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A simple evaluation of the quartic integral

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Abstract. The integral

$$I(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}$$

is evaluated by elementary methods.

1. Introduction

The interest of the first author in the evaluation began with a statement from a (former) graduate student saying that he was able to evaluate the integral

$$I(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}, \tag{1.1}$$

with $m \in \mathbb{N}$ and $a > -1$. This story has been told in [**8**]. Obtaining the result stated in (1.2) and (1.3) below, was a combination of good luck, ignorance of the field of special functions and the perseverance of George Boros. The first proof of the identity

$$I(a;m) = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a) \tag{1.2}$$

appeared in [**4**]. Here

$$P_m(a) = 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} (a+1)^k \tag{1.3}$$

is a polynomial in a of degree m. Since then many other proofs have appeared in the literature [**1, 2, 3, 5, 6**].

The goal of this short note is to present a small modification of Hirschhorn's proof [**7**], perhaps the simplest to date, offering an alternative that might prove to be useful to beginning students as well as instructors.

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2. The proof

Start with the factorization

$$\frac{1}{(x^4+2ax^2+1)^{m+1}} = \frac{1}{x^{2(m+1)}} \times \frac{1}{((x-1/x)^2+2(a+1))^{m+1}}, \tag{2.1}$$

make the change of variables $t = \frac{1}{\sqrt{2(a+1)}}(x-1/x)$ and introduce the notation $b=(a+1)/2$ to produce

$$I(a;m) = \frac{1}{2^{2m+2}b^{m+1/2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{1+bt^2}(\sqrt{b}t+\sqrt{1+bt^2})^{2m+1}} \frac{dt}{(t^2+1)^{m+1}}. \tag{2.2}$$

Then use the fact that

$$\left(\sqrt{b}t+\sqrt{1+bt^2}\right) \times \left(-\sqrt{b}t+\sqrt{1+bt^2}\right) = 1 \tag{2.3}$$

to write (2.2) as

$$I(a;m) = \frac{1}{2^{2m+2}b^{m+1/2}} \int_{-\infty}^{\infty} \frac{\left(-\sqrt{b}t+\sqrt{1+bt^2}\right)^{2m+1}}{\sqrt{1+bt^2}} \frac{dt}{(t^2+1)^{m+1}}. \tag{2.4}$$

The change of variables $t \mapsto -t$ gives

$$I(a;m) = \frac{1}{2^{2m+2}b^{m+1/2}} \int_{-\infty}^{\infty} \frac{\left(\sqrt{b}t+\sqrt{1+bt^2}\right)^{2m+1}}{\sqrt{1+bt^2}} \frac{dt}{(t^2+1)^{m+1}}. \tag{2.5}$$

Adding (2.4) and (2.5) gives

$$2I(a;m) = \frac{1}{2^{2m+2}b^{m+1/2}} \int_{-\infty}^{\infty} \frac{\left(\sqrt{b}t+\sqrt{1+bt^2}\right)^{2m+1} + \left(-\sqrt{b}t+\sqrt{1+bt^2}\right)^{2m+1}}{\sqrt{1+bt^2}} \frac{dt}{(t^2+1)^{m+1}}. \tag{2.6}$$

Now expand the integrand in (2.6) to produce

$$\left(\sqrt{b}t+\sqrt{1+bt^2}\right)^{2m+1} + \left(-\sqrt{b}t+\sqrt{1+bt^2}\right)^{2m+1} = 2\sum_{k=0}^{m} \binom{2m+1}{2k} b^k t^{2k} (1+bt^2)^{m-k+1/2}.$$

Expanding the power $(1+bt^2)^{m-k}$ leads to

$$I(a;m) = \frac{1}{2^{m+3/2}(a+1)^{m+1/2}} \sum_{k=0}^{m}\sum_{j=0}^{k} \left(\frac{a+1}{2}\right)^k \binom{m-j}{k-j}\binom{2m+1}{2j} \int_{-\infty}^{\infty} \frac{t^{2k}\,dt}{(1+t^2)^{m+1}}. \tag{2.7}$$

The next lemma gives the value of the integrals appearing in (2.7).

LEMMA 2.1. *Let $m \in \mathbb{N}$ and $0 \leqslant k \leqslant m$. Then*

$$\int_{-\infty}^{\infty} \frac{t^{2k}\,dt}{(1+t^2)^{m+1}} = \frac{(2k)!(m-k)!}{k!m!2^{2m}} \binom{2m-2k}{m-k}\pi. \tag{2.8}$$

PROOF. Integration by parts gives

$$\int_{-\infty}^{\infty} \frac{t^{2k}\,dt}{(1+t^2)^{m+1}} = \frac{2k(2k-1)}{4mk} \int_{-\infty}^{\infty} \frac{t^{2k-2}\,dt}{(1+t^2)^m}, \tag{2.9}$$

and iterating this identity yields

$$\int_{-\infty}^{\infty} \frac{t^{2k}\,dt}{(1+t^2)^{m+1}} = \frac{(2k)!(m-k)!}{k!m!2^{2k}} \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{m+1-k}}. \tag{2.10}$$

The conclusion follows from Wallis' formula

$$\int_{-\infty}^{\infty} \frac{dx}{(1+t^2)^{m+1}} = \frac{\pi}{2^{2m}}\binom{2m}{m}. \tag{2.11}$$

The reader will find in [**9**] a detailed discussion of this classic fiormula. □

Using the result of the Lemma in the formula (2.7) produces
(2.12)

$$I(a;m) = \frac{\pi}{m!(2\sqrt{2(a+1)})^{2m+1}} \sum_{i=0}^{m}\sum_{k=0}^{i} \frac{(a+1)^i}{2^i}\binom{m-k}{i-k}\binom{2m+1}{2k}\frac{(2i)!}{i!}\frac{(2m-2i)!}{(m-i)!}.$$

This proves (1.2) with the alternative expression

$$P_m(a) = \frac{1}{m!2^{2m}} \sum_{i=0}^{m}\sum_{k=0}^{i} \frac{(a+1)^i}{2^i}\binom{m-k}{i-k}\binom{2m+1}{2k}\frac{(2i)!}{i!}\frac{(2m-2i)!}{(m-i)!}. \tag{2.13}$$

The next section is devoted to proving that this polynomial is exactly the one stated in (1.2).

3. The original expression for the polynomial P_m

The goal of this section is to produce a new expression for the polynomial $P_m(a)$. The identity (2.6) is written as

$$I(a;m) = \frac{1}{2(\sqrt{2(a+1)})^{m+1}} \int_{-\infty}^{\infty} X_m(a,t)\frac{dt}{(1+t^2)^{m+1}}, \tag{3.1}$$

with $X_m(a,t) = Y_m\left(\sqrt{\frac{a+1}{2}}t\right)$ and

$$Y_m(x) = \frac{(x+\sqrt{1+x^2})^{2m+1} + (-x+\sqrt{1+x^2})^{2m+1}}{2\sqrt{1+x^2}}. \tag{3.2}$$

The first step is to establish a recurrence for $Y_m(x)$. Using this recurrence and the initial values $Y_0(x) = 1$ and $Y_1(x) = 1 + 4x^2$ it becomes clear that $Y_m(x)$ is a polynomial in x. An expression for it is given in Theorem 3.1.

LEMMA 3.1. *The function $Y_m(x)$ satisfies the recurrence*

$$Y_{m+1}(x) = 2(1+2x^2)Y_m(x) - Y_{m-1}(x), \quad \text{for } m \geqslant 1. \tag{3.3}$$

PROOF. The key point of the proof is the identity

$$(x+\sqrt{1+x^2})\times(-x+\sqrt{1+x^2}) = 1. \tag{3.4}$$

Define $Z_m(x) = 2\sqrt{1+x^2}Y_m(x)$, then the result follows from

$$\begin{aligned} Z_{m+1}(x)+Z_{m-1}(x) &= (x+\sqrt{1+x^2})^{2m+3}+(-x+\sqrt{1+x^2})^{2m+3} \\ &\quad + (x+\sqrt{1+x^2})^{2m-1}+(-x+\sqrt{1+x^2})^{2m-1} \\ &= (x+\sqrt{1+x^2})^2(x+\sqrt{1+x^2})^{2m+1} \\ &\qquad +(-x+\sqrt{1+x^2})^2(-x+\sqrt{1+x^2})^{2m+1} \\ &\quad + (x+\sqrt{1+x^2})^{-2}(x+\sqrt{1+x^2})^{2m+1} \\ &\qquad +(-x+\sqrt{1+x^2})^{-2}(-x+\sqrt{1+x^2})^{2m+1} \\ &= \left[(x+\sqrt{1+x^2})^2+(-x+\sqrt{1+x^2})^2\right]\times \\ &\qquad \left[(x+\sqrt{1+x^2})^{2m+1}+(-x+\sqrt{1+x^2})^{2m+1}\right]. \end{aligned}$$

The proof is complete. □

The recurrence (3.3) is now used to get a new expression for the polynomial Y_m.

THEOREM 3.1. *The polynomial $Y_m(x)$ is given by*

$$Y_m(x) = \sum_{k=0}^{m}\binom{m+k}{m-k}(2x)^{2k}. \tag{3.5}$$

PROOF. Assuming the result holds for m and $m-1$ it follows that

$$2(1+2x^2)Y_m(x)-Y_{m-1}(x) = 2(1+2x^2)\sum_{k=0}^{m}\binom{m+k}{m-k}2^{2k}x^{2k}-\sum_{k=0}^{m-1}\binom{m-1+k}{m-1-k}2^{2k}x^{2k}.$$

Now separate on the right hand side with exponents $2m+2$, $2m$ and the constant term to reduce it to

$$\begin{aligned} &2^{2m+2}x^{2m+2}+(2m+1)2^{2m}x^{2m}+1 \\ &\qquad +\sum_{k=1}^{m-1}\left[2\binom{m+k}{m-k}+\binom{m+k-1}{m-k+1}-\binom{m-1+k}{m-1-k}\right]2^{2k}x^{2k}. \end{aligned} \tag{3.6}$$

The result now follows from the identity

$$2\binom{m+k}{m-k}+\binom{m+k-1}{m-k+1}-\binom{m-1+k}{m-1-k} = \binom{m+1+k}{m+1-k}, \tag{3.7}$$

which is easily verified. □

The next step is to replace the result in Theorem 3.1 in (3.1). A direct calculation, using the evaluation in Lemma 2.1 agrees with (1.2) with the formula (1.3) for the polynomial $P_m(a)$. All the details of the proof are finally in place.

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References

[1] T. Amdeberhan and V. Moll. A formula for a quartic integral: a survey of old proofs and some new ones. *The Ramanujan Journal*, 18:91–102, 2009.
[2] T. Amdeberhan, V. Moll, and C. Vignat. The evaluation of a quartic integral via Schwinger, Schur and Bessel. *The Ramanujan Journal*, 28:1–14, 2012.
[3] M. Apagodu. Series evaluation of a quartic integral. *The Ramanujan Journal*, 24:147–150, 2011.
[4] G. Boros and V. Moll. An integral hidden in Gradshteyn and Ryzhik. *Jour. Comp. Applied Math.*, 106:361–368, 1999.
[5] G. Boros, V. Moll, and S. Riley. An elementary evaluation of a quartic integral. *Scientia*, 11:1–12, 2005.
[6] I. Gonzalez and V. Moll. Definite integrals by the method of brackets. Part 1. *Adv. Appl. Math.*, 45:50–73, 2010.
[7] M. Hirschhorn. An interesting integral. *Math. Gazette*, 95:90–91, 2011.
[8] V. Moll. The evaluation of integrals: a personal story. *Notices of the AMS*, 49:311–317, 2002.
[9] V. Moll. The integral of Wallis. *Scientia*, 33:13–20, 2023.

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