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CBMO Estimates for Multilinear Commutator of Marcinkiewicz Operator in Herz and Morrey-Herz Spaces

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Abstract. In this paper, we establish CBMO estimates for the multilinear commutator related to the Marcinkiewicz operator in Herz and Morrey-Herz spaces.

1. Introduction.

Let $b \in BMO(R^n)$ and I_α be the fractional operator, the commutator $[b, I_\alpha]$ generated by b and I_α is defined by

$$[b, I_\alpha](f) = b(x)I_\alpha(f)(x) - I_\alpha(bf).$$

A result of Chanillo (see [3]) proved that the commutator $[b, I_\alpha]$ is bounded from $L^p(R^n)$ to $L^q(R^n)$, where $1 < p < q < \infty$ and $1/p - 1/q = \alpha/n$. Lu and Yang (see [5]) introduced the central BMO space that is CBMO space. Since it is obvious that $BMO(R^n) \subsetneqq CBMO_q(R^n)$ for all $1 \leqslant q < \infty$. However, we know the (L^p, L^q) boundedness fails with only the assumption $b \in CBMO_q(R^n)$. Instead, certain boundedness properties on Herz spaces and Morrey-Herz spaces can be proved. The purpose of this paper is to introduce the multilinear operator associated to the Marcinkiewicz operator and establish CBMO estimates for the multilinear commutator in Herz and Morrey-Herz spaces.

2. Preliminaries and Theorems

First, let us introduce some notations.

Definition 1. Let $1 \leqslant q < \infty$. A function $f \in L^q_{loc}(R^n)$ is said to belong to the space $CBMO_q(R^n)$ if

$$||f||_{CBMO_q} = \sup_{r>0} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^q dx \right)^{1/q} < \infty,$$

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where, $B = B(0,r) = \{x \in R^n : |x| < r\}$ and $f_{B(0,r)}$ is the mean value of f on $B(0,r)$.

Let $\vec{q} = (q'_1, \cdots q'_j)$, for $b_j \in CBMO_{q'_j}(R^n)(j = 1, \cdots, m)$, set

$$||\vec{b}||_{CBMO_{\vec{q}}} = \prod_{j=1}^{m} ||b_j||_{CBMO_{q'_j}}.$$

Given a positive integer m and $1 \leqslant j \leqslant m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{CBMO_{\vec{q}}} = ||b_{\sigma(1)}||_{CBMO_{q'_1}} \cdots ||b_{\sigma(j)}||_{CBMO_{q'_j}}$.

Definition 2. Let $\alpha \in R$, $0 < p \leqslant \infty$ and $0 < q < \infty$. For $k \in Z$, set $B_k = \{x \in R^n : |x| \leqslant 2^k\}$ and $A_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of A_k and χ_0 the characteristic function of B_0.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,\ p}(R^n) = \{f \in L_{loc}^q(R^n \setminus \{0\}) : ||f||_{\dot{K}_q^{\alpha,\ p}} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,\ p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p\right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,\ p}(R^n) = \{f \in L_{loc}^q(R^n) : ||f||_{K_q^{\alpha,\ p}} < \infty\},$$

where

$$||f||_{K_q^{\alpha,\ p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p + ||f\chi_{B_0}||_{L^q}^p\right]^{1/p};$$

And the usual modification is made when $p = q = \infty$.

Definition 3. Let $\alpha \in R$, $0 \leqslant \lambda < \infty$, $0 < p \leqslant \infty$ and $0 < q < \infty$. The homogeneous Morrey-Herz space $M\dot{K}_{p,q}^{\alpha,\lambda}(R^n)$ is defined by

$$M\dot{K}_{p,q}^{\alpha,\lambda}(R^n) = \{f \in L_{loc}^q(R^n \setminus \{0\}) : ||f||_{M\dot{K}_{p,q}^{\alpha,\lambda}} < \infty\},$$

where

$$||f||_{M\dot{K}_{p,q}^{\alpha,\lambda}} = \sup_{k_0 \in \mathbf{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} ||f\chi_k||_{L^q}^p\right)^{1/p}$$

with the usual modifications made when $p = \infty$.

Remark 1. Compare the homogeneous Morrey-Herz space $M\dot{K}_{p,q}^{\alpha,\lambda}(R^n)$ with the homogeneous Herz space $\dot{K}_q^{\alpha,p}(R^n)$ and the Morrey space $M_q^\lambda(R^n)$(see [6]), obviously, $M\dot{K}_{p,q}^{\alpha,0}(R^n) = \dot{K}_q^{\alpha,p}(R^n)$ and $M_q^\lambda(R^n) \subset M\dot{K}_{p,q}^{\alpha,0}(R^n)$. We can see that when $\lambda = 0$, $M\dot{K}_{p,q}^{\alpha,0}(R^n)$ is just the homogeneous Herz space.

Definition 4. Let $0 < \delta < n$, $0 < \gamma \leqslant 1$ and Ω be homogeneous of degree zero on R^n such that $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \leqslant M|x-y|^{\gamma}$. The Marcinkiewicz multilinear commutator is defined by

$$\mu_{\Omega,\delta}^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_{t,\delta}^{\vec{b}}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{t,\delta}^{\vec{b}}(f)(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} \left[\prod_{j=1}^m (b_j(x) - b_j(y))\right] f(y)dy.$$

Set

$$F_{t,\delta}(f)(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy,$$

we also define that

$$\mu_{\Omega,\delta}(f)(x) = \left(\int_0^\infty |F_{t,\delta}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

which is the Marcinkiewicz operator(see [12]).

Let H be the space $H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 \frac{dt}{t^3}\right)^{1/2} < \infty\right\}$. Then, it is clear that

$$\mu_{\Omega,\delta}(f)(x) = ||F_{t,\delta}(f)(x)|| \text{ and } \mu_{\Omega,\delta}^{\vec{b}}(f)(x) = ||F_{t,\delta}^{\vec{b}}(f)(x)||.$$

Note that when $b_1 = \cdots = b_m$, $\mu_{\Omega,\delta}^{\vec{b}}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1-3][7][10]). The purpose of this paper is to study the boundedness properties for the multilinear commutator $\mu_{\Omega,\delta}^{\vec{b}}$ in Herz spaces and Morrey-Herz spaces .

Now we state our theorems as following.

Theorem 1. Let $1 < q < \infty$, $b \in CBMO_q(R^n)$, and $\mu_{\Omega,\delta}^{\vec{b}}$ be defined as in Definition 4 with $0 < \delta < n$, $1 < q_1 < n/\delta$, $1 < q_2 < \infty$. If $0 < p \leqslant \infty$, $\frac{1}{q_2} = \frac{1}{q_1} + \frac{1}{q} - \frac{\delta}{n}$, where $1/q = 1/q_1' + \cdots 1/q_j'$, and $\frac{1}{t} = \frac{1}{q_1} + \frac{1}{q} < 1$, $\frac{1}{u} = \frac{1}{q_1} - \frac{\delta}{n}$, $\delta - \frac{n}{q_1} < \alpha_1 < n - \frac{n}{q_1}$ and $\alpha_2 = \alpha_1 - n/q$, then

$$||\mu_{\Omega,\delta}^{\vec{b}}||_{\dot{K}_{q_2}^{\alpha_2,p}} \leqslant C||\vec{b}||_{CBMO_{\vec{q}}}||f||_{\dot{K}_{q_1}^{\alpha_1,p}}.$$

Theorem 2. Let $\lambda \geqslant 0$, $1 < q < \infty$, $b \in CBMO_q(R^n)$, and $\mu_{\Omega,\delta}^{\vec{b}}$ be defined as in Definition 4 with $0 < \delta < n$, $1 < q_1 < n/\delta$, $1 < q_2 < \infty$. If $0 < p \leqslant \infty$, $\frac{1}{q_2} = \frac{1}{q_1} + \frac{1}{q} - \frac{\delta}{n}$, where $1/q = 1/q_1' + \cdots 1/q_j'$, and $\frac{1}{t} = \frac{1}{q_1} + \frac{1}{q} < 1$, $\frac{1}{u} = \frac{1}{q_1} - \frac{\delta}{n}$, $\lambda + \delta - \frac{n}{q_1} < \alpha_1 < n - \frac{n}{q_1} + \lambda$ and $\alpha_2 = \alpha_1 - n/q$, then

$$||\mu_{\Omega,\delta}^{\vec{b}}||_{M\dot{K}_{p,q_2}^{\alpha_2,\lambda}} \leqslant C||\vec{b}||_{CBMO_{\vec{q}}}||f||_{M\dot{K}_{p,q_1}^{\alpha_1,\lambda}}.$$

3. Proof of Theorems.

To prove the theorems, we need the following lemmas.

Lemma 1.(see [7]) Suppose that $f \in CBMO_q(R^n)$, $1 \leqslant q < \infty$ and $r_1, r_2 > 0$. Then

$$\left(\frac{1}{|B(0,r_1)|}\int_{B(0,r_1)} |f(x) - f_{B(0,r_2)})|^q dx\right)^{1/q} \leqslant \left(1 + \left|\ln\left(\frac{r_1}{r_2}\right)\right|\right)||f||_{CBMO_q}.$$

Lemma 2.(see [11]) Let $0 < \delta < n$, $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then $\mu_{\Omega,\delta}$ is bounded from $L^p(R^n)$ to $L^q(R^n)$.

Proof of Theorem 1. We only consider the case $0 < p < \infty$. Let $f \in \dot{K}_{q_1}^{\alpha_1,p}(R^n)$ and decompose f into

$$f(x) = \sum_{l=-\infty}^{\infty} f(x)\chi_l(x) \equiv \sum_{l=-\infty}^{\infty} f_l(x).$$

When $m = 1$, we consider

$$\begin{aligned}
||\mu_{\Omega,\delta}^{b_1}(f)||_{\dot{K}_{q_2}^{\alpha_2,p}(R^n)} &= \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}||\mu_{\Omega,\delta}^{b_1}(f_l)\chi_k||_{L^{q_2}(R^n)}^p\right)^{1/p} \\
&\leqslant C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=-\infty}^{k-3} ||\mu_{\Omega,\delta}^{b_1}(f_l)\chi_k||_{L^{q_2}(R^n)}\right)^p\right\}^{1/p} \\
&+ C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2} ||\mu_{\Omega,\delta}^{b_1}(f_l)\chi_k||_{L^{q_2}(R^n)}\right)^p\right\}^{1/p} \\
&+ C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=k+3}^{\infty} ||\mu_{\Omega,\delta}^{b_1}(f_l)\chi_k||_{L^{q_2}(R^n)}\right)^p\right\}^{1/p} \\
&= E_1 + E_2 + E_3.
\end{aligned}$$

Let us first estimate E_2, note that

$$\mu_{\Omega,\delta}^{b_1}(f_l)\chi_k = (b - b_{B_k})\mu_{\Omega,\delta}(f_l)\chi_k + \mu_{\Omega,\delta}((b - b_{B_k})f_l)\chi_k.$$

We have

$$||\mu_{\Omega,\delta}^{b_1}(f_l)\chi_k||_{L^{q_2}} \leqslant ||(b - b_{B_k})\mu_{\Omega,\delta}(f_l)\chi_k||_{L^{q_2}} + ||\mu_{\Omega,\delta}((b - b_{B_k})f_l)\chi_k||_{L^{q_2}} = J_1 + J_2.$$

For J_1, by Hölder's inequality , Lemma 1 and the boundedness of $\mu_{\Omega,\delta}$ from $L^{q_1}(R^n)$ to $L^u(R^n)$, we have

$$\begin{aligned}
J_1 &\leqslant C\left(\int_{B_k} |b_1(x) - b_{B_k}|^q dx\right)^{1/q}\left(\int_{B_k} |\mu_{\Omega,\delta}(f_l)|^u dx\right)^{1/u} \\
&\leqslant C|B_k|^{1/q}||b_1||_{CBMO_q}||f_l||_{L^{q_1}}.
\end{aligned}$$

For J_2, by Hölder's inequality, Lemma 1 and the boundedness of $\mu_{\Omega,\delta}$ from $L^t(R^n)$ to $L^{q_2}(R^n)$, we have

$$\begin{aligned} J_2 &\leqslant C\left(\int_{B_k}|(b-b_{B_k})f_l|^t dx\right)^{1/t} \\ &\leqslant C\left(\int_{B_k}|b_1(x)-b_{B_k}|^q dx\right)^{1/q}\left(\int_{B_k}|f_l|^{q_1}dx\right)^{1/q_1} \\ &\leqslant C\left(\int_{B_l}|b_1(x)-b_{B_k}|^q dx\right)^{1/q}||f_l||_{L^{q_1}} \\ &\leqslant C|B_l|^{1/q}||b_1||_{CBMO_q}||f_l||_{L^{q_1}}. \end{aligned}$$

Therefore

$$E_2 \leqslant C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2}J_1\right)^p\right\}^{1/p}+C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2}J_2\right)^p\right\}^{1/p}=I_1+I_2.$$

For I_1, if $1<p<\infty$, by Minkowski's inequality and if $0<p\leqslant 1$, by the inequality $(\sum a_i)^p\leqslant\sum|a_i|^p$, we have

$$\begin{aligned} I_1 &\leqslant C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2}2^{kn/q}||b_1||_{CBMO_q}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p} \\ &\leqslant C||b_1||_{CBMO_q} \\ &\times\begin{cases}\left[\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||^p_{L^{q_1}}\sum_{k=l-2}^{l+2}2^{(k-l)\alpha_1 p}\right]^{1/p}, & 0<p\leqslant 1\\ \left[\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||^p_{L^{q_1}}\left(\sum_{k=l-2}^{l+2}2^{(k-l)\alpha_1 p}\right)\left(\sum_{k=l-2}^{l+2}2^{(k-l)\alpha_1 p}\right)^{p/p'}\right]^{1/p}, \\ 1<p<\infty.\end{cases} \\ &\leqslant C||b_1||_{CBMO_q} \\ &\times\begin{cases}\left[\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||^p_{L^{q_1}}\sum_{k=l-2}^{l+2}2^{(k-l)\alpha_1 p}\right]^{1/p}, & 0<p\leqslant 1\\ \left[\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||^p_{L^{q_1}}\left(\sum_{k=l-2}^{l+2}2^{(k-l)\alpha_1 p/2}\right)\left(\sum_{k=l-2}^{l+2}2^{(k-l)\alpha_1 p'/2}\right)^{p/p'}\right]^{1/p}, \\ 1<p<\infty.\end{cases} \\ &\leqslant C||b_1||_{CBMO_q}\left(\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||^p_{L^{q_1}}\right)^{1/p} \\ &\leqslant C||b_1||_{CBMO_q}||f||_{\dot{K}^{\alpha_1,p}_{q_1}}. \end{aligned}$$

For I_2, similarly to the method estimating for I_1, we have

$$\begin{aligned}
I_2 &\leqslant C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2} 2^{ln/q}||b_1||_{CBMO_q}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p} \\
&\leqslant C||b_1||_{CBMO_q}\begin{cases}\left[\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||^p_{L^{q_1}} \sum_{k=l-2}^{l+2} 2^{(k-l)(\alpha_1-\frac{n}{q})p}\right]^{1/p}, & 0<p\leqslant 1 \\ \left[\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||^p_{L^{q_1}}\left(\sum_{k=l-2}^{l+2} 2^{(k-l)(\alpha_1-\frac{n}{q})p/2}\right)\right. & \\ \left.\times\left(\sum_{k=l-2}^{l+2} 2^{(k-l)(\alpha_1-\frac{n}{q})p'/2}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty.\end{cases} \\
&\leqslant C||b_1||_{CBMO_q}\begin{cases}\left[\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||^p_{L^{q_1}} \sum_{k=l-2}^{l+2} 2^{(k-l)(\alpha_1-\frac{n}{q})p}\right]^{1/p}, & 0<p\leqslant 1 \\ \left[\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||^p_{L^{q_1}}\left(\sum_{k=l-2}^{l+2} 2^{(k-l)(\alpha_1-\frac{n}{q})p/2}\right)\right. & \\ \left.\times\left(\sum_{k=l-2}^{l+2} 2^{(k-l)(\alpha_1-\frac{n}{q})p'/2}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty.\end{cases} \\
&\leqslant C||b_1||_{CBMO_q}\left(\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||^p_{L^{q_1}}\right)^{1/p} \\
&\leqslant C||b_1||_{CBMO_q}||f||_{\dot{K}^{\alpha_1,p}_{q_1}}.
\end{aligned}$$

Thus, we deduce

$$E_2 \leqslant C||b_1||_{CBMO_q}||f||_{\dot{K}^{\alpha_1,p}_{q_1}}.$$

Now let us turn to estimate E_1, choosing $(b_1)_B = |B|^{-1}\int_B b_1(x)dx$, by Minkowski's inequality, note that $x \in A_k$, $y \in A_l$, we have

$$\begin{aligned}
&||\mu^{b_1}_{\Omega,\delta}(f_l)\chi_k||_{L^{q_2}} \\
\leqslant&\left\{\int_{A_k}\left[\int_0^\infty\left|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}(b_1(x)-b_1(y))f(y)dy\right|^2\frac{dt}{t^3}\right]^{q_2/2}dx\right\}^{1/q_2} \\
\leqslant&\left\{\int_{A_k}\left[\int_{R^n}(b_1(x)-b_1(y))\frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}}f(y)\left(\int_{|x-y|\leqslant t}\frac{dt}{t^3}\right)^{1/2}dy\right]^{q_2}dx\right\}^{1/q_2} \\
\leqslant&\left\{\int_{A_k}\left[\int_{R^n}(b_1(x)-b_1(y))\frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}}f(y)\left|\frac{1}{|x-y|^2}\right|^{1/2}dy\right]^{q_2}dx\right\}^{1/q_2} \\
\leqslant&\left\{\int_{A_k}\left[\int_{R^n}(b_1(x)-b_1(y))\frac{|\Omega(x-y)|}{|x-y|^{n-\delta}}f(y)dy\right]^{q_2}dx\right\}^{1/q_2}
\end{aligned}$$

$$\begin{aligned}
&\leqslant C|B_k|^{\delta/n-1}\left\{\int_{A_k}\left(\int_{A_l}|b_1(x)-b_1(y)||f(y)|dy\right)^{q_2}dx\right\}^{1/q_2}\\
&\leqslant C|B_k|^{\delta/n-1}\left(\int_{A_k}|b_1(x)-(b_1)_B|^{q_2}\left(\int_{A_l}|f(y)|dy\right)^{q_2}dx\right)^{1/q_2}\\
&+ C|B_k|^{\delta/n-1}\left(\int_{A_k}\int_{A_l}|b_1(y)-(b_1)_B||f(y)|dydx\right)^{1/q_2}\\
&\leqslant C|B_k|^{\delta/n-1}||f_l||_{L^1}\left(\int_{A_k}|b_1(x)-(b_1)_B|^{q_2}dx\right)^{1/q_2}\\
&+ C|B_k|^{\delta/n-1+1/q_2}\int_{A_l}|b_1(y)-(b_1)_B||f(y)|dy\\
&\leqslant C|B_k|^{\delta/n-1}||f_l||_{L^{q_1}}\mu(B_l)^{1-1/q_1}\left(\int_{A_k}|b_1(x)-(b_1)_B|^q dx\right)^{1/q}|B_k|^{1/q_2-1/q}\\
&+ C|B_k|^{\delta/n-1+1/q_2}\left(\int_{A_l}|b_1(y)-(b_1)_B|^q dy\right)^{1/q}\left(\int_{A_l}|f(y)|^{q_1}dy\right)^{1/q_1}|B_l|^{1-1/q-1/q_1}\\
&\leqslant C|B_k|^{\delta/n-1+1/q_2}|B_l|^{1-1/q_1}||f_l||_{L^{q_1}}||b_1||_{CBMO_q}.
\end{aligned}$$

Therefore, we know

$$E_1\leqslant C||b_1||_{CBMO_q}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\left(\sum_{l=-\infty}^{k-3}2^{nk(\delta/n-1+1/q_2)}2^{nl(1-1/q_1)}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p}$$

$$\begin{aligned}
&\leqslant C||b_1||_{CBMO_q}\\
&\times\begin{cases}\left[\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p\sum_{k=l+3}^{\infty}\left(2^{nk(\delta/n-1+1/q_2-1/q)}2^{nl(1-1/q_1)}2^{(k-l)\alpha_1}\right)^p\right]^{1/p}, \\ 0<p\leqslant 1\\ \left[\sum_{l=-\infty}^{\infty}|B_l|^{\alpha_1 p}||f_l||_{L^{q_1}}^p\left(\sum_{k=l+3}^{\infty}\left(2^{nk(\delta/n-1+1/q_2-1/q)}2^{nl(1-1/q_1)}2^{(k-l)\alpha_1}\right)^{p/2}\right)\right.\\ \left.\times\left(\sum_{k=l+3}^{\infty}\left(2^{nk(\delta/n-1+1/q_2-1/q)}2^{nl(1-1/q_1)}2^{(k-l)\alpha_1}\right)^{p'/2}\right)^{p/p'}\right]^{1/p},\ 1<p<\infty.\end{cases}\\
&\leqslant C||b_1||_{CBMO_q}\\
&\times\begin{cases}\left[\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p\sum_{k=l+3}^{\infty}\left(2^{nk(1/q_1-1)}2^{nl(1-1/q_1)}2^{(k-l)\alpha_1}\right)^p\right]^{1/p}, \quad 0<p\leqslant 1\\ \left[\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p\left(\sum_{k=l+3}^{\infty}\left(2^{nk(1/q_1-1)}2^{nl(1-1/q_1)}2^{(k-l)\alpha_1}\right)^{p/2}\right)\right.\\ \left.\times\left(\sum_{k=l+3}^{\infty}\left(2^{nk(1/q_1-1)}2^{nl(1-1/q_1)}2^{(k-l)\alpha_1}\right)^{p'/2}\right)^{p/p'}\right]^{1/p},\ 1<p<\infty.\end{cases}
\end{aligned}$$

$$\leqslant C||b_1||_{CBMO_q}\begin{cases}\left[\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p\sum_{k=l+3}^{\infty}2^{(k-l)(n/q_1-n+\alpha_1)p}\right]^{1/p}, & 0<p\leqslant 1\\ \left[\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p\left(\sum_{k=l+3}^{\infty}2^{(k-l)(\alpha_1-1+\frac{n}{q_1})p/2}\right)\right. & \\ \left.\times\left(\sum_{k=l+3}^{\infty}2^{(k-l)(\alpha_1-1+\frac{n}{q_1})p'/2}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty.\end{cases}$$

$$\leqslant C||b_1||_{CBMO_q}\left(\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p\right)^{1/p}$$

$$\leqslant C||b_1||_{CBMO_q}||f||_{\dot{K}_{q_1}^{\alpha_1,p}}.$$

Now let us turn to estimate E_3, by Hölder's inequality, we have

$$||\mu_{\Omega,\delta}^{b_1}(f_l)\chi_k||_{L^{q_2}}$$

$$\leqslant \left\{\int_{A_k}\left[\int_0^\infty\left|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}(b_1(x)-b_1(y))f(y)dy\right|^2\frac{dt}{t^3}\right]^{q_2/2}dx\right\}^{1/q_2}$$

$$\leqslant \left\{\int_{A_k}\left[\int_{R^n}(b_1(x)-b_1(y))\frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}}f(y)\left(\int_{|x-y|\leqslant t}\frac{dt}{t^3}\right)^{1/2}dy\right]^{q_2}dx\right\}^{1/q_2}$$

$$\leqslant \left\{\int_{A_k}\left[\int_{R^n}(b_1(x)-b_1(y))\frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}}f(y)\left|\frac{1}{|x-y|^2}\right|^{1/2}dy\right]^{q_2}dx\right\}^{1/q_2}$$

$$\leqslant \left\{\int_{A_k}\left[\int_{R^n}(b_1(x)-b_1(y))\frac{|\Omega(x-y)|}{|x-y|^{n-\delta}}f(y)dy\right]^{q_2}dx\right\}^{1/q_2}$$

$$\leqslant C|B_l|^{\delta/n-1}\left\{\int_{A_k}\left(\int_{A_l}|b_1(x)-b_1(y)||f(y)|dy\right)^{q_2}dx\right\}^{1/q_2}$$

$$\leqslant C|B_l|^{\delta/n-1}\left(\int_{A_k}|b_1(x)-(b_1)_B|^{q_2}\left(\int_{A_l}|f(y)|dy\right)^{q_2}dx\right)^{1/q_2}$$

$$+\ C|B_l|^{\delta/n-1}\left(\int_{A_k}\int_{A_l}|b_1(y)-(b_1)_B||f(y)|dydx\right)^{1/q_2}$$

$$\leqslant C|B_l|^{\delta/n-1}||f_l||_{L^1}\left(\int_{A_k}|b_1(x)-(b_1)_B|^{q_2}dx\right)^{1/q_2}$$

$$+\ C|B_l|^{\delta/n-1}|B_k|^{1/q_2}\int_{A_l}|b_1(y)-(b_1)_B||f(y)|dy$$

$$\leqslant C|B_l|^{\delta/n-1}||f_l||_{L^{q_1}}|B_l|^{1-1/q_1}\left(\int_{A_k}|b_1(x)-(b_1)_B|^q dx\right)^{1/q}|B_k|^{1/q_2-1/q}$$

$$+\ C|B_l|^{\delta/n-1}|B_k|^{1/q_2}\left(\int_{A_l}|b_1(y)-(b_1)_B|^q dy\right)^{1/q}\left(\int_{A_l}|f(y)|^{q_1}dy\right)^{1/q_1}$$

$$\times\ |B_l|^{1-1/q-1/q_1}$$

$$\leqslant C|B_l|^{\delta/n-1/q_1}|B_k|^{1/q_2}||f_l||_{L^{q_1}}||b_1||_{CBMO_q}.$$

Thus, in this case, we obtain

$$
\begin{aligned}
E_3 &\leqslant C||b_1||_{CBMO_q}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\left(\sum_{l=k+3}^{\infty}2^{ln(\delta/n-1/q_1)}2^{kn(1/q_2)}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p}\\
&\leqslant C||b_1||_{CBMO_q}\\
&\times\begin{cases}\left[\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p\sum_{k=-\infty}^{l-3}\left(2^{ln(\delta/n-1/q_1)}2^{kn(1/q_2-1/q)}2^{(k-l)\alpha_1}\right)^p\right]^{1/p}, \\ 0<p\leqslant 1\\ \left[\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p\left(\sum_{k=-\infty}^{l-3}\left(2^{ln(\delta/n-1/q_1)}2^{kn(1/q_2-1/q)}2^{(k-l)\alpha_1}\right)^{p/2}\right)\right.\\ \left.\times\left(\sum_{k=-\infty}^{l-3}\left(2^{ln(\delta/n-1/q_1)}2^{kn(1/q_2-1/q)}2^{(k-l)\alpha_1}\right)^{p'/2}\right)^{p/p'}\right]^{1/p}, \ 1<p<\infty.\end{cases}\\
&\leqslant C||b_1||_{CBMO_q}\begin{cases}\left[\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p\sum_{k=-\infty}^{l-3}2^{(l-k)(\delta-\alpha_1-\frac{n}{q_1})p}\right]^{1/p}, \quad 0<p\leqslant 1\\ \left[\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p\left(\sum_{k=-\infty}^{l-3}2^{(l-k)(\delta-\alpha_1-\frac{n}{q_1})p/2}\right)\right.\\ \left.\times\left(\sum_{k=-\infty}^{l-3}2^{(l-k)(\delta-\alpha_1-\frac{n}{q_1})p'/2}\right)^{p/p'}\right]^{1/p}, \ 1<p<\infty.\end{cases}\\
&\leqslant C||b_1||_{CBMO_q}\left(\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p\right)^{1/p}\\
&\leqslant C||b_1||_{CBMO_q}||f||_{\dot K_{q_1}^{\alpha_1,p}}.
\end{aligned}
$$

This completes the proof of the case $m=1$.

Now, we consider the case $m\geqslant 2$. We write,

$$
\begin{aligned}
||\mu_{\Omega,\delta}^{\vec b}(f)||_{\dot K_{q_2}^{\alpha_2,p}(X)} &= \left(\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}||\mu_{\Omega,\delta}^{\vec b}(f_l)\chi_k||_{L^{q_2}(R^n)}^p\right)^{1/p}\\
&\leqslant C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\left(\sum_{l=-\infty}^{k-3}||\mu_{\Omega,\delta}^{\vec b}(f_l)\chi_k||_{L^{q_2}(R^n)}\right)^p\right\}^{1/p}\\
&+ C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2}||\mu_{\Omega,\delta}^{\vec b}\chi_k||_{L^{q_2}(R^n)}\right)^p\right\}^{1/p}\\
&+ C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\left(\sum_{l=k+3}^{\infty}||\mu_{\Omega,\delta}^{\vec b}\chi_k||_{L^{q_2}(R^n)}\right)^p\right\}^{1/p}\\
&= G_1+G_2+G_3.
\end{aligned}
$$

Let us first estimate G_2, set $\vec{b}_B=((b_1)_B,\cdots,(b_m)_B)$, where $(b_j)_B=|B|^{-1}\int_B|b_j(x)|dx$, $1\leqslant j\leqslant m$, we have

$$
\begin{aligned}
&F^{\vec{b}}_{t,\delta}(f)(x)=\int_{|x-y|\leqslant t}\left[\prod_{j=1}^m((b_j(x)-(b_j)_B)-(b_j(y)-(b_j)_B))\right]f(y)\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}dy\\
=&\sum_{j=0}^m\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-b_B)_\sigma\int_{|x-y|\leqslant t}(b(y)-b_B)_{\sigma^c}f(y)\Omega(x-y)|x-y|^{1-n+\delta}dy\\
=&\prod_{j=1}^m(b_j(x)-(b_j)_BF_{t,\delta}(f)(x)+(-1)^mF_{t,\delta}(\prod_{j=1}^m(b_j(y)-(b_j)_B)f)(x)\\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-b_B)_\sigma F^{\vec{b}_{\sigma^c}}_{t,\delta}(f)(x),
\end{aligned}
$$

thus

$$
\begin{aligned}
&\mu^{\vec{b}}_{\Omega,\delta}(f_l)(x)=||F^{\vec{b}}_{t,\delta}(f_l)(x)||\\
\leqslant\ &||\prod_{j=1}^m(b_j(x)-(b_j)_B)F_{t,\delta}(f_l)(x)||+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||(b(x)-(b)_B)_\sigma F^{\vec{b}_{\sigma^c}}_{t,\delta}(f_l)(x)||\\
+\ &||F_{t,\delta}(\prod_{j=1}^m(b_j-(b_j)_B)f_l)(x)||\\
\leqslant\ &\prod_{j=1}^m(b_j(x)-(b_j)_B))\mu_{\Omega,\delta}(f_l)(x)+(-1)^m\mu_{\Omega,\delta}(\prod_{j=1}^m(b_j-(b_j)_B))_B)f_l)(x)\\
+\ &\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(\vec{b}(x)-\vec{b}_B)_\sigma\mu_{\Omega,\delta}((\vec{b}-\vec{b}_B)_{\sigma^c}f_l)(x)\\
=\ &H_1+H_2+H_3.
\end{aligned}
$$

For H_1, taking $1<q_1<n/\delta$ and u such that $1/u=1/q_1-\delta/n$, choosing $1/q=1/q_1'+\cdots 1/q_j'$, by Hölder's inequality and the boundedness of $\mu_{\Omega,\delta}$ from $L^{q_1}(R^n)$ to $L^u(R^n)$, we have

$$
\begin{aligned}
&||\prod_{j=1}^m(b_j(x)-(b_j)_B)\mu_{\Omega,\delta}(f_l)(x)\chi_k||_{L^{q_2}}\\
\leqslant\ &C\Big(\int_{A_k}\Big|\prod_{j=1}^m(b_j(x)-(b_j)_B)\Big|^q dx\Big)^{1/q}\Big(\int_{A_k}|\mu_{\Omega,\delta}f(x)|^u dx\Big)^{1/u}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C\prod_{j=1}^{m}\Big(\int_{A_k}|b_j(x)-(b_j)_B|^{q_j'}dx\Big)^{1/q_j'}\Big(\int_{A_k}|f_l(x)|^{q_1}dx\Big)^{1/q_1}\\
&\leqslant C|B_k|^{1/q_1'+\cdots 1/q_j'}\prod_{j=1}^{m}\Big(\frac{1}{|B_k|}\int_{B_k}|b_j(x)-(b_j)_B|^{q_j'}dx\Big)^{1/q_j'}\Big(\int_{A_k}|f_l(x)|^{q_1}dx\Big)^{1/q_1}\\
&\leqslant C|B_k|^{1/q}\prod_{j=1}^{m}||b_j||_{CBMO_{q_j}}||f_l||_{L^{q_1}}\\
&\leqslant C|B_k|^{1/q}||\vec{b}||_{CBMO_{\vec{q}}}||f_l||_{L^{q_1}}.
\end{aligned}
$$

For H_2, taking $1<t<n/\delta$ and u such that $1/q_2=1/t-\delta/n$, choosing $1/t=1/q+1/q_1$, by Hölder's inequality and the boundedness of $\mu_{\Omega,\delta}$ from $L^t(R^n)$ to $L^{q_2}(R^n)$, we have

$$
\begin{aligned}
&||(-1)^m\mu_{\Omega,\delta}(\prod_{j=1}^{m}(b_j-(b_j)_B)))(x)\chi_k||_{L^{q_2}}\\
\leqslant\ & C||\prod_{j=1}^{m}(b_j-(b_j)_B))f_l\chi_k||_{L^t}\\
\leqslant\ & C\Big(\int_{A_k}\Big|\prod_{j=1}^{m}(b_j(x)-(b_j)_B))\Big|^q dx\Big)^{1/q}\Big(\int_{A_k}|f_l(x)|^{q_1}dx\Big)^{1/q_1}\\
\leqslant\ & C|B_k|^{1/q}||\vec{b}||_{CBMO_{\vec{q}}}||f_l||_{L^{q_1}}.
\end{aligned}
$$

For H_3, choosing $1/q_2=1/q_1'+1/\omega$ and $1/\omega=1/q_2'+1/q_1-\delta/n$, using Hölder's inequality and the boundedness of $\mu_{\Omega,\delta}$, we have

$$
\begin{aligned}
&||\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-b_B)_\sigma\mu_{\Omega,\delta}((b-b_B)_{\sigma^c}f_l)(x)\chi_k||_{L^{q_2}}\\
\leqslant\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\Big(\int_{A_k}|(b(x)-b_B)_\sigma|^{q_1'}dx\Big)^{1/q_1'}\Big(\int_{A_k}|\mu_{\Omega,\delta}((b-b_B)_{\sigma^c}f_l)(x)|^{\omega}dx\Big)^{1/\omega}\\
\leqslant\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\Big(\int_{A_k}|(b(x)-b_B)_\sigma|^{q_1'}dx\Big)^{1/q_1'}\Big(\int_{A_k}|(b(x)-b_B)_{\sigma^c}|^{q_2'}dx\Big)^{1/q_2'}\\
\times\ & \Big(\int_{A_k}|f_l(x)|^{p_1}dx\Big)^{1/q_1}\\
\leqslant\ & C|B_k|^{1/q_1'}||b_\sigma||_{CBMO_q}|B_k|^{1/q_2'}||b_{\sigma^c}||_{CBMO_{\vec{q}}}||f_l||_{L^{q_1}}\\
\leqslant\ & C|B_k|^{1/q}||\vec{b}||_{CBMO_{\vec{q}}}||f_l||_{L^{q_1}}.
\end{aligned}
$$

Then, similarly to the method estimating for I_1, we get $G_2 \leqslant C||\vec{b}||_{CBMO_q}||f||_{\dot{K}^{\alpha_1,p}_{q_1}}$. Next, we estimate G_1, let $\tau,\tau' \in \mathbf{N}$ such that $\tau+\tau'=m$, we have

$$\begin{aligned}
&||\mu^{\vec{b}}_{\Omega,\delta}(f_l)\chi_k||_{L^{q_2}}\\
\leqslant\ & \left\{\int_{A_k}\left[\int_0^\infty\left|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}\prod_{j=1}^m|b_j(x)-b_j(y)f(y)dy\right|^2\frac{dt}{t^3}\right]^{q_2/2}dx\right\}^{1/q_2}\\
\leqslant\ & \left\{\int_{A_k}\left[\int_{R^n}\prod_{j=1}^m|b_j(x)-b_j(y)\frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}}f(y)\left(\int_{|x-y|\leqslant t}\frac{dt}{t^3}\right)^{1/2}dy\right]^{q_2}dx\right\}^{1/q_2}\\
\leqslant\ & \left\{\int_{A_k}\left[\int_{R^n}\prod_{j=1}^m|b_j(x)-b_j(y)\frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}}f(y)\left|\frac{1}{|x-y|^2}\right|^{1/2}dy\right]^{q_2}dx\right\}^{1/q_2}\\
\leqslant\ & \left\{\int_{A_k}\left[\int_{R^n}\prod_{j=1}^m|b_j(x)-b_j(y)\frac{|\Omega(x-y)|}{|x-y|^{n-\delta}}f(y)dy\right]^{q_2}dx\right\}^{1/q_2}\\
\leqslant\ & C|B_k|^{\delta/n-1}\sum_{j=0}^m\sum_{\sigma\in C_j^m}\left(\int_{A_k}|(b(x)-b_B)_\sigma|^{q_2}dx\right)^{1/q_2}\int_{A_l}|(b(y)-b_B)_{\sigma^c}||f(y)|dy\\
\leqslant\ & C|B_k|^{\delta/n-1}\sum_{j=0}^m\sum_{\sigma\in C_j^m}\sum_{\tau+\tau'=m}\left(\int_{A_k}|(b(x)-b_B)_\sigma|^{\tau}dx\right)^{1/\tau}|B_k|^{1/q_2-1/\tau}\\
\times\ & \left(\int_{A_l}|(b(x)-b_B)_{\sigma^c}|^{\tau'}dx\right)^{1/\tau'}\left(\int_{A_l}|f(y)|^{q_1}dy\right)^{1/q_1}|B_l|^{1-1/\tau'-1/q_1}\\
\leqslant\ & C|B_k|^{\delta/n-1+1/q_2}|B_l|^{1-1/q_1}||\vec{b}||_{CBMO_{\vec{q}}}||f_l||_{L^{q_1}}.
\end{aligned}$$

Then, similarly to the method estimating for E_1, we can obtain $G_1 \leqslant C||\vec{b}||_{CBMO_{\vec{q}}}||f||_{\dot{K}^{\alpha_1,p}_{q_1}}$. Finally, let us estimate G_3, since

$$\begin{aligned}
&||\mu^{\vec{b}}_{\Omega,\delta}(f_l)\chi_k||_{L^{q_2}}\\
\leqslant\ & \left\{\int_{A_k}\left[\int_0^\infty\left|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}\prod_{j=1}^m|b_j(x)-b_j(y)f(y)dy\right|^2\frac{dt}{t^3}\right]^{q_2/2}dx\right\}^{1/q_2}\\
\leqslant\ & \left\{\int_{A_k}\left[\int_{R^n}\prod_{j=1}^m|b_j(x)-b_j(y)\frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}}f(y)\left(\int_{|x-y|\leqslant t}\frac{dt}{t^3}\right)^{1/2}dy\right]^{q_2}dx\right\}^{1/q_2}\\
\leqslant\ & \left\{\int_{A_k}\left[\int_{R^n}\prod_{j=1}^m|b_j(x)-b_j(y)\frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}}f(y)\left|\frac{1}{|x-y|^2}\right|^{1/2}dy\right]^{q_2}dx\right\}^{1/q_2}\\
\leqslant\ & \left\{\int_{A_k}\left[\int_{R^n}\prod_{j=1}^m|b_j(x)-b_j(y)\frac{|\Omega(x-y)|}{|x-y|^{n-\delta}}f(y)dy\right]^{q_2}dx\right\}^{1/q_2}
\end{aligned}$$

$$\begin{aligned}
&\leqslant C|B_l|^{\delta/n-1}\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}\left(\int_{A_k}|(b(x)-b_B)_\sigma|^{q_2}dx\right)^{1/q_2}\int_{A_l}|(b(x)-b_B)_{\sigma^c}||f(y)|dy\\
&\leqslant C|B_l|^{\delta/n-1}\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}\sum_{\tau+\tau'=m}\left(\int_{A_k}|(b(x)-b_B)_\sigma|^{\tau}dx\right)^{1/\tau}|B_k|^{1/q_2-1/\tau}\\
&\times\left(\int_{A_l}|(b(y)-b_B)_{\sigma^c}|^{\tau'}dx\right)^{1/\tau'}\left(\int_{A_l}|f(y)|^{q_1}dy\right)^{1/q_1}|B_l|^{1-1/\tau'-1/q_1}\\
&\leqslant C|B_l|^{\delta/n-1/q_1}|B_k|^{1/q_2}||\vec{b}||_{CBMO_{\vec{q}}}||f_l||_{L^{q_1}}.
\end{aligned}$$

Then, similarly to the method estimating for E_3, we can get $G_3\leqslant C||\vec{b}||_{CBMO_{\vec{q}}}||f||_{\dot{K}_{q_1}^{\alpha_1,p}}$. This completes the proof of Theorem 1.

Proof of Theorem 2. Let $f\in M\dot{K}_{p,q_1}^{\alpha_1,\lambda}(R^n)$ and decompose f into

$$f(x)=\sum_{l=-\infty}^{\infty}f(x)\chi_l(x)\equiv\sum_{l=-\infty}^{\infty}f_l(x).$$

When $m=1$, we consider

$$\begin{aligned}
||\mu_{\Omega,\delta}^{b_1}(f)||_{M\dot{K}_{p,q_2}^{\alpha_2,\lambda}(R^n)} &= \sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha_2p}||\mu_{\Omega,\delta}^{b_1}(f_l)\chi_k||_{L^{q_2}(R^n)}^p\right)^{1/p}\\
&\leqslant C\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}2^{k\alpha_2p}\left(\sum_{l=-\infty}^{k-3}||\mu_{\Omega,\delta}^{b_1}(f_l)\chi_k||_{L^{q_2}(R^n)}\right)^p\right\}^{1/p}\\
&+ C\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}2^{k\alpha_2p}\left(\sum_{l=k-2}^{k+2}||\mu_{\Omega,\delta}^{b_1}(f_l)\chi_k||_{L^{q_2}(R^n)}\right)^p\right\}^{1/p}\\
&+ C\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}2^{k\alpha_2p}\left(\sum_{l=k+3}^{\infty}||\mu_{\Omega,\delta}^{b_1}(f_l)\chi_k||_{L^{q_2}(R^n)}\right)^p\right\}^{1/p}\\
&= U_1+U_2+U_3.
\end{aligned}$$

Let us first estimate U_2, similarly to the method estimating for E_2, we have

$$\begin{aligned}
U_2 &\leqslant C\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}2^{k\alpha_2p}\left(\sum_{l=k-2}^{k+2}J_1\right)^p\right\}^{1/p}\\
&\quad+C\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}2^{k\alpha_2p}\left(\sum_{l=k-2}^{k+2}J_2\right)^p\right\}^{1/p}\\
&= V_1+V_2.
\end{aligned}$$

For V_1, we have

$$\begin{aligned}
V_1 &\leqslant C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2}|B_k|^{1/q}||b_1||_{CBMO_q}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p}\\
&\leqslant C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k(\alpha_1-n/q)p}\left(\sum_{l=k-2}^{k+2}2^{kn/q}||b_1||_{CBMO_q}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p}\\
&\leqslant C||b_1||_{CBMO_q} \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\\
&\times \left\{\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left[\sum_{l=k-2}^{k+2}2^{(k-l)\alpha_1}2^{(l-k)\lambda}2^{-l\lambda}\left(\sum_{i=-\infty}^{l}2^{i\alpha_1 p}||f_i||^p_{L^{q_1}}\right)^{1/p}\right]^p\right\}^{1/p}\\
&\leqslant C||b_1||_{CBMO_q} \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left[\sum_{l=k-2}^{k+2}2^{(k-l)(\alpha_1-\lambda)}||f||_{M\dot{K}^{\alpha_1,p}_{p,q_1}}\right]^p\right\}^{1/p}\\
&\leqslant C||b_1||_{CBMO_q}||f||_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}}.
\end{aligned}$$

Similarly, for V_2, we have

$$\begin{aligned}
V_2 &\leqslant C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2}|B_l|^{1/q}||b_1||_{CBMO_q}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p}\\
&\leqslant C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k(\alpha_1-n/q)p}\left(\sum_{l=k-2}^{k+2}2^{ln/q}||b_1||_{CBMO_q}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p}\\
&\leqslant C||b_1||_{CBMO_q} \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\\
&\times \left\{\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left[\sum_{l=k-2}^{k+2}2^{(k-l)\alpha_1}2^{(l-k)n/q}2^{(l-k)\lambda}2^{-l\lambda}\left(\sum_{i=-\infty}^{l}2^{i\alpha_1 p}||f_i||^p_{L^{q_1}}\right)^{1/p}\right]^p\right\}^{1/p}\\
&\leqslant C||b_1||_{CBMO_q} \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left[\sum_{l=k-2}^{k+2}2^{(k-l)(\alpha_1-n/q-\lambda)}||f||_{M\dot{K}^{\alpha_1,p}_{p,q_1}}\right]^p\right\}^{1/p}\\
&\leqslant C||b_1||_{CBMO_q}||f||_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}}.
\end{aligned}$$

Therefore $U_2 \leqslant C||b_1||_{CBMO_q}||f||_{M\dot{K}^{\alpha_1,p}_{p,q_1}}$.
Then, let us estimate U_1, similar to E_1, we get

$$\begin{aligned}
U_1 &\leqslant C||b_1||_{CBMO_q} \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\\
&\times \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=-\infty}^{k-3}|B_k|^{\delta/n-1+1/q_2}|B_l|^{1-1/q_1}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C||b_1||_{CBMO_q}\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\\
&\times \left\{\sum_{k=-\infty}^{\infty}2^{k(\alpha_1-n/q)p}\left(\sum_{l=-\infty}^{k-3}2^{kn(\delta/n-1+1/q_2)}2^{ln(1-1/q_1)}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p}\\
&\leqslant C||b_1||_{CBMO_q}\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\\
&\times \left\{\sum_{k=-\infty}^{k_0}2^{k\lambda p}\left[\sum_{l=-\infty}^{k-3}2^{(k-l)(n/q_1-n)}2^{(k-l)\alpha_1}2^{(l-k)\lambda}2^{-l\lambda}\left(\sum_{i=-\infty}^{l}2^{i\alpha_1 p}||f_i||^p_{L^{q_1}}\right)^{1/p}\right]^p\right\}^{1/p}\\
&\leqslant C||b_1||_{CBMO_q}\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}2^{k\lambda p}\left[\sum_{l=-\infty}^{k-3}2^{(k-l)(\alpha_1-n+n/q_1-\lambda)}||f||_{M\dot{K}^{\alpha_1,p}_{p,q_1}}\right]^p\right\}^{1/p}\\
&\leqslant C||b_1||_{CBMO_q}||f||_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}}.
\end{aligned}
$$

Last, let us estimate U_3, similar to E_3, we get

$$
\begin{aligned}
U_3 &\leqslant C||b_1||_{CBMO_q}\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\left(\sum_{l=k+3}^{\infty}|B_l|^{\delta/n-1/q_1}|B_k|^{1/q_2}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p}\\
&\leqslant C||b_1||_{CBMO_q}\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\\
&\times \left\{\sum_{k=-\infty}^{\infty}2^{k(\alpha_1-n/q)p}\left(\sum_{l=k+3}^{\infty}2^{ln(\delta/n-1/q_1)}2^{kn(1/q_2)}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p}\\
&\leqslant C||b_1||_{CBMO_q}\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\\
&\times \left\{\sum_{k=-\infty}^{k_0}2^{k\lambda p}\left[\sum_{l=k+3}^{\infty}2^{(l-k)(\delta-n/q_1)}2^{(k-l)\alpha_1}2^{(l-k)\lambda}2^{-l\lambda}\left(\sum_{i=-\infty}^{l}2^{i\alpha_1 p}||f_i||^p_{L^{q_1}}\right)^{1/p}\right]^p\right\}^{1/p}\\
&\leqslant C||b_1||_{CBMO_q}\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}2^{k\lambda p}\left[\sum_{l=k+3}^{\infty}2^{(l-k)(\delta-n/q_1-\alpha_1+\lambda)}||f||_{M\dot{K}^{\alpha_1,p}_{p,q_1}}\right]^p\right\}^{1/p}\\
&\leqslant C||b_1||_{CBMO_q}||f||_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}}.
\end{aligned}
$$

This completes the proof of the case $m=1$.
When $m>1$, we consider

$$
||\mu^{\vec{b}}_{\Omega,\delta}(f)||_{M\dot{K}^{\alpha_2,\lambda}_{p,q_2}(R^n)} = \sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha_2 p}||\mu^{\vec{b}}_{\Omega,\delta}(f_l)\chi_k||^p_{L^{q_2}(R^n)}\right)^{1/p}
$$

$$\begin{aligned}
&\leqslant C\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}2^{k\alpha_2 p}\left(\sum_{l=-\infty}^{k-3}||\mu_{\Omega,\delta}^{\vec{b}}(f_l)\chi_k||_{L^{q_2}(R^n)}\right)^p\right\}^{1/p}\\
&+ C\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2}||\mu_{\Omega,\delta}^{\vec{b}}(f_l)\chi_k||_{L^{q_2}(R^n)}\right)^p\right\}^{1/p}\\
&+ C\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}2^{k\alpha_2 p}\left(\sum_{l=k+3}^{\infty}||\mu_{\Omega,\delta}^{\vec{b}}(f_l)\chi_k||_{L^{q_2}(R^n)}\right)^p\right\}^{1/p}\\
&= W_1+W_2+W_3.
\end{aligned}$$

For W_2, similar to G_2, we have

$$\begin{aligned}
W_2 &\leqslant C||\vec{b}||_{CBMO_{\vec{q}}}\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\Bigg\{\sum_{k=-\infty}^{k_0}2^{k\alpha_2 p}\Bigg(\sum_{l=k-2}^{k+2}|B_k|^{1/q}||f_l||_{L^{q_1}}\Bigg)^p\Bigg\}^{1/p}\\
&\leqslant C||\vec{b}||_{CBMO_{\vec{q}}}\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\Bigg\{\sum_{k=-\infty}^{k_0}2^{k\lambda p}\Bigg[\sum_{l=k-2}^{k+2}2^{(k-l)(\alpha_1-\lambda)}||f||_{M\dot{K}^{\alpha_1,p}_{p,q_1}}\Bigg]^p\Bigg\}^{1/p}\\
&\leqslant C||\vec{b}||_{CBMO_{\vec{q}}}||f||_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}}.
\end{aligned}$$

For W_1, similar to G_1, we have

$$\begin{aligned}
W_1 &\leqslant C||\vec{b}||_{CBMO_{\vec{q}}}\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\\
&\times \Bigg\{\sum_{k=-\infty}^{k_0}2^{k\alpha_2 p}\Bigg(\sum_{l=-\infty}^{k-3}|B_k|^{\delta/n-1+1/q_2}|B_l|^{1-1/q_1}||f_l||_{L^{q_1}}\Bigg)^p\Bigg\}^{1/p}\\
&\leqslant C||\vec{b}||_{CBMO_{\vec{q}}}\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\Bigg\{\sum_{k=-\infty}^{k_0}2^{k\lambda p}\Bigg[\sum_{l=-\infty}^{k-3}2^{(k-l)(\alpha_1-n+n/q_1-\lambda)}||f||_{M\dot{K}^{\alpha_1,p}_{p,q_1}}\Bigg]^p\Bigg\}^{1/p}\\
&\leqslant C||\vec{b}||_{CBMO_{\vec{q}}}||f||_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}}.
\end{aligned}$$

For W_3, similar to G_3, we have

$$\begin{aligned}
W_3 &\leqslant C||\vec{b}||_{CBMO_{\vec{q}}}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\left(\sum_{l=k+3}^{\infty}|B_l|^{\delta/n-1/q_1}|B_k|^{1/q_2}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p}\\
&\leqslant C||\vec{b}||_{CBMO_{\vec{q}}}\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\\
&\times \Bigg\{\sum_{k=-\infty}^{k_0}2^{k\lambda p}\Bigg[\sum_{l=k+3}^{\infty}2^{(l-k)(\delta-n/q_1-\alpha_1+\lambda)}2^{-l\lambda}\Bigg(\sum_{i=-\infty}^{l}2^{i\alpha_1 p}||f_i||^p_{L^{q_1}}\Bigg)^{1/p}\Bigg]^p\Bigg\}^{1/p}\\
&\leqslant C||\vec{b}||_{CBMO_{\vec{q}}}\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\Bigg\{\sum_{k=-\infty}^{k_0}2^{k\lambda p}\Bigg[\sum_{l=k+3}^{\infty}2^{(l-k)(\alpha_1-n+n/q_1-\lambda)}||f||_{M\dot{K}^{\alpha_1,p}_{p,q_1}}\Bigg]^p\Bigg\}^{1/p}\\
&\leqslant C||\vec{b}||_{CBMO_{\vec{q}}}||f||_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}}.
\end{aligned}$$

This completes the proof of Theorem 2.

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