)} s)}$$

$$\times \frac{\prod_{j'=1}^{n'} \Gamma(1 - a_{j'}^{(j)} + \alpha_{j'}^{(j)} s)}{\prod_{j'=n+1}^{p} \Gamma(a_{j'}^{(j)} - \alpha_{j'}^{(j)} s)} \frac{(-1)^r}{r!\beta_h} \int_0^1 (1-t)^{\alpha-1}\, t^{\beta-\tau_j s-1} \left[1 - \left(\frac{\delta_i - \rho_i}{\delta_i} \right) t \right]^{-\sigma_i} dt. \quad (3.1)$$

On employing the formula [24, p.163]

$$\frac{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} F_D^{(n)} [\alpha, \beta_1, ..., \beta_n;\, \gamma\,;\, x_1, ..., x_n]$$

$$= \int_0^1 v^{\alpha-1}(1-v)^{\gamma-\alpha-1}(1-vx_1)^{-\beta_1} ... (1-vx_n)^{-\beta_n} dv, \quad (3.2)$$

where $Re(\gamma) > 0, Re(\alpha) > 0$, (3.1) becomes

$$H_{(\sigma_j,k_j)}^{(\alpha,\beta)} [(\rho_i),\, (\delta_i)\,;\, \lambda_j]$$

$$= \prod_{i=1}^{m} (\delta_i)^{-\sigma_i} \prod_{j=1}^{n} \left(\frac{2k_j^2}{1-k_j^2} \right)^{-\tau_j s} \frac{\Gamma(c_j)}{\Gamma(a_j)\,\Gamma(b_j)} \sum_{h=1}^{m'} \sum_{r=0}^{\infty} \frac{\prod_{j'=1,j'\neq h}^{m'} \Gamma(b_{j'}^{(j)} - \beta_{j'}^{(j)} s)}{\prod_{j'=m'+1}^{q} \Gamma(1 - b_{j'}^{(j)} + \beta_{j'}^{(j)} s)}$$

$$\times \frac{\prod_{j'=1}^{n'} \Gamma(1 - a_{j'}^{(j)} + \alpha_{j'}^{(j)} s)}{\prod_{j'=n'+1}^{p} \Gamma(a_{j'}^{(j)} - \alpha_{j'}^{(j)} s)} \frac{(-1)^r}{r!\beta_h} \frac{\Gamma(\alpha)\,\Gamma(\beta - \tau_j s)}{\Gamma(\alpha + \beta - \tau_j s)}$$

$$\times F_D^{(m)} \left[\alpha, \sigma_1, ..., \sigma_m;\, \alpha + \beta - \tau_j s\,;\, \frac{\delta_{1-}\rho_1}{\delta_1}\,, ...,\, \frac{\delta_m - \rho_m}{\delta_m} \right], \quad (3.3)$$

where $F_D^{(m)}(.)$ is the Lauricella function of m variables, $\min[Re(\alpha),\ Re(\beta-\tau_j s)] > 0\,; \sigma_i, \tau_j \in C$; $\max\left\{ |\,\rho_j\,|\,,\, |\delta_j|\,,\, \left| \frac{\delta_i - \rho_i}{\delta_i} \right| \right\} < 1$ $(i = 1,..., m \text{ and } j = 1,..., n)$.

We will now derive the asymptotic expansion of the generalized elliptic type integral (1.11). Expressing the Lauricella hypergeometric type function $F_D^{(m)}(.)$ in terms of Gauss hypergeometric function, we obtain

$$F_D^{(m)} \left[\alpha, \sigma_1, ..., \sigma_m;\, \alpha + \beta - \tau_j s; \frac{\delta_1 - \rho_1}{\delta_1}\,, ...,\, \frac{\delta_m - \rho_m}{\delta_m} \right]$$

$$= \sum_{n_1,...,n_{m-1}=0}^{\infty} \frac{(\alpha)_{n_1+...+n_{m-1}}}{(\alpha+\beta-\tau_j s)_{n_1+...+n_{m-1}}} \frac{(b_1)_{n_1} ...(b_{m-1})_{n_{m-1}}}{n_1! ... n_{m-1}!} \left(\frac{\delta_1 - \rho_1}{\delta_1} \right)^{n_1} ...$$

$$\left(\frac{\delta_{m-1}-\rho_{m-1}}{\delta_{m-1}}\right)^{n_{m-1}} {}_2F_1\left[\alpha+n_1+...+n_{m-1},\sigma_m;\,\alpha+\beta-\tau_n s+n_1+...+n_{m-1};\,\frac{\delta_m-\rho_m}{\delta_m}\right]. \tag{3.4}$$

If $\alpha-\sigma_m$ is not an integer, then by an appeal to the analytic continuation formula for the Gauss hypergeometric function [25, p.559, equation 15.3.7] namely

$$\begin{aligned}{}_2F_1\,(a,b;c;z) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\,\Gamma(c-a)}\,(-z)^{-a}\,{}_2F_1\left[a,1-c+a;1-\ b+a;\frac{1}{z}\right]\\ &+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)}\,(-z)^{-b}\,{}_2F_1\left[b,1-c+b;1-a+b;\frac{1}{z}\right],\end{aligned} \tag{3.5}$$

where $|arg(-z)|<\Pi$, we have

$$\begin{aligned}&{}_2F_1\left[\alpha+n_1+...+n_{m-1},\sigma_m;\,\alpha+\beta-\tau_j s+n_1+...+n_{m-1};\,\frac{\delta_m-\rho_m}{\delta_m}\right]\\ &=\frac{\Gamma(\alpha+\beta-\tau_j s+n_1+...+n_{m-1})\,\Gamma(\sigma_m-\alpha-n_1-...-n_{m-1})}{\Gamma(\sigma_m)\,\Gamma(\beta-\tau_j s)}\left(\frac{\delta_m}{\rho_m-\delta_m}\right)^{\alpha+n_1+...+n_{m-1}}\\ &\times{}_2F_1\left[\alpha+n_1+...+n_{m-1},1-\beta+\tau_n s;1-\sigma_m+\alpha+n_1+...+n_{m-1};\,\frac{\delta_m}{\delta_m-\rho_m}\right]\\ &+\frac{\Gamma(\alpha+\beta-\tau_j s+n_1+...+n_{m-1})\,\Gamma(\alpha-\sigma_m+n_1+...+n_{m-1})}{\Gamma(\alpha+n_1+...+n_{m-1})\,\Gamma(\alpha+\beta-\tau_j s-\sigma_m+n_1+...+n_{m-1})}\left(\frac{\delta_m}{\rho_m-\delta_m}\right)^{\sigma_m}\\ &\times{}_2F_1\left[\sigma_m,1-\alpha-\beta+\tau_j s+n_1+...+n_{m-1};1-\alpha+\sigma_m-n_1-...-n_{m-1};\,\frac{\delta_m}{\delta_m-\rho_m}\right]\end{aligned} \tag{3.6}$$

$$\begin{aligned}&=\frac{\Gamma(\alpha+\beta-\tau_j s)\,\Gamma(\alpha-\sigma_m)}{\Gamma(\sigma_m)\,\Gamma(\beta-\tau_j s)}\left(\frac{\delta_m}{\rho_m-\delta_m}\right)^{\alpha}\frac{(\alpha+\beta-\tau_j s)_{n_1+...+n_{m-1}}}{(1+\alpha-\sigma_m)_{n_1+...+n_{m-1}}}\\ &\left(\frac{\delta_m}{\delta_m-\rho_m}\right)^{n_1+...+n_{m-1}}{}_2F_1\left[\alpha+n_1+...+n_{m-1},1-\beta+\tau_j s;1-\sigma_m+\alpha+n_1+...+n_{m-1};\,\frac{\delta_m}{\delta_m-\rho_m}\right]\\ &+\frac{\Gamma(\alpha+\beta-\tau_j s)\,\Gamma(\alpha-\sigma_m)}{\Gamma(\alpha)\,\Gamma(\alpha+\beta-\tau_j s{-}s_m)}\left(\frac{\delta_m}{\rho_m-\delta_m}\right)^{\sigma_m}\frac{(\alpha+\beta-\tau_j s)_{n_1+...+n_{m-1}}(\alpha-\sigma_m)_{n_1+...+n_{m-1}}}{(\alpha)_{n_1+...+n_{m+1}}(\alpha+\beta-\tau_j s-\sigma_m)_{n_1+...+n_{m+1}}}\\ &\times{}_2F_1\left[\sigma_m,1-\alpha-\beta+\tau_j s+\sigma_m+n_1+...+n_{m-1};1-\alpha+\sigma_m-n_1-...-n_{m-1};\,\frac{\delta_m}{\delta_m-\rho_m}\right].\end{aligned} \tag{3.7}$$

If we combine (3.3), (3.4) and (3.7), then we have the result

$$H^{(\alpha,\beta)}_{(\sigma_i,k_j)}\left[(\rho_i),\,(\delta_i)\,;\,\lambda_j\right]$$

$$=\prod_{i=1}^{m}(\delta_i)^{-\sigma_i}\prod_{j=1}^{n}\left(\frac{2k_j^2}{1-k_j^2}\right)^{-\tau_j s}\frac{\Gamma(c_j)}{\Gamma(a_j)\,\Gamma(b_j)}\sum_{h=1}^{m'}\sum_{r=0}^{\infty}\frac{\prod\limits_{j'=1,j'\neq h}^{m'}\Gamma(b^{(j)}_{j'}-\beta^{(j)}_{j'}s)}{\prod\limits_{j'=m'+1}^{q}\Gamma(1-b^{(j)}_{j'}+\beta^{(j)}_{j'}s)}$$

$$\times \frac{\prod_{j'=1}^{n'} \Gamma(1-a_{j'}^{(j)}+\alpha_{j'}^{(j)}s)}{\prod_{j'=n+1}^{p} \Gamma(a_{j'}^{(j)}-\alpha_{j'}^{(j)}s)} \frac{(-1)^r}{r!\beta_h} \frac{\Gamma(\alpha)\Gamma(\alpha-\sigma_m)}{\Gamma(\sigma_m)} \left(\frac{\delta_m}{\rho_m-\delta_m}\right)^{\alpha} \sum_{n_1,\ldots,n_{m-1}=0}^{\infty}$$

$$\times \frac{(\alpha)_{n_1+\ldots+n_{m-1}}}{(1+\alpha-\sigma_n)_{n_1+\ldots+n_{m-1}}} \frac{(b_1)_{n_1}\ldots(b_{m-1})_{n_{m-1}}}{n_1!\ldots n_{m-1}!} \prod_{k=1}^{m-1}\left(\frac{\delta_k-\rho_k}{\delta_k}\right)^{n_k}\left(\frac{\delta_m}{\delta_m-\rho_m}\right)^{n_1+\ldots+n_{m-1}}$$

$$\times {}_2F_1\left[\alpha+n_1+\ldots+n_{m-1}, 1-\beta+\tau_j s; 1-\sigma_m+\alpha+n_1+\ldots+n_{m-1}; \frac{\delta_m}{\delta_m-\rho_m}\right] + \prod_{i=1}^{m}(\delta_i)^{-\sigma_i}$$

$$\times \prod_{j=1}^{n}\left(\frac{2k_j^2}{1-k_j^2}\right)^{-\tau_j s} \frac{\Gamma(c_j)}{\Gamma(a_j)\,\Gamma(b_j)} \sum_{h=1}^{m'}\sum_{r=0}^{\infty} \frac{\prod_{j'=1,j'\neq h}^{m'} \Gamma(b_{j'}^{(j)}-\beta_{j'}^{(j)}s)}{\prod_{j'=m'+1}^{q} \Gamma(1-b_{j'}^{(j)}+\beta_{j'}^{(j)}s)}$$

$$\times \frac{\prod_{j'=1}^{n'} \Gamma(1-a_{j'}^{(j)}+\alpha_{j'}^{(j)}s)}{\prod_{j'=n+1}^{p} \Gamma(a_{j'}^{(j)}-\alpha_{j'}^{(j)}s)} \frac{(-1)^r}{r!\beta_h} \frac{\Gamma(\beta-\tau_j s)\Gamma(\alpha-\sigma_m)}{\Gamma(\alpha+\beta-\tau_j s-\sigma_m)} \left(\frac{\delta_m}{\rho_m-\delta_m}\right)^{\sigma_m}$$

$$\times \sum_{n_1,\ldots,n_{m-1}=0}^{\infty} \frac{(b_1)_{n_1}\ldots(b_{m-1})_{n_{m-1}}}{n_1!\ldots n_{m-1}!} \prod_{k=1}^{m-1}\left(\frac{\delta_k-\rho_k}{\delta_k}\right)^{n_k} \frac{(\alpha-\sigma_m)_{n_1+\ldots+n_{m-1}}}{(\alpha+\beta-\tau_j s-\sigma_m)_{n_1+\ldots+n_{m-1}}}$$

$$\times {}_2F_1\left[\sigma_m, 1-\alpha-\beta+\tau_j s+\sigma_m+n_1+\ldots+n_{m-1}; 1-\alpha+\sigma_m-n_1-\ldots-n_{m-1}; \frac{\delta_m}{\delta_m-\rho_m}\right], \tag{3.8}$$

(3.8) may be regarded as the asymptotic series for $H_{(\sigma_i,k_j)}^{(\alpha,\beta)}[(\rho_i),\,(\delta_i)\,;\,\lambda_j]$ as $k_m^2 \to 1$. Next we consider the expansion of

$${}_2F_1\left[\alpha+n_1+\ldots+n_{m-1}, \sigma_n;\, \alpha+\beta-\tau_j s+n_1+\ldots+n_{m-1};\, \frac{\delta_m-\rho_m}{\delta_m}\right]$$

appearing in (3.4), when its upper parameters differ by integers. The following two cases arise:

(i) when $\sigma_m = \alpha-\mu$, $\mu = 0, 1, 2, \ldots$ and

(ii) when $\sigma_m = \alpha+\mu$, $\mu = 0, 1, 2, \ldots$.

In both the cases the results are derived by Al-Zamel et al. [18, p.21-23, equation (4.4), (4.6) and (4.7)]. By making use of results of Al-Zamel et al. [18], the asymptotic expansion can be easily derived in the above two cases. Here we are discussing only first case, second case can be treated on the same lines.

When $\sigma_m = \alpha-\mu$; $\mu = 0, 1, 2, \ldots$ then by virtue of the results given by Al-Zamel [18, p. 21-23], we find that

$$H_{(\sigma_i,k_j)}^{(\alpha,\beta)}[(\rho_i),\,(\delta_i)\,;\,\lambda_j]$$

$$= \prod_{i=1}^{m} (\delta_i)^{-\sigma_i} \prod_{j=1}^{n} \left(\frac{2k_j^2}{1-k_j^2} \right)^{-\tau_j s} \frac{\Gamma(c_j)}{\Gamma(a_j)\,\Gamma(b_j)} \sum_{h=1}^{m'} \sum_{r=0}^{\infty} \frac{\prod_{j'=1, j' \neq h}^{m'} \Gamma(b_{j'}^{(j)} - \beta_{j'}^{(j)} s)}{\prod_{j'=m'+1}^{q} \Gamma(1 - b_{j'}^{(j)} + \beta_{j'}^{(j)} s)}$$

$$\times \frac{\prod_{j'=1}^{n'} \Gamma(1 - a_{j'}^{(j)} + \alpha_{j'}^{(j)} s)}{\prod_{j'=n'+1}^{p} \Gamma(a_{j'}^{(j)} - \alpha_{j'}^{(j)} s)} \frac{(-1)^r}{r! \beta_h} \frac{\Gamma(\alpha)\Gamma(\beta - \tau_j s)}{\Gamma(\alpha + \beta - \tau_j s)} \sum_{n_1, \ldots, n_{m-1}=0}^{\infty}$$

$$\times \frac{(\alpha)_{n_1+\ldots+n_{m-1}}}{(\alpha+\beta-\tau_j s)_{n_1+\ldots+n_{m-1}}} \frac{(b_1)_{n_1} \ldots (b_{m-1})_{n_{m-1}}}{n_1! \ldots n_{m-1}!} \left(\frac{\delta_1 - \rho_1}{\delta_1} \right)^{n_1} \ldots \left(\frac{\delta_{m-1} - \rho_{m-1}}{\delta_{m-1}} \right)^{n_{m-1}}$$

$$\times \frac{\Gamma(\alpha + \beta - \tau_j s + n_1 + \ldots + n_{m-1})}{\Gamma(\alpha + n_1 + \ldots + n_{m-1})\,\Gamma(\beta - \tau_j s)} \left(\frac{\delta_m}{\rho_m - \delta_m} \right)^{\alpha + n_1 + \ldots + n_{m-1}}$$

$$\times \sum_{x=0}^{\infty} \frac{(n_1 + \ldots + n_{m-1} + \mu)_{x + n_1 + \ldots + n_{m-1} + \mu}}{x!(x + n_1 + \ldots + n_{m-1} + \mu)\,!} \left(\frac{\delta_m}{\delta_m - \rho_m} \right)^{x}$$

$$\times [\, \ell n(\rho_m - \delta_m) - \ell n(\delta_m) + \psi(1 + n_1 + \ldots + n_{m-1} + \mu + x) + \psi(1 + x) - \psi(\alpha - \mu + n_1 + \ldots$$

$$+ n_{m-1} + \mu + x) - \psi(\beta - \tau_j s - x)] + \prod_{i=1}^{m} (\delta_i)^{-\sigma_i} \prod_{j=1}^{n} \left(\frac{2k_j^2}{1-k_j^2} \right)^{-\tau_j s} \frac{\Gamma(c_j)}{\Gamma(a_j)\,\Gamma(b_j)} \sum_{h=1}^{m'} \sum_{r=0}^{\infty}$$

$$\frac{\prod_{j'=1, j' \neq h}^{m'} \Gamma(b_{j'}^{(j)} - \beta_{j'}^{(j)} s) \prod_{j'=1}^{n'} \Gamma(1 - a_{j'}^{(j)} + \alpha_{j'}^{(j)} s)}{\prod_{j'=m'+1}^{p} \Gamma(1 - b_{j'}^{(j)} + \beta_{j'}^{(j)} s) \prod_{j'=n+1}^{p} \Gamma(a_{j'}^{(j)} - \alpha_{j'}^{(j)} s)} \frac{(-1)^r}{r! \beta_h} \frac{\Gamma(\alpha)\Gamma(\beta - \tau_j s)}{\Gamma(\alpha + \beta - \tau_j s)} \sum_{n_1, \ldots, n_{m-1}=0}^{\infty}$$

$$\times \frac{(\alpha)_{n_1+\ldots+n_{m-1}}}{(\alpha+\beta-\tau_j s)_{n_1+\ldots+n_{m-1}}} \frac{(b_1)_{n_1} \ldots (b_{m-1})_{n_{m-1}}}{n_1! \ldots n_{m-1}!} \left(\frac{\delta_1 - \rho_1}{\delta_1} \right)^{n_1} \ldots \left(\frac{\delta_{m-1} - \rho_{m-1}}{\delta_{m-1}} \right)^{n_{m-1}}$$

$$\times \left(\frac{\delta_m}{\rho_m - \delta_m} \right)^{\alpha - \mu} \frac{\Gamma(\alpha + \beta - \tau_j s + n_1 + \ldots + n_{m-1})}{\Gamma(\alpha + n_1 + \ldots + n_{m-1})} \sum_{x=0}^{n_1 + \ldots + n_{m-1} + \mu - 1} \frac{\Gamma(\alpha + n_1 + \ldots + n_{m-1} - x)}{x!}$$

$$\times \frac{(\alpha - \mu)_x}{\Gamma(\beta - \tau_j s + \mu + n_1 + \ldots + n_{m-1} - x)} \left[\frac{\delta_m}{\delta_m - \rho_m} \right]^{x} \tag{3.9}$$

$$= \prod_{i=1}^{m} (\delta_i)^{-\sigma_i} \prod_{j=1}^{n} \left(\frac{2k_j^2}{1-k_j^2} \right)^{-\tau_j s} \frac{\Gamma(c_j)}{\Gamma(a_j)\,\Gamma(b_j)} \sum_{h=1}^{m'} \sum_{r=0}^{\infty} \frac{\prod_{j'=1, j' \neq h}^{m'} \Gamma(b_{j'}^{(j)} - \beta_{j'}^{(j)} s)}{\prod_{j'=m'+1}^{q} \Gamma(1 - b_{j'}^{(j)} + \beta_{j'}^{(j)} s)}$$

$$\times \frac{\prod_{j'=1}^{n'} \Gamma(1 - a_{j'}^{(j)} + \alpha_{j'}^{(j)} s)}{\prod_{j'=n'+1}^{p} \Gamma(a_{j'}^{(j)} - \alpha_{j'}^{(j)} s)} \frac{(-1)^r}{r! \beta_h} \sum_{n_1, \ldots, n_{m-1}=0}^{\infty} \frac{\prod_{\ell=1}^{m-1} \left[(b_\ell)_{n_\ell} \left(\frac{\delta_\ell - \rho_\ell}{\delta_\ell} \right)^{n_\ell} \right]}{n_1! \ldots n_{m-1}\,!}$$

$$\times\left(\frac{\delta_m}{\rho_m-\delta_m}\right)^{\alpha+n_1+\ldots+n_{m-1}}\sum_{x=0}^{\infty}\frac{(n_1+n_2+\ldots+n_{m-1}+\mu)_{x+n_1+\ldots+n_{m-1}+\mu}}{x!(x+n_1+\ldots+n_{m-1}+\mu)\,!}$$

$$\times\left(\frac{\delta_m}{\delta_m-\rho_m}\right)^{x}[\ell n(\rho_m-\delta_m)-\ell n(\delta_m)+\psi(1+n_1+\ldots+n_{m-1}+\mu+x)+\psi(1+x)$$

$$-\psi\,(\alpha-\mu+n_1+\ldots+n_{m-1}+\mu+x)-\psi\,(\beta-\tau_j s-x)]$$

$$+\prod_{i=1}^{m}(\delta_i)^{-\sigma_i}\prod_{j=1}^{n}\left(\frac{2k_j^2}{1-k_j^2}\right)^{-\tau_j s}\frac{\Gamma(c_j)}{\Gamma(a_j)\,\Gamma(b_j)}\sum_{h=1}^{m'}\sum_{r=0}^{\infty}\frac{\prod\limits_{j'=1,j'\neq h}^{m'}\Gamma(b_{j'}^{(j)}-\beta_{j'}^{(j)}s)}{\prod\limits_{j'=m'+1}^{q}\Gamma(1-b_{j'}^{(j)}+\beta_{j'}^{(j)}s)}$$

$$\times\frac{\prod\limits_{j'=1}^{n'}\Gamma(1-a_{j'}^{(j)}+\alpha_{j'}^{(j)}s)}{\prod\limits_{j'=n'+1}^{p}\Gamma(a_{j'}^{(j)}-\alpha_{j'}^{(j)}s)}\;\frac{(-1)^r\Gamma(\beta-\tau_j s)}{r!\beta_h}\sum_{n_1,\ldots,n_{m-1}=0}^{\infty}\frac{\prod\limits_{\ell=1}^{m-1}\left[(b_\ell)_{n_\ell}\left(\frac{\delta_\ell-\rho_\ell}{\delta_\ell}\right)^{n_\ell}\right]}{n_1!\ldots n_{m-1}\,!}$$

$$\times\left(\frac{\delta_m}{\rho_m-\delta_m}\right)^{\alpha-\mu}\sum_{x=0}^{n_1+\ldots+n_{m-1}+\mu-1}\frac{\Gamma(\alpha+n_1+\ldots+n_{m-1}-x)\,(\alpha-\mu)_x}{x!\,\Gamma\,(\beta-\tau_j s+\mu+n_1+\ldots+n_{m-1}-x)}\left(\frac{\delta_m}{\delta_m-\rho_m}\right)^{x}. \tag{3.10}$$

Following the same procedure one can establish asymptotic formulas for other families of elliptic-type integrals.

References

[1] E.L. Kaplan, Multiple elliptic integrals, *J. Math. Phys.* **29** (1950), 69-75.

[2] M.J. Berger, J.C. Lamkin, Sample calculation of gamma ray penetration into shelters, contribution of sky shine and roof contamination, *J.Res. N.B.S.* **60** (1958), 109-116.

[3] Charles Fox, The *G* and *H*-functions as symmetrical Fourier kernels, *Trans. Amer. Math. Soc.* **98** (1961), 395-429.

[4] J.H. Hubbell, R.L. Bach, R.J. Herbold, Radiation field from a circular disk source, *J. Res. N.B.S.* **65** (1961), 249-264.

[5] L.F. Epstein, J.H. Hubbell, Evalulation of a generalized elliptic-type integrals, *J. Res. N.B.S.* **67** (1963), 1-17.

[6] S.L. Kalla, Results on generalized elliptic-type integrals, *Mathematical Structure Computational Mathematics-Mathematical Modelling (Edited by B.I. Sendov), Special Vol., Bolg. Acad. Sci.*, (1984), 216-219.

[7] S.L. Kalla, S. Conde, J.H. Hubbell, Some results on generalized elliptic type integrals, *Appl. Anal.* **22** (1986), 273-287.

[8] S.L. Kalla, B. Al-Saqabi, On a generalized elliptic-type integral, *Rev. Bra. Fis.* **16** (1986), 145-156.

[9] S.L. Kalla, C. Leubner, J.H. Hubbel, Further results on generalized elliptic-type integrals, *Appl. Anal.* **25** (1987), 269-274.

[10] B.N. Al-Saqabi, A generalization of elliptic-type integrals, *Hadronic J.* **10** (1987), 331-337.

[11] M.L. Glasser, S.L. Kalla, Recursion relations for a class of generalized elliptic-type integrals, *Rev. Tec. Ing. Univ. Zulia* **12**(1989), 47-50.

[12] R.N. Siddiqi, On a class of generalized elliptic-type integrals, *Rev. Brasileira Fis.* **19** (1989), 137-147.

[13] J.D. Evans, J.H. Hubbell and V.D. Evans, Exact series solution to the Epstein-Hubbell generalized elliptic-type integral using complex variable residue theory, *Appl. Math. Comp.* **53**(1993), 173-189.

[14] S.L. Kalla, The Hubbell rectangular source integral and its generalizations, *Radiat. Phys. Chem.*, **41** (1993), 775-781.

[15] H.M. Srivastava, S. Bromberg, Some families of generalized elliptic-type integrals, *Math. Comput. Modelling* **21(3)** (1995), 29-38.
[16] H.M. Srivastava, R.N. Siddiqi, A unified presentation of certain families of elliptic-type integrals related to radiation field problems, *Radiat. Phys. Chem.* **46** (1995), 303-315.
[17] S.L. Kalla, V.K. Tuan, Asymptotic formulas for generalized elliptic-type integrals, *Comput. Math. Appl.* **32** (1996), 49-55.
[18] A. Al-Zamel, V.K. Tuan, S.L. Kalla, Generalized elliptic-type integrals and asymptotic formulas, *Appl. Math. Comput.*, **114** (2000), 13-25.
[19] R.K. Saxena, S.L. Kalla, A new method for evaluating Epstein-Hubbell generalized elliptic-type integrals, *Int. J. Appl. Math.* **2** (2000), 732-742.
[20] R.K. Saxena, S.L. Kalla, J.H. Hubbell, Asymptotic expansion of a unified elliptic-type integrals, *Math. Balkanica*, **15** (2001), 387-396.
[21] J. Matera, L. Galue, S.L. Kalla, Asymptotic expansions for some elliptic-type integrals, *Raj. Acad. Phy. Sci.* **1(2)** (2002), 71-82.
[22] Mridula Garg, Vimal Katta, S.L. Kalla, Study of a class of generalized elliptic-type integrals, *Appl. Math. Comput.*, **131** (2002), 607-613.
[23] R.K. Saxena, M.A. Pathan, Asymptotic formulas for unified elliptic-type integrals, *Demonstratio Math.* **36(3)** (2003), 579-589.
[24] A.M. Mathai, R.K. Saxena, The H-function with Application in statistics and other disciplines, Halsted Press, New York (1978).
[25] M. Abramowitz, I. Stegun, Hand book of Mathematical Functions, Dover, New York (1972).

Received 20 10 2009, revised 03 09 2010

*Department of Mathematics,
University of Rajasthan,
Jaipur - 302055, India.
E-mail address: choursiavbl@yahoo.com

**Department of Mathematics,
Jaipur National University
Jaipur-302022, India.
E-mail address: anilsharma84@gmail.com