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A COMPARISON OF THE METHOD OF BRACKETS WITH THE METHOD OF NEGATIVE DIMENSION

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Abstract. Negative dimensional integration is a method of evaluating Feynman diagrams. The method of brackets is an improvement on existing algorithms. These methods are compared for the "flying saucer" diagram.

1. Introduction

The goal of this work is to compare a variety of methods for the evaluation of a definite integral, all being variations of the so-called `method of negative dimension`, with an alternative presented in [5] under the name of `method of brackets`.

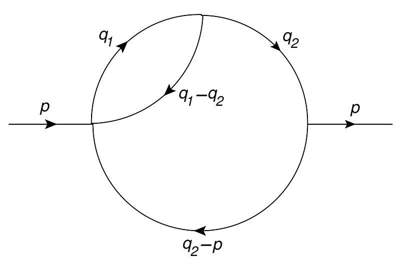

Figure 1. The Feynman diagram for the flying saucer

These methods are compared in terms of efficiency of a single integral, the one associated to the Feynman diagram of the `flying saucer`, shown in Figure 1.

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Feynman studied the interaction among elementary particles via his now famous diagrams. The literature on this subject is immense. The reader will find [10, 11, 13, 18] good introductions to several parts of this topic.

From our point of view, a `Feynman diagram` is a graph with a collection of vertices $\mathcal{V}$ and two types of oriented edges: the first type is a semi-infinite edge ending at a single vertex; the second type are finite edges connecting two vertices. These two vertices need not be distinct, so loops are allowed. The semi-infinite edges are also known as `external edges` and the other type are `internal edges`.

The example shown in Figure 1 has 3 vertices, 2 external edges and 4 internal ones. The edges are oriented with a label assigned to them. The initial assignments of the six edges is modified using the condition that, at each vertex, the sum of the assignments is zero. This gives a new total of three parameters p, q_1, q_2.

The Feynman diagram represents the integral of the product of the propagators over all $\mathbb{R}^4$, where the propagators are the reciprocals of the squares of the momenta that are labeled on the internal edges of the diagram. The details may be found in [16]. The integral becomes a function of the external parameter p and the remaining propagators parameters q_1, q_2 run over all $\mathbb{R}^4$. The corresponding integral is

$$J = \int d^4q_1 d^4q_2 \frac{1}{q_1^2(q_2-p)^2q_2^2(q_1-q_2)^2}. \tag{1.1}$$

The following notation is used throughout: for a 4-vector $q = q^\mu = (q^1, q^2, q^3, q^4) \in \mathbb{R}^4$, the symbol q^2 is the euclidean norm $(q^1)^2 + (q^2)^2 + (q^3)^2 + (q^4)^2$.

In order to illustrate the methods of integration discussed here, it is convenient to extend the dimension of the variables q_j from 4 to a complex-valued parameter D. A second generalization is to introduce powers ν_j on the propagators. This yields the integral

$$I_{fs}(\vec{\nu}, D; p) = \int d^Dq_1 d^Dq_2 \frac{1}{(q_1^2)^{\nu_1}[(q_2-p)^2]^{\nu_2}(q_2^2)^{\nu_3}[(q_1-q_2)^2]^{\nu_4}}. \tag{1.2}$$

`Notation`. A variable with a multi-index notation, refers to the sum of the variable with each individual summand as an index. Examples include

$$\begin{aligned} n_{123} &= n_1 + n_2 + n_3 \\ \nu_{123} &= \nu_1 + \nu_2 + \nu_3. \end{aligned} \tag{1.3}$$

All D-dimensional integrals over q_j variables are calculated over $\mathbb{R}^D$. All integrals over the x-parameters are calculated in one dimension, over the positive real numbers. All contour integrals are calculated over a small circle containing the origin.

2. The computation of gaussian integrals

A classical statement of elementary integration theory is that the function $\exp(-x^2)$ does not admit a primitive in the class of elementary function. In spite of this, the complete integral can be evaluated as

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \Gamma(\tfrac{1}{2}), \tag{2.1}$$

where $\Gamma(1/2) = \sqrt{\pi}$. Details on this classical formula appear in [4, Chapter 8]. A simple scaling, with $A > 0$, gives

$$\int_{-\infty}^{\infty} e^{-Ax^2}\, dx = \frac{\Gamma(\frac{1}{2})}{A^{1/2}}. \tag{2.2}$$

This result is now used to establish the formula

$$\int_{\mathbb{R}^N} \exp\left(-A_1x_1^2 - \cdots - A_Nx_N^2\right)\, dx_1 \cdots dx_N = \frac{\Gamma\left(\frac{1}{2}\right)^N}{[A_1 \cdots A_N]^{1/2}}, \tag{2.3}$$

simply from the observation that

$$\exp\left(-\sum_{j=1}^{N} A_jx_j^2\right) = \prod_{j=1}^{N} \exp(-A_jx_j^2). \tag{2.4}$$

In the special case $A_i = A$ for $1 \leqslant i \leqslant N$ this yields

$$\int_{\mathbb{R}^N} \exp(-A(x_1^2 + \cdots + x_N^2))dx_1 \ldots dx_N = \frac{\Gamma(\frac{1}{2})^N}{(A^N)^{1/2}}, \tag{2.5}$$

which may be written as

$$\int_{\mathbb{R}^N} \exp(-Ax^2)d^Nx = \frac{\Gamma(\frac{1}{2})^N}{A^{N/2}}, \tag{2.6}$$

using the notation $x^2 = x_1^2 + \cdots + x_N^2$. The identity (2.6) is generalized to the situation where the integrand contains a quadratic form by the formula

$$\int_{\mathbb{R}^N} \exp(-x^tAx)\, d^Nx = \frac{\Gamma(\frac{1}{2})^N}{|\det(A)|^{1/2}}. \tag{2.7}$$

At this point, one makes an analytic continuation of the parameter D, initially in $\mathbb{N}$, to a complex-valued parameter D. Then (2.6) is written as

$$\int_{\mathbb{R}^D} \exp(-Ax^2)d^Dx = \frac{\Gamma(\frac{1}{2})^D}{A^{D/2}}. \tag{2.8}$$

3. The Schwinger parametrization

The result presented next allows to convert powers of a real variable into an integral containing exponentials. This Schwinger parametrization appears in the literature in several forms.

The parametrization used in [1, 2, 15, 16] is given first.

Lemma 3.1. *For $A > 0$ and $\nu \in \mathbb{R}$,*

$$\frac{1}{A^\nu} = \frac{1}{\Gamma(\nu)} \int_{\mathbb{R}^+} x^{\nu-1} \exp(-xA)\, dx. \tag{3.1}$$

Proof. Start with the integral representation of the gamma function

$$\Gamma(\nu) = \int_0^\infty x^{\nu-1}e^{-x}\, dx \tag{3.2}$$

and scale to produce

$$\frac{1}{A^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty x^{\nu-1} e^{-Ax}\, dx, \text{ for } A > 0. \tag{3.3}$$

□

The parametrization used in [9, 12] is given next.

Lemma 3.2. *For* A, $\nu \in \mathbb{R}$,

$$\frac{1}{A^\nu} = \frac{(-1)^\nu \Gamma(1-\nu)}{2\pi i} \oint x^{\nu-1} \exp(-Ax)\, dx. \tag{3.4}$$

Proof. Start with the representation of the gamma function using a Hankel type contour

$$\Gamma(\nu) = -\frac{1}{2i \sin(\pi\nu)} \oint (-x)^{\nu-1} e^{-x}\, dx. \tag{3.5}$$

Now use the identity

$$\Gamma(\nu)\Gamma(1-\nu) = \frac{\pi}{\sin \pi\nu} \tag{3.6}$$

and scale the integrand by $x \mapsto Ax$ to obtain the result. □

The reader will find in [3] information for the type of contour used above. Observe that, changing A to $-A$, gives the equivalent form

$$\frac{1}{A^\nu} = \frac{\Gamma(1-\nu)}{2\pi i} \oint x^{\nu-1} \exp(Ax)\, dx. \tag{3.7}$$

This is used in Section 6.

4. Integration of the internal variables

The Schwinger parametrization converts the integral in (1.2) into

$$\begin{aligned} I_{fs}(\vec{\nu}, D; p) = & \prod_{j=1}^4 (-1)^{\nu_j} \Gamma(1-\nu_j) \frac{1}{(2\pi i)^4} \oint \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3} \frac{dx_4}{x_4} x_1^{\nu_1} x_2^{\nu_2} x_3^{\nu_3} x_4^{\nu_4} \\ & \times \int_{\mathbb{R}^{2D}} d^D q_1 d^D q_2 \exp\left([x_1 q_1^2 + x_2(q_2-p)^2 + x_3 q_2^2 + x_4(q_1-q_2)^2] \right). \end{aligned} \tag{4.1}$$

The goal of this section is to describe a procedure to evaluate the integrals in the q_j variables, corresponding to the momenta on the loops. These will be referred as the `internal variables`. This computation uses the result in (2.8).

`Step 1`. Compute the integral in q_1. Completing the square leads to

(4.2)

$$\begin{aligned} x_1 q_1^2 + x_2(q_2-p)^2 + x_3 q_2^2 + x_4(q_1-q_2)^2 &= (x_1+x_4)\left(q_1 - \frac{x_4 q_2}{x_1+x_4} \right)^2 + V \\ &= x_{14} \left(q_1 - \frac{x_4 q_2}{x_{14}} \right)^2 + V \end{aligned}$$

with

$$V = -\frac{x_4^2 q_2^2}{x_1+x_4} + x_2(q_2-p)^2 + (x_3+x_4)q_2^2 \tag{4.3}$$

$$= \left[-\frac{x_4^2}{x_1+x_4} + x_2 + x_3 + x_4 \right] q_2^2 - 2x_2 p\cdot q_2 + x_2p^2.$$

$$= \left[-\frac{x_4^2}{x_{14}} + x_{234} \right] q_2^2 - 2x_2 p\cdot q_2 + x_2p^2.$$

(4.4)

Then (2.8) gives

$$\int_{\mathbb{R}^D} d^D q_1 \exp\left([x_1q_1^2 + x_2(q_2-p)^2 + x_3q_2^2 + x_4(q_1-q_2)^2] \right) = \frac{\Gamma\left(\frac{1}{2}\right)^D}{(x_1+x_4)^{D/2}} \times e^V. \tag{4.5}$$

`Step 2`. Now integrate with respect to q_2. As before, complete the square and write

$$V = -W\left(q_2 + \frac{x_2p}{W} \right)^2 + T \tag{4.6}$$

with

$$W = \frac{x_4^2}{x_1+x_4} - x_2 - x_3 - x_4 \quad \text{and} \quad T = \frac{x_2^2p^2}{W} + x_2p^2. \tag{4.7}$$

This gives

$$\int_{\mathbb{R}^{2D}} \exp\left([x_1q_1^2 + x_2(q_2-p)^2 + x_3q_2^2 + x_4(q_1-q_2)^2] \right) dq_1\, dq_2 = \tag{4.8}$$

$$\frac{\Gamma\left(\frac{1}{2}\right)^{2D}}{(x_1+x_4)^{D/2}(\frac{x_4^2}{x_1+x_4} - x_2 - x_3 - x_4)^{D/2}} \exp\left(\frac{x_2^2p^2}{\frac{x_4^2}{x_1+x_4} - x_2 - x_3 - x_4} + x_2p^2 \right).$$

With the notation introduced in (4.7), the expression above can be written as

$$\frac{\Gamma\left(\frac{1}{2}\right)^{2D}}{((x_1+x_4)W)^{D/2}} e^T = \frac{\Gamma\left(\frac{1}{2}\right)^{2D}}{(-x_4(x_2+x_3) - x_1(x_2+x_3+x_4))^{D/2}} e^T \tag{4.9}$$

Introduce the notation

$$F = (x_1x_2x_3 + x_1x_2x_4 + x_2x_3x_4)p^2 \tag{4.10}$$

$$U = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_4 + x_3x_4.$$

Then (4.8) is written as

(4.11)

$$\int_{\mathbb{R}^{2D}} \exp\left([x_1q_1^2 + x_2(q_2-p)^2 + x_3q_2^2 + x_4(q_1-q_2)^2] \right) dq_1\, dq_2 = \frac{\Gamma\left(\frac{1}{2}\right)^{2D}}{(-U)^{D/2}} \exp\left(\frac{F}{U} \right).$$

Definition 4.1. The polynomials F and U are called the `Symanzik polynomials` corresponding to the diagram appearing in Figure 1.

The final product of this section is that the integral in (1.2) has been reduced to:

$$I_{fs}(\vec{\nu}, D; p) = \prod_{j=1}^{4} (-1)^{\nu_j} \Gamma(1-\nu_j) \frac{1}{(2\pi i)^4} \oint \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3} \frac{dx_4}{x_4} x_1^{\nu_1} x_2^{\nu_2} x_3^{\nu_3} x_4^{\nu_4} \times \frac{\Gamma\left(\frac{1}{2}\right)^{2D}}{(-1)^{D/2}} \frac{1}{U^{D/2}} \exp\left(\frac{F}{U}\right), \tag{4.12}$$

in terms of the Symanzik polynomials F and U.

Note 4.2. The next five sections present different methods in the literature to evaluate integrals associated to Feynman diagrams. The methods are used to evaluate the flying saucer integral.

5. Ricotta method without parametrization

This section presents the evaluation of

$$I_{fs}(\vec{\nu}, D; p) = \int d^D q_1 \, d^D q_2 \frac{1}{(q_1^2)^{\nu_1} [(q_2-p)^2]^{\nu_2} (q_2^2)^{\nu_3} [(q_1-q_2)^2]^{\nu_4}}, \tag{5.1}$$

where $\int$ is an abbreviation of $\int_{\mathbb{R}^D} \int_{\mathbb{R}^D}$. The method discussed here appears in [12]. In her thesis, the author discusses two approaches to this evaluation. In the first one, discussed in this Section, there are some gaussian integrals appearing as intermediate steps are evaluated directly. This method is referred to as `Ricotta method without parametrization`. In the second approach, the author uses the Schwinger parameterization as an intermediate step. This is discussed in Section 6 and is called `Ricotta method with parametrization`.

`Step 1`. The binomial expressions $[(q_2-p)^2]^{-\nu_2}$ and $[(q_1-q_2)^2]^{-\nu_4}$ are expanded using the multinomial theorem. For n, $m \in \mathbb{N}$, recall the classical expansion

$$(y_1 + \cdots + y_m)^n = \sum_{|\alpha|=n} \binom{n}{\alpha} y^\alpha \tag{5.2}$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$, $y^\alpha = y_1^{\alpha_1} \cdots y_m^{\alpha_m}$, the sum runs over all multi-indices α with $|\alpha| = \alpha_1 + \cdots + \alpha_m = n$ and the binomial coefficient given by

$$\binom{n}{\alpha} = \frac{n!}{\alpha_1! \cdots \alpha_m!}. \tag{5.3}$$

In the case with three summands[1] and real exponent, the expansion (5.2) becomes

$$(y_1 + y_2 + y_3)^{-\nu} = \Gamma(1-\nu) \sum_{\substack{n_1, n_2, n_3 \\ n_1+n_2+n_3=-\nu}} \frac{y_1^{n_1}}{n_1!} \frac{y_2^{n_2}}{n_2!} \frac{y_3^{n_3}}{n_3!} \tag{5.4}$$

[1]This is the case appearing in the calculations discussed here.

The sum can be written without restrictions on the indices by introducing the Kronecker delta

$$\delta_{x,y} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases} \tag{5.5}$$

and then (5.4) becomes

$$(y_1 + y_2 + y_3)^{-\nu} = \Gamma(1-\nu) \sum_{n_1,n_2,n_3} \frac{y_1^{n_1}}{n_1!} \frac{y_2^{n_2}}{n_2!} \frac{y_3^{n_3}}{n_3!} \times \delta_{n_1+n_2+n_3,-\nu}. \tag{5.6}$$

In the expression (5.1) use the expansions

(5.7)

$$\begin{aligned} ((q_2-p)^2)^{-\nu_2} &= (q_2^2 - 2q_2\cdot p + p^2)^{-\nu_2} \\ &= \Gamma(1-\nu_2) \sum_{n_1,n_2,n_3} (q_2^2)^{n_1} (2p\cdot q_2)^{n_2} (p^2)^{n_3} \frac{(-1)^{n_2}}{n_1!n_2!n_3!} \delta_{n_1+n_2+n_3,-\nu_2} \\ ((q_1-q_2)^2)^{-\nu_4} &= (q_1^2 - 2q_1\cdot q_2 + q_2^2)^{-\nu_4} \\ &= \Gamma(1-\nu_4) \sum_{n_4,n_5,n_6} (q_1^2)^{n_4} (2q_1\cdot q_2)^{n_5} (q_2^2)^{n_6} \frac{(-1)^{n_5}}{n_4!n_5!n_6!} \delta_{n_4+n_5+n_6,-\nu_4} \end{aligned}$$

in (5.1) to obtain

(5.8)

$$\begin{aligned} I_{fs}(\vec{\nu}, D; p) = \Gamma(1-\nu_2)\Gamma(1-\nu_4) &\sum_{n_1,\ldots,n_6} \frac{(-1)^{n_2+n_5}}{n_1!n_2!n_3!n_4!n_5!n_6!} (p^2)^{n_3} \delta_{n_1+n_2+n_3,-\nu_2} \delta_{n_4+n_5+n_6,-\nu_4} \\ &\times \int d^Dq_1 d^Dq_2\, (q_1^2)^{-\nu_1+n_4} (q_2^2)^{-\nu_3+n_1+n_6} (2p\cdot q_2)^{n_2} (2q_1\cdot q_2)^{n_5}. \end{aligned}$$

`Step 2`. Derive first an identity for general integrals of the type

$$\int d^Dq_1 d^Dq_2 (q_1^2)^{k_1} (q_2^2)^{k_2} (2p\cdot q_1)^{k_3} (2p\cdot q_2)^{k_4} (2q_1\cdot q_2)^{k_5} \tag{5.9}$$

by using the associated gaussian

(5.10)

$$E = \int d^Dq_1 d^Dq_2 \exp\left(-\alpha_1 q_1^2 - \alpha_2 q_2^2 + \alpha_3(2p\cdot q_1) + \alpha_4(2p\cdot q_2) + \alpha_5(2q_1\cdot q_2)\right)$$

with α_i, $1 \leqslant i \leqslant 5$ are arbitrary parameters. The evaluation of (5.8) will employ this identity.

Start by evaluating E using the method of completing squares detailed in Section 3. The result is

$$E = \pi^D (\alpha_1\alpha_2 - \alpha_5^2)^{-D/2} \exp\left[p^2 \left(\frac{\alpha_1\alpha_4^2 + \alpha_2\alpha_3^2 + 2\alpha_3\alpha_4\alpha_5}{\alpha_1\alpha_2 - \alpha_5^2}\right)\right]. \tag{5.11}$$

Now expand the equations (5.10) and (5.11) in powers of the parameters α_i and match corresponding coefficients to produce the identity

$$
\begin{aligned}
(5.12)\quad & \int d^D q_1 d^D q_2 (q_1^2)^{k_1} (q_2^2)^{k_2} (2p\cdot q_1)^{k_3} (2p\cdot q_2)^{k_4} (2q_1\cdot q_2)^{k_5} \\
&= \sum_{m_1,\ldots,m_5} \frac{\pi^D (-1)^{k_1+k_2} 2^{m_3} k_1! k_2! k_3! k_4! k_5! \Gamma(1+m_4+m_5)(p_2)^{m_1+m_2+m_3}}{m_1! m_2! m_3! m_4! m_5!} \times \\
& \delta_{m_1+m_2+m_3+m_4+m_5,-\frac{1}{2}D} \delta_{k_1,m_1+m_4} \delta_{k_2,m_2+m_4} \delta_{k_3,2m_2+m_3} \delta_{k_4,2m_1+m_3} \delta_{k_5,m_3+2m_5}.
\end{aligned}
$$

The constraints given by the Kronecker deltas lead to the system of constraints

$$
\begin{aligned}
m_1 + m_4 &= k_1 \\
m_2 + m_4 &= k_2 \\
2m_2 + m_3 &= k_3 \\
2m_1 + m_3 &= k_4 \\
m_3 + 2m_5 &= k_5 \\
m_1 + m_2 + m_3 + m_4 + m_5 &= -\frac{D}{2}.
\end{aligned}
\tag{5.13}
$$

The corresponding matrix is of rank 4 and the system is consistent only if the conditions

$$
2k_2 + k_4 + k_5 = -D \quad \text{and} \quad 2k_1 + k_3 + k_5 = -D \tag{5.14}
$$

are satisfied. These constraints will be imposed by including the Kronecker deltas $\delta_{2k_2+k_4+k_5,-D}$ and $\delta_{2k_1+k_3+k_5,-D}$ in the series giving the value of the integral I. Since the rank is 4 there is a single choice for the free index m_j and the value of the integral is independent of which choice of index is made.

Choose m_3 as the free index, then impose the condition (5.14) and solve the system to obtain

$$
\begin{aligned}
m_1 &= \tfrac{1}{2}(-m_3 + k_4) \\
m_2 &= \tfrac{1}{2}(-m_3 + k_3), \\
m_4 &= \tfrac{1}{2}(m_3 + 2k_2 - k_3) \\
m_5 &= \tfrac{1}{2}(-m_3 + k_5).
\end{aligned}
\tag{5.15}
$$

Note that the Kronecker deltas $\delta_{k_1,m_1+m_4}\delta_{k_2,m_2+m_4}\delta_{k_3,2m_2+m_3}\delta_{k_4,2m_1+m_3}\delta_{k_5,m_3+2m_5}$ appearing in (5.12) imply that the solutions m_j, $j \neq 3$, are integers. For instance, the term $\delta_{k_3,2m_2+m_3}$ imposes $k_3 = 2m_2 + m_3$, so that $k_3 \equiv m_3 \bmod 2$ and this shows that m_2 is an integer. The sign condition on these solutions are discussed below.

With this information (5.12) becomes, with m_3 written simply as m,

$$
\int d^Dq_1d^Dq_2(q_1^2)^{k_1}(q_2^2)^{k_2}(2p\cdot q_1)^{k_3}(2p\cdot q_2)^{k_4}(2q_1\cdot q_2)^{k_5} \tag{5.16}
$$

$$
\sum_m \frac{\pi^D(-1)^{-k_1-k_2}2^m k_1!k_2!k_3!k_4!k_5!\Gamma(1+k_2-\frac{1}{2}k_3+\frac{1}{2}k_5)(p^2)^{\frac{1}{2}(k_3+k_4)}}{\Gamma(1-\frac{1}{2}m+\frac{1}{2}k_3)\Gamma(1-\frac{1}{2}m+\frac{1}{2}k_4)\Gamma(1-\frac{1}{2}m+\frac{1}{2}k_5)} \times \frac{\delta_{2k_2+k_4+k_5,-D}\,\delta_{2k_1+k_3+k_5,-D}}{\Gamma(1+\frac{1}{2}m+k_2-\frac{1}{2}k_3)\Gamma(1+m)}
$$

In the previous sum one has to impose restrictions of the index m so that the variables m_1, m_2, m_4, m_5 are non-negative. This requires m to range from $\max\{0,-2k_2+k_3\}$ to $\min\{k_3,k_4,k_5\}$.

`Step 3` is to use (5.16) to evaluate (5.1)

$$
I_{fs}(\vec{\nu},D;p) = \int d^Dq_1\,d^Dq_2\frac{1}{(q_1^2)^{\nu_1}[(q_2-p)^2]^{\nu_2}(q_2^2)^{\nu_3}[(q_1-q_2)^2]^{\nu_4}}, \tag{5.17}
$$

that has been written in (5.8) as

$$
I_{fs}(\vec{\nu},D;p) = \Gamma(1-\nu_2)\Gamma(1-\nu_4)\sum_{n_1,\ldots,n_6}\frac{(-1)^{n_2+n_5}}{n_1!n_2!n_3!n_4!n_5!n_6!}(p^2)^{n_3}\delta_{n_1+n_2+n_3,-\nu_2}\delta_{n_4+n_5+n_6,-\nu_4} \times\int d^Dq_1d^Dq_2\,(q_1^2)^{-\nu_1+n_4}(q_2^2)^{-\nu_3+n_1+n_6}(2p\cdot q_2)^{n_2}(2q_1\cdot q_2)^{n_5}. \tag{5.18}
$$

Comparing with (5.16) it follows that

$$
k_1=-\nu_1+n_4,\; k_2=-\nu_3+n_1+n_6,\; k_3=0,\; k_4=n_2,\; k_5=n_5 \tag{5.19}
$$

and then (5.18) becomes

$$
I_{fs}(\vec{\nu},D;p) = \sum_{n_1,\ldots,n_6}\sum_{m=\max\{0,2\nu_3-2n_{16}\}}^{\min\{0,n_2,n_5\}}\frac{\pi^D2^m(-1)^{-\nu_{13}+n_{12456}}(p^2)^{\frac{1}{2}n_2+n_3}\Gamma(1-\nu_2)}{n_1!n_3!n_4!n_6!m!\Gamma(1-\nu_3+n_{16}+\frac{1}{2}m)} \times\frac{\Gamma(1-\nu_4)\Gamma(1-\nu_1+n_4)\Gamma(1-\nu_3+n_{16})\Gamma(1-\nu_3+n_{16}+\frac{1}{2}n_5)}{\Gamma(1-\frac{m}{2})\Gamma(1+\frac{n_2}{2}-\frac{m}{2})\Gamma(1+\frac{n_5}{2}-\frac{m}{2})} \times\delta_{n_{123},-\nu_2}\delta_{n_{456},-\nu_4}\delta_{2n_{16}+n_{25},2\nu_3-D}\delta_{2n_4+n_5,2\nu_1-D}. \tag{5.20}
$$

Now observe that n_2, $n_5\in\mathbb{N}$, so that $\min\{0,\,n_2,\,n_5\}=0$. Therefore the inner sum is trivial if $2\nu_3-2n_{16}>0$. The only contribution comes from the case $2\nu_3-2n_{16}\leqslant 0$ and then only $m=0$ contributes to the sum. In addition, the Kronecker deltas appearing in the derivation of the identity require that $\frac{n_2}{2}\pm\frac{m}{2}$ and $\frac{n_5}{2}\pm\frac{m}{2}$ to be integers. The conclusion is that n_2, n_5 and m must be of the same parity. Since the only contribution is from $m=0$, it follows that n_2 and n_5 must be even. Replacing

(n_2, n_5) by $(\frac{n_2}{2}, \frac{n_5}{2})$ transforms the sum to one over all integers, not just even ones. This is now written as

(5.21)
$$I_{fs}(\vec{\nu}, D; p) = \sum_{n_1 \ldots n_6 *} \frac{\pi^D (p^2)^{n_{23}} \Gamma(1-\nu_2)\Gamma(1-\nu_4)\Gamma(1-\nu_1+n_4)\Gamma(1-\nu_3+n_{156})}{n_1! n_2! n_3! n_4! n_5! n_6!} \times (-1)^{-\nu_{1234}-n_3} \delta_{n_{13}+2n_2, -\nu_2} \delta_{n_{46}+2n_5, -\nu_4} \delta_{n_{1256}, \nu_3 - \frac{D}{2}} \delta_{n_{45}, \nu_1 - \frac{D}{2}}$$

where the star in the sum indicates that the constraint $\nu_3 - n_{16} \leqslant 0$ must be satisfied.

`Step 4` is to use the constraint on the indices to write the sum in (5.21) with a minimal number of them.

The constraints in (5.21) are

(5.22)
$$\begin{aligned} n_1 + 2n_2 + n_3 &= -\nu_2 \\ n_4 + 2n_5 + n_6 &= -\nu_4 \\ n_1 + n_2 + n_5 + n_6 &= -\tfrac{D}{2} + \nu_3 \\ n_4 + n_5 &= -\tfrac{D}{2} + \nu_1. \end{aligned}$$

There are four constraints and six indices, therefore there will be two free indices and at most $\binom{6}{2} = 15$ solutions. The first two equations in (5.22) show that the sets $\{n_1, n_2, n_3\}$ and $\{n_4, n_5, n_6\}$ must contain a free index each. This lowers the total number of solutions to at most 9. One of these examples is completely discussed below, the remaining eight are evaluated in a similar form, so the details are omitted.

`Case 1 :` n_1 `and` n_4 `as free indices`. Solving the system of constraints (5.22) in terms of the free parameters gives

(5.23)
$$\begin{aligned} n_2 &= -D + \nu_{134} - n_1 \\ n_3 &= 2D - 2\nu_{134} - \nu_2 + n_1 \\ n_5 &= -\tfrac{D}{2} + \nu_1 - n_4 \\ n_6 &= D - 2\nu_1 - \nu_4 + n_4. \end{aligned}$$

Replacing in (5.21) gives

(5.24)
$$S_{14} = \sum_{n_1, n_4 *} \frac{\pi^D (-1)^{2D+\nu_{134}+n_1} (p^2)^{D-\nu_{1234}} \Gamma(1-\nu_2)\Gamma(1-\nu_4)}{n_1! n_4! \Gamma(1+2D-2\nu_{134}-\nu_2+n_1)\Gamma(1+D-2\nu_1-\nu_4+n_4)} \times \frac{\Gamma(1-\nu_1+n_4)\Gamma(1+\frac{D}{2}-\nu_{134}+n_1)}{\Gamma(1-D+\nu_{134}-n_1)\Gamma(1-\frac{D}{2}+\nu_1-n_4)},$$

where the star in the sum indicates that the constrain $n_1 + n_4 \geqslant -D + 2\nu_1 + \nu_3 + \nu_4$.

The expression for S_{14} given above depends on the parameters D, ν_1, ν_2, ν_3 and ν_4, originally assumed integers.

In order to write S_{14} in more compact form, introduce the notation

(5.25) $\rho = -D + 2\nu_1 + \nu_3 + \nu_4,\ \gamma_1 = 1 - \nu_1,\ \gamma_2 = 1 + \frac{D}{2} - \nu_{134},$
$\gamma_3 = 1 + 2D - 2\nu_{134} - \nu_2,\ \gamma_4 = 1 + D - 2\nu_1 - \nu_4,\ \gamma_5 = 1 - D + \nu_{134},\ \gamma_6 = 1 - \frac{D}{2} + \nu_1$

to obtain

$$S_{14} = \pi^D (-1)^{2D+\nu_{134}} (p^2)^{D-\nu_{1234}} \Gamma(1-\nu_2)\Gamma(1-\nu_4) \sum_{n_1=0}^{\infty} \sum_{n_4 \geqslant \rho - n_1} \frac{(-1)^{n_1} \Gamma(\gamma_1 + n_4)\Gamma(\gamma_2 + n_1)}{n_1! n_4! \Gamma(\gamma_3 + n_1)\Gamma(\gamma_4 + n_4)\Gamma(\gamma_5 - n_1)\Gamma(\gamma_6 - n_4)}. \tag{5.26}$$

Note 5.1. In the special case $\rho < 0$, the lower limit of summation in the index n_4 can be reduced to 0. Then the double series simplifies and one obtains

$$S_{14} = \pi^D (-1)^{2D+\nu_{134}} (p^2)^{D-\nu_{1234}} \Gamma(1-\nu_2)\Gamma(1-\nu_4) \times \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}\Gamma(\gamma_2 + n_1)}{n_1!\Gamma(\gamma_3 + n_1)\Gamma(\gamma_5 - n_1)} \sum_{n_4=0}^{\infty} \frac{\Gamma(\gamma_1 + n_4)}{n_4!\Gamma(\gamma_4 + n_4)\Gamma(\gamma_6 - n_4)}. \tag{5.27}$$

Simplify these expressions using

$$\Gamma(a+n) = \Gamma(a)(a)_n \quad \text{and} \quad \Gamma(a-n) = \frac{(-1)^n \Gamma(a)}{(1-a)_n}, \tag{5.28}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol. The first series $S_{14}^{(1)}$ in (5.27) becomes

$$\begin{aligned} S_{14}^{(1)} &= \frac{\Gamma(\gamma_2)}{\Gamma(\gamma_3)\Gamma(\gamma_5)} \sum_{n_1=0}^{\infty} \frac{(\gamma_2)_{n_1} (1-\gamma_5)_{n_1}}{n_1!(\gamma_3)_{n_1}} \\ &= \frac{\Gamma(\gamma_2)}{\Gamma(\gamma_3)\Gamma(\gamma_5)} {}_2F_1\left(\begin{matrix} \gamma_2 & 1-\gamma_5 \\ & \gamma_3 \end{matrix} \middle| 1 \right). \end{aligned} \tag{5.29}$$

Similarly, the second series $S_{14}^{(2)}$ is

$$S_{14}^{(2)} = \frac{\Gamma(\gamma_1)}{\Gamma(\gamma_4)\Gamma(\gamma_6)} {}_2F_1\left(\begin{matrix} \gamma_1 & 1-\gamma_6 \\ & \gamma_4 \end{matrix} \middle| -1 \right). \tag{5.30}$$

Therefore

$$S_{14} = \pi^D (-1)^{2D+\nu_{134}} (p^2)^{D-\nu_{1234}} \Gamma(1-\nu_2)\Gamma(1-\nu_4) \times \frac{\Gamma(\gamma_2)}{\Gamma(\gamma_3)\Gamma(\gamma_5)} \frac{\Gamma(\gamma_1)}{\Gamma(\gamma_4)\Gamma(\gamma_6)} {}_2F_1\left(\begin{matrix} \gamma_2 & 1-\gamma_5 \\ & \gamma_3 \end{matrix} \middle| 1 \right) {}_2F_1\left(\begin{matrix} \gamma_1 & 1-\gamma_6 \\ & \gamma_4 \end{matrix} \middle| -1 \right). \tag{5.31}$$

In summary: the `Ricotta method without parametrization` gives 9 solutions of the type given in (5.26). Under certain restriction on the parameter ρ, it is possible that the terms in these sums could be products of two hypergeometric ${}_2F_1$-functions. However, the results may not always allow simplification.

6. Ricotta method with parametrization

This section presents a method to evaluate the integral

$$I_{fs}(\vec{\nu}, D; p) = \int d^D q_1 d^D q_2 \frac{1}{(q_1^2)^{\nu_1}[(q_2-p)^2]^{\nu_2}(q_2^2)^{\nu_3}[(q_1-q_2)^2]^{\nu_4}}, \tag{6.1}$$

which uses the Schwinger parametrization in the alternate form given in Lemma 3.2 as an initial step.

`Step 1`. Use the Schwinger parametrization

$$\frac{1}{A^\nu} = \frac{\Gamma(1-\nu)}{2\pi i} \oint x^\nu \exp(xA)\, \frac{dx}{x} \tag{6.2}$$

to rewrite (6.1) as

$$\begin{aligned} I_{fs}(\vec{\nu}, D; p) = {} & \frac{1}{(2\pi i)^4} \prod_{j=1}^{4} \Gamma(1-\nu_j) \oint \frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{dx_3}{x_3}\frac{dx_4}{x_4} x_1^{\nu_1}x_2^{\nu_2}x_3^{\nu_3}x_4^{\nu_4} \\ & \times \int d^D q_1 d^D q_2 \exp\left[x_1 q_1^2 + x_2(q_2-p)^2 + x_3 q_2^2 + x_4(q_1-q_2)^2\right]. \end{aligned} \tag{6.3}$$

`Step 2`. Evaluate the gaussian integral using the formula, discussed in Section 2,

$$\int d^D q_1 \ldots d^D q_L \exp\left(-\sum_{i=1}^{N} x_i A_i\right) = \frac{(-1)^{DL/2}\pi^{DL/2}}{U^{D/2}} \exp\left(\frac{F}{U}\right), \tag{6.4}$$

where F, U are determined in Section 3. The result is

$$\begin{aligned} I_{fs}(\vec{\nu}, D; p) = {} & \frac{1}{(2\pi i)^4} \prod_{j=1}^{4} \Gamma(1-\nu_j) \oint \frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{dx_3}{x_3}\frac{dx_4}{x_4} x_1^{\nu_1}x_2^{\nu_2}x_3^{\nu_3}x_4^{\nu_4} \\ & \times \frac{(-1)^D \pi^D}{U^{D/2}} \exp\left(\frac{F}{U}\right). \end{aligned} \tag{6.5}$$

`Step 3a`. Expand the exponentials in powers of x_j and U to obtain

$$\begin{aligned} I_{fs}(\vec{\nu}, D; p) = {} & \frac{\Gamma(1-\nu_1)\Gamma(1-\nu_2)\Gamma(1-\nu_3)\Gamma(1-\nu_4)}{(2\pi i)^4} \oint \frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{dx_3}{x_3}\frac{dx_4}{x_4} \\ & \times \sum_{n_1,n_2,n_3} \frac{\pi^D (-1)^D (p^2)^{n_{123}}}{n_1! n_2! n_3!} x_1^{n_{12}+\nu_1} x_2^{n_{123}+\nu_2} x_3^{n_{13}+\nu_3} x_4^{n_{23}+\nu_4} U^{-D/2-n_{123}}. \end{aligned} \tag{6.6}$$

`Step` 3b. Expand the multinomial U to express the integral $I_{fs}(\vec{\nu}, D; p)$ in terms of the parameters x_j. This produces

(6.7)
$$I_{fs}(\vec{\nu}, D; p) = \sum_{n_1 \cdots n_8} \frac{\pi^D (-1)^D (p^2)^{n_{123}} \Gamma(1+n_{45678}) \Gamma(1-\nu_1)\Gamma(1-\nu_2)\Gamma(1-\nu_3)\Gamma(1-\nu_4)}{n_1! n_2! n_3! n_4! n_5! n_6! n_7! n_8!}$$
$$\times \frac{1}{2\pi i} \oint \frac{dx_1}{x_1} x^{\nu_1 + n_{12456}} \frac{1}{2\pi i} \oint \frac{dx_2}{x_2} x^{\nu_2 + n_{12347}} \frac{1}{2\pi i} \oint \frac{dx_3}{x_3} x_3^{\nu_3 + n_{1358}}$$
$$\times \frac{1}{2\pi i} \oint \frac{dx_4}{x_4} x_4^{\nu_4 + n_{23678}} \delta_{n_{12345678}, -D/2}.$$

`Step` 4. Use the identity

$$\frac{1}{2\pi i} \oint \frac{dx}{x} x^n = \delta_{n,0} \tag{6.8}$$

to evaluate the integrals involving x_j. This gives

(6.9)
$$I_{fs}(\vec{\nu}, D; p) = \sum_{n_1 \cdots n_8} \frac{\pi^D (-1)^D (p^2)^{n_{123}} \Gamma(1+n_{45678}) \Gamma(1-\nu_1)\Gamma(1-\nu_2)\Gamma(1-\nu_3)\Gamma(1-\nu_4)}{n_1! n_2! n_3! n_4! n_5! n_6! n_7! n_8!}$$
$$\times \delta_{n_{12456}, -\nu_1} \delta_{n_{12347}, -\nu_2} \delta_{n_{1358}, -\nu_3} \delta_{n_{23678}, -\nu_4} \delta_{n_{12345678}, -D/2}.$$

`Step` 5 is to solve the system of constraints to reduce the numbers of indices. This system is

$$\begin{aligned} n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8 &= -\tfrac{1}{2} D \\ n_1 + n_2 + n_4 + n_5 + n_6 &= -\nu_1 \\ n_1 + n_2 + n_3 + n_4 + n_7 &= -\nu_2 \\ n_1 + n_3 + n_5 + n_8 &= -\nu_3 \\ n_2 + n_3 + n_6 + n_7 + n_8 &= -\nu_4. \end{aligned} \tag{6.10}$$

There are 5 equations with 8 indices, so there may be as many as $\binom{8}{5} = 56$ solutions. It turns out that 20 choices of a set of three free indices do not lead to a solution. For instance, choosing n_1, n_2, n_3 as the free parameters, leads to a 5×5 system of rank 4. Gaussian elimination on this system leads to the relation

$$n_1 + n_2 + n_3 = D - \nu_{1234}, \tag{6.11}$$

contradicting the fact that n_1, n_2, n_3 are free variables.

Take n_1, n_2, n_4 to be the free variables. Solving the system of equations,

$$\begin{aligned} n_3 &= D - \nu_{1234} - n_{12} \\ n_5 &= -\frac{D}{2} + \nu_4 - n_{14} \\ n_6 &= \frac{D}{2} - \nu_{14} - n_2 \\ n_7 &= -D + \nu_{134} - n_4 \\ n_8 &= D - \nu_{1234} - n_{12} \end{aligned} \tag{6.12}$$

The next step is to replace these values in (6.9) to get

$$I_{fs}(\vec{\nu}, D; p) = \sum_{n_1,n_2,n_4} \frac{\pi^D(-1)^D(p^2)^{D-\nu_{1234}}\Gamma(1-\nu_1)\Gamma(1-\nu_2)\Gamma(1-\nu_3)\Gamma(1-\nu_4)}{n_1!n_2!n_4!\Gamma(1+D-\nu_{1234}-n_{12})\Gamma(1-D+\nu_{134}-n_4)} \times \frac{\Gamma\left(1-\frac{3D}{2}+\nu_{1234}\right)}{\Gamma\left(1-\frac{D}{2}+\nu_4-n_{14}\right)\Gamma\left(1+\frac{D}{2}-\nu_{14}-n_2\right)\Gamma\left(1-\frac{D}{2}+\nu_{12}+n_{124}\right)}. \tag{6.13}$$

This solution may be written as a product of three ${}_2F_1$ hypergeometric functions. Now use Gauss's formula for the ${}_2F_1$ at $z = 1$

$${}_2F_1\left(\begin{matrix} a \ \ b \\ c \end{matrix}\middle| 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{6.14}$$

to arrive at the solution

$$I_{fs}(\vec{\nu}, D; p) = \frac{\pi^D(-1)^D(p^2)^{D-\nu_{1234}}\Gamma(1-\nu_1)\Gamma(1-\nu_2)\Gamma(1-\nu_4)}{\Gamma(1+D-\nu_{1234})\Gamma\left(1-\frac{D}{2}+\nu_4\right)\Gamma\left(1+\frac{D}{2}-\nu_{14}\right)} \times \frac{\Gamma\left(1-\frac{3D}{2}+\nu_{1234}\right)\Gamma\left(1+\frac{D}{2}-\nu_{134}\right)\Gamma(1-D+\nu_{14})}{\Gamma(1-D+\nu_{134})\Gamma\left(1-\frac{D}{2}+\nu_2\right)\Gamma\left(1-\frac{D}{2}+\nu_1\right)}. \tag{6.15}$$

Each of the 36 sets of free indices lead to the same solution when hypergeometric identities are used, so there is a 36-fold degeneracy in the solutions for the flying saucer diagram using the Ricotta method with Schwinger parametrization. The solutions are classified in Tables 1 and 2

n_1, n_2, n_3	n_1, n_5, n_7	n_2, n_5, n_8	n_3, n_7, n_8
n_1, n_2, n_6	n_2, n_3, n_6	n_2, n_6, n_7	n_4, n_5, n_7
n_1, n_3, n_6	n_2, n_4, n_6	n_2, n_6, n_8	n_4, n_6, n_7
n_1, n_4, n_5	n_2, n_4, n_7	n_3, n_4, n_7	n_4, n_7, n_8
n_1, n_4, n_7	n_2, n_5, n_6	n_3, n_4, n_8	n_5, n_6, n_8

TABLE 1. Sets of free indices leading to no solution

n_1, n_2, n_4	n_1, n_5, n_6	n_2, n_4, n_5	n_3, n_6, n_7
n_1, n_2, n_5	n_1, n_6, n_7	n_2, n_4, n_8	n_3, n_6, n_8
n_1, n_2, n_7	n_1, n_6, n_8	n_2, n_5, n_7	n_4, n_5, n_6
n_1, n_2, n_8	n_1, n_7, n_8	n_2, n_7, n_8	n_4, n_5, n_8
n_1, n_3, n_4	n_2, n_3, n_4	n_3, n_4, n_5	n_4, n_6, n_8
n_1, n_3, n_7	n_2, n_3, n_5	n_3, n_4, n_6	n_5, n_6, n_7
n_1, n_4, n_6	n_2, n_3, n_7	n_3, n_5, n_6	n_5, n_7, n_8
n_1, n_4, n_8	n_2, n_3, n_8	n_3, n_5, n_7	n_6, n_7, n_8
n_1, n_3, n_5	n_1, n_3, n_8	n_1, n_5, n_8	n_3, n_5, n_8

TABLE 2. Sets of free indices leading to a solution

`Step 6`: Analytic continuation to positive values of D, ν_1, ν_2, ν_3, and ν_4.

The solution contains poles coming from the gamma factors. This last step is necessary to eliminate as many them as possible. The gamma functions for which analytic continuation is necessary are determined by the values of D, ν_1, ν_2, ν_3, and ν_4. Assuming that $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 1$, one can see that at least three gamma factors will require continuation.

The formula for analytic continuation of a ratio of gamma factors, given as equation 2.33 in [1], is

$$\prod_{i=1}^{n} \frac{\Gamma(\alpha_i)}{\Gamma(\beta_i)} = (-1)^{\sum_{i=1}^{n}(\beta_i - \alpha_i)} \prod_{i=1}^{n} \frac{\Gamma(1-\beta_i)}{\Gamma(1-\alpha_i)}. \tag{6.16}$$

The gamma function has poles at 0 and at negative integers, so estimating $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 1$, one can see that at least the gamma factors $\Gamma(1-\nu_1)$, $\Gamma(1-\nu_2)$, and $\Gamma(1-\nu_4)$ will require continuation. Further estimating that $D = 4$, one can see that all gamma factors in the numerator will require continuation. These six factors must be balanced by the six gamma factors of the denominator. The result is

$$I_{fs}(\vec{\nu}, D; p) = \frac{\pi^D (p^2)^{D-\nu_{1234}} \Gamma\left(\frac{D}{2} - \nu_1\right) \Gamma\left(\frac{D}{2} - \nu_2\right) \Gamma\left(\frac{D}{2} - \nu_4\right) \Gamma\left(-\frac{D}{2} + \nu_{14}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_4)\Gamma\left(\frac{3D}{2} - \nu_{1234}\right) \Gamma\left(-\frac{D}{2} + \nu_{134}\right) \Gamma(D - \nu_{14}) \times \Gamma(-D + \nu_{1234})\Gamma(D - \nu_{134})} \tag{6.17}$$

7. The method of Anastasiou

In this method, one begins with the scaled version of the integral

$$I(\nu_1,\nu_2,\nu_3,\nu_4) = \int \frac{d^Dq_1}{i\pi^{\frac{D}{2}}} \frac{d^Dq_2}{i\pi^{\frac{D}{2}}} \frac{1}{(q_1^2)^{\nu_1}[(q_2-p)^2]^{\nu_2}(q_2^2)^{\nu_3}[(q_1-q_2)^2]^{\nu_4}}, \tag{7.1}$$

with the extra factors depending on the dimension D. The dependence on p is dropped from the notation.

`Step 1`: Use the Schwinger parametrization

$$\frac{1}{A^\nu} = \frac{1}{\Gamma(\nu)} \int_{\mathbb{R}^+} x^{\nu-1}\exp(-xA)\,dx \tag{7.2}$$

given in (3.1) and then change the order of integration. This yields

$$I(\nu_1,\nu_2,\nu_3,\nu_4) = \int \mathfrak{D}x \int \frac{d^Dq_1}{i\pi^{\frac{D}{2}}} \frac{d^Dq_2}{i\pi^{\frac{D}{2}}} \exp\left[-(x_1q_1^2 + x_2(q_2-p)^2 + x_3q_2^2 + x_4(q_1-q_2)^2)\right], \tag{7.3}$$

where

$$\int \mathfrak{D}x = \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)} \int x_1^{\nu_1}x_2^{\nu_2}x_3^{\nu_3}x_4^{\nu_4} \frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{dx_3}{x_3}\frac{dx_4}{x_4}$$

is a shorthand notation for the quantity which will appear on both sides of the equation in Step 4 below.

`Step 2`: Use the identity (2.8) to evaluate the gaussian of (7.3). Here F and U are the Symanzik polynomials determined in Section 4:

$$I(\nu_1,\nu_2,\nu_3,\nu_4) = \int \mathfrak{D}x \frac{(-1)^D}{U^{\frac{D}{2}}} \exp\left(\frac{F}{U}\right). \tag{7.4}$$

The next part of the process is to evaluate the integral $I(\nu_1,\nu_2,\nu_3,\nu_4)$ in (7.3) in two different forms and match corresponding results.

`Step 3a`: Expand the exponential of (7.4) and the resulting monomials. This is the first method of evaluation of (7.3). One obtains the relation

$$I(\nu_1,\nu_2,\nu_3,\nu_4) = \int \mathfrak{D}x \sum_{n_1\cdots n_8} \frac{(-1)^D\Gamma(1+n_{45678})(p^2)^{D-\nu_{1234}}}{n_1!n_2!n_3!n_4!n_5!n_6!n_7!n_8!} \times \delta_{n_{12345678},-\frac{D}{2}} x_1^{n_{12456}}x_2^{n_{12347}}x_3^{n_{1358}}x_4^{n_{23678}}. \tag{7.5}$$

`Step 3b`: Expand the exponentials of (7.3) in powers of the propagators and rewrite in terms of the original integral. This is the second method of evaluation.

$$I(\nu_1,\nu_2,\nu_3,\nu_4) = \int \mathfrak{D}x \sum_{m_1\cdots m_4} \frac{x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}}{m_1!m_2!m_3!m_4!} I(-m_1,-m_2,-m_3,-m_4). \tag{7.6}$$

`Step 4`: Equate the integrands of the two expressions for $I(\nu_1,\nu_2,\nu_3,\nu_4)$ obtained in Steps $3a$ and $3b$. This gives

$$\sum_{n_1\cdots n_8}\frac{(-1)^D\Gamma(1+n_{45678})(p^2)^{D-\nu_{1234}}}{n_1!n_2!n_3!n_4!n_5!n_6!n_7!n_8!}\delta_{n_{12345678},-\frac{D}{2}}x_1^{n_{12456}}x_2^{n_{12347}}x_3^{n_{1358}}x_4^{n_{23678}} = \sum_{m_1\cdots m_4}\frac{x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}}{m_1!m_2!m_3!m_4!}I(-m_1,-m_2,-m_3,-m_4). \tag{7.7}$$

This implies relations on the parameters m_j and n_ℓ. For example, matching the power of x_1 gives

$$m_1 = n_{12456}. \tag{7.8}$$

This condition is now imposed by putting an extra Kronecker delta on the sum of the left-hand side and write

$$\sum_{n_1\cdots n_8}\frac{(-1)^D\Gamma(1+n_{45678})(p^2)^{D-\nu_{1234}}}{n_1!n_2!n_3!n_4!n_5!n_6!n_7!n_8!}\delta_{n_{12345678},-\frac{D}{2}}\delta_{n_{12456},m_1}x_2^{n_{12347}}x_3^{n_{1358}}x_4^{n_{23678}} = \sum_{m_1\cdots m_4}\frac{x_2^{m_2}x_3^{m_3}x_4^{m_4}}{m_1!m_2!m_3!m_4!}I(-m_1,-m_2,-m_3,-m_4). \tag{7.9}$$

Continuing this procedure with x_2, x_3, x_4 gives

$$\sum_{n_1\cdots n_8}\frac{(-1)^D\Gamma(1+n_{45678})(p^2)^{D-\nu_{1234}}}{n_1!n_2!n_3!n_4!n_5!n_6!n_7!n_8!}\delta_{n_{12345678},-\frac{D}{2}}\delta_{n_{12456},m_1}\delta_{n_{12347},m_2}\delta_{n_{1358},m_3}\delta_{n_{23678},m_4} = \frac{I(-m_1,-m_2,-m_3,-m_4)}{m_1!m_2!m_3!m_4!}. \tag{7.10}$$

and this can be written as

$$I(-m_1,-m_2,-m_3,-m_4) = \sum_{n_1\cdots n_8}\frac{(-1)^D\Gamma(1+n_{45678})(p^2)^{D-\nu_{1234}}}{n_1!n_2!n_3!n_4!n_5!n_6!n_7!n_8!}\times m_1!m_2!m_3!m_4! \; \delta_{n_{12345678},-\frac{D}{2}}\delta_{n_{12456},m_1}\delta_{n_{12347},m_2}\delta_{n_{1358},m_3}\delta_{n_{23678},m_4}. \tag{7.11}$$

`Step 5`: Replace m_i by $-\nu_i$ for $i=1$ to 4 in order to obtain a new expression equivalent to $I(\nu_1,\nu_2,\nu_3,\nu_4)$.

$$I(\nu_1,\nu_2,\nu_3,\nu_4) = \sum_{n_1\cdots n_8}\frac{(-1)^D\Gamma(1-\nu_1)\Gamma(1-\nu_2)\Gamma(1-\nu_3)\Gamma(1-\nu_4)\Gamma(1+n_{45678})(p^2)^{n_{123}}}{n_1!n_2!n_3!n_4!n_5!n_6!n_7!n_8!} \times \delta_{n_{12345678},-\frac{D}{2}}\delta_{n_{12456},-\nu_1}\delta_{n_{12347},-\nu_2}\delta_{n_{1358},-\nu_3}\delta_{n_{23678},-\nu_4}. \tag{7.12}$$

This expression is valid for ν_j a negative integer.

`Step 6`: Solve the system of constraints to reduce the number of indices of the sum.

The constraints given by the Kronecker deltas are

$$\begin{aligned} n_1+n_2+n_3+n_4+n_5+n_6+n_7+n_8 &= -\tfrac{1}{2}D \\ n_1+n_2+n_4+n_5+n_6 &= -\nu_1 \\ n_1+n_2+n_3+n_4+n_7 &= -\nu_2 \\ n_1+n_3+n_5+n_8 &= -\nu_3 \\ n_2+n_3+n_6+n_7+n_8 &= -\nu_4. \end{aligned} \tag{7.13}$$

There are 5 equations with 8 indices, so there may be as many as $\binom{8}{5} = 56$ solutions. It turns out that 20 choices of a set of three free indices do not lead to a solution. The remaining 36 sets all lead to the same solution when hypergeometric identities are used. The solution is:

$$I = \frac{\pi^D(-1)^D(p^2)^{D-\nu_{1234}}\Gamma(1-\nu_1)\Gamma(1-\nu_2)\Gamma(1-\nu_4)}{\Gamma(1+D-\nu_{1234})\Gamma\left(1-\frac{D}{2}+\nu_4\right)\Gamma\left(1+\frac{D}{2}-\nu_{14}\right)} \times \frac{\Gamma\left(1-\frac{3D}{2}+\nu_{1234}\right)\Gamma\left(1+\frac{D}{2}-\nu_{134}\right)\Gamma(1-D+\nu_{14})}{\Gamma\left(1-D+\nu_{134}\right)\Gamma\left(1-\frac{D}{2}+\nu_2\right)\Gamma\left(1-\frac{D}{2}+\nu_1\right)}. \tag{7.14}$$

There are 36 sets of free indices which lead to the solution, so there is a 36-fold degeneracy in the solutions for the flying saucer diagram using the Anastasiou method. The sets of free indices leading to the solution are listed in Table 2

`Step 7`: Analytic continuation to positive values of D, ν_1, ν_2, ν_3, and ν_4.

This step is required in order to minimize the number of poles in the solution. The gamma factors to be analytically continued is determined by the values of D, ν_1, ν_2, ν_3, and ν_4. In the special case $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 1$, one can see that at least three gamma factors require continuation. This is true in general.

The formula for analytic continuation of a ratio of gamma factors, given as equation 2.33 in [1], is

$$\prod_{i=1}^{n} \frac{\Gamma(\alpha_i)}{\Gamma(\beta_i)} = (-1)^{\sum_{i=1}^{n}(\beta_i-\alpha_i)} \prod_{i=1}^{n} \frac{\Gamma(1-\beta_i)}{\Gamma(1-\alpha_i)}. \tag{7.15}$$

The gamma function has poles at 0 and at negative integers, so estimating $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 1$, one can see that at least the gamma factors $\Gamma(1-\nu_1)$, $\Gamma(1-\nu_2)$, and $\Gamma(1-\nu_4)$ will require continuation. Further estimating that $D = 4$, one can see that all gamma factors in the numerator will require continuation. These six factors must be balanced by the six gamma factors of the denominator. The result is

$$I_{fs}(\vec{\nu}, D; p) = \frac{\pi^D (p^2)^{D-\nu_{1234}} \Gamma\left(\frac{D}{2} - \nu_1\right) \Gamma\left(\frac{D}{2} - \nu_2\right) \Gamma\left(\frac{D}{2} - \nu_4\right) \Gamma\left(-\frac{D}{2} + \nu_{14}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_4)\Gamma\left(\frac{3D}{2} - \nu_{1234}\right) \Gamma\left(-\frac{D}{2} + \nu_{134}\right) \Gamma(D - \nu_{14}) \times \Gamma(-D + \nu_{1234})\Gamma(D - \nu_{134})} \tag{7.16}$$

8. The method of Suzuki

This method begins with the integral

$$I_{fs}(\vec{\nu}, D; p) == \int d^D q_1 d^D q_2 \frac{1}{(q_1^2)^{\nu_1} [(q_2 - p)^2]^{\nu_2} (q_2^2)^{\nu_3} [(q_1 - q_2)^2]^{\nu_4}}. \tag{8.1}$$

The discussion is adapted from [15, 16].

`Step 1`: Write the gaussian integral corresponding to the diagram, which takes the form

$$\int d^D q \exp\left(\sum_{i=1}^{N} -x_i A_i \right), \tag{8.2}$$

Here, A_i is the propagator associated to the i^{th} internal line of the diagram.

For the flying saucer diagram, the gaussian integral is

$$G = \int d^D q_1 d^D q_2 \exp\left[-x_1 q_1^2 - x_2 (q_2 - p)^2 - x_3 q_2^2 - x_4 (q_1 - q_2)^2) \right]. \tag{8.3}$$

`Step 2a`. The first method of evaluating $I_{fs}(\vec{\nu}, D; p)$ starts with the computation of G using

$$\int d^D q_1 \cdots d^D q_L \exp\left(-\sum_{i=1}^{N} x_i A_i \right) = \frac{\pi^{\frac{DL}{2}}}{U^{\frac{D}{2}}} \exp\left(-\frac{F}{U} \right), \tag{8.4}$$

where F and U are the Symanzik polynomials determined in Section 4. This first method to evaluate G gives

$$G = \frac{\pi^D}{U^{\frac{D}{2}}} \exp\left(-\frac{F}{U} \right). \tag{8.5}$$

`Step 2b`. Complete the method in Step 2a by expanding the exponential in (8.5) and the resulting multinomials in powers of x_1, x_2, x_3, and x_4. This yields

$$G = \sum_{n_1, \cdots, n_8} \frac{\pi^D (-1)^{n_{123}} (p^2)^{n_{123}} \Gamma\left(1 - \frac{D}{2} - \nu_{123}\right)}{n_1! n_2! n_3! n_4! n_5! n_6! n_7! n_8!} \times x_1^{n_{12456}} x_2^{n_{12347}} x_3^{n_{1358}} x_4^{n_{23678}} \delta_{n_{12345678}, -\frac{D}{2}}. \tag{8.6}$$

`Step 3`. The second method of evaluating $I_{fs}(\vec{\nu}, D; p)$ starts by expanding the exponential of (8.3) in powers of the propagators.

The result is

$$G = \int d^D q_1 d^D q_2 \sum_{m_1, \cdots, m_4} \frac{(-1)^{m_{1234}}}{m_1! m_2! m_3! m_4!} x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4} \times (q_1^2)^{m_1} \left[(q_2 - p)^2 \right]^{m_2} (q_2^2)^{m_3} \left[(q_1 - q_2)^2 \right]^{m_4}. \tag{8.7}$$

Exchanging the order of the sum and the integral shows that the integral is (8.1) with the powers of the propagators changed. The integral G can be rewritten as

$$G = \sum_{m_1, \cdots, m_4} \frac{(-1)^{m_{1234}}}{m_1! m_2! m_3! m_4} x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4} I(-m_1, -m_2, -m_3, -m_4). \tag{8.8}$$

`Step 4`: Let $m_i = -\nu_i$ for $i = 1$ to 4 and equate coefficients in the two expressions for G. Solve for $I = I(\nu_1, \nu_2, \nu_3, \nu_4)$ to get

$$I = \sum_{n_1, \cdots, n_8} \frac{(-1)^D \pi^D (p^2)^{n_{123}} \Gamma(1-\nu_1)\Gamma(1-\nu_2)\Gamma(1-\nu_3)\Gamma(1-\nu_4)}{n_1! n_2! n_3! n_4! n_5! n_6! n_7! n_8!} \times \Gamma\left(1 - \frac{D}{2} - n_{123}\right) \delta_{n_{12456}, -\nu_1} \delta_{n_{12347}, -\nu_2} \delta_{n_{1358}, -\nu_3} \delta_{n_{23678}, -\nu_4} \delta_{n_{12345678}, -\frac{D}{2}}. \tag{8.9}$$

This is the presolution.

`Step 5`: Solve the system of constraints to reduce the number of indices of the sum.

The constraints given by the Kronecker deltas are

$$\begin{aligned} n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8 &= -\frac{D}{2} \\ n_1 + n_2 + n_4 + n_5 + n_6 &= -\nu_1 \\ n_1 + n_2 + n_3 + n_4 + n_7 &= -\nu_2 \\ n_1 + n_3 + n_5 + n_8 &= -\nu_3 \\ n_2 + n_3 + n_6 + n_7 + n_8 &= -\nu_4. \end{aligned} \tag{8.10}$$

There are 5 equations with 8 indices, so there may be as many as $\binom{8}{5} = 56$ solutions. It turns out that 20 choices of a set of three free indices do not lead to a solution. The remaining 36 sets all lead to the same solution when hypergeometric identities are used. The solution is:

$$I = \frac{\pi^D (-1)^D (p^2)^{D-\nu_{1234}} \Gamma(1-\nu_1)\Gamma(1-\nu_2)\Gamma(1-\nu_4)}{\Gamma(1+D-\nu_{1234})\Gamma\left(1-\frac{D}{2}+\nu_4\right)\Gamma\left(1+\frac{D}{2}-\nu_{14}\right)} \times \frac{\Gamma\left(1-\frac{3D}{2}+\nu_{1234}\right)\Gamma\left(1+\frac{D}{2}-\nu_{134}\right)\Gamma(1-D+\nu_{14})}{\Gamma\left(1-D+\nu_{134}\right)\Gamma\left(1-\frac{D}{2}+\nu_2\right)\Gamma\left(1-\frac{D}{2}+\nu_1\right)}. \tag{8.11}$$

There are 36 sets of free indices which lead to the solution, so there is a 36-fold degeneracy in the solutions for the flying saucer diagram using the Anastasiou method. The sets of free indices leading to the solution are listed in Tables 1 and 2

`Step 7`: Analytic continuation to positive values of D, ν_1, ν_2, ν_3, and ν_4.

This step is required in order to minimize the number of poles in the solution. The gamma factors to be analytically continued is determined by the values of D, ν_1, ν_2, ν_3, and ν_4. Assuming that $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 1$, one can see that at least three gamma factors require continuation.

The formula for analytic continuation of a ratio of gamma factors, given as equation 2.33 in [1], is

$$\prod_{i=1}^{n} \frac{\Gamma(\alpha_i)}{\Gamma(\beta_i)} = (-1)^{\sum_{i=1}^{n}(\beta_i - \alpha_i)} \prod_{i=1}^{n} \frac{\Gamma(1-\beta_i)}{\Gamma(1-\alpha_i)}. \tag{8.12}$$

The gamma function has poles at 0 and at negative integers, so estimating $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 1$, one can see that at least the gamma factors $\Gamma(1-\nu_1)$, $\Gamma(1-\nu_2)$, and $\Gamma(1-\nu_4)$ will require continuation. Further estimating that $D = 4$, one can see that all gamma factors in the numerator will require continuation. These six factors must be balanced by the six gamma factors of the denominator. The result is

$$I_{fs}(\vec{\nu}, D; p) = \frac{\pi^D (p^2)^{D-\nu_{1234}} \Gamma\left(\frac{D}{2} - \nu_1\right) \Gamma\left(\frac{D}{2} - \nu_2\right) \Gamma\left(\frac{D}{2} - \nu_4\right) \Gamma\left(-\frac{D}{2} + \nu_{14}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_4)\Gamma\left(\frac{3D}{2} - \nu_{1234}\right) \Gamma\left(-\frac{D}{2} + \nu_{134}\right) \Gamma(D-\nu_{14}) \times \Gamma(-D+\nu_{1234})\Gamma(D-\nu_{134})} \tag{8.13}$$

9. The method of brackets

Start with $I = \int \frac{d^D q_1}{i\pi^{\frac{D}{2}}} \frac{d^D q_2}{i\pi^{\frac{D}{2}}} \frac{1}{(q_1^2)^{\nu_1}[(q_2-p)^2]^{\nu_2}(q_2^2)^{\nu_3}[(q_1-q_2)^2]^{\nu_4}}$.

`Step 1`: Use the Schwinger parametrization

$$\frac{1}{A^\nu} = \frac{1}{\Gamma(\nu)} \int dx \cdot x^{\nu-1} \exp(-xA) \tag{9.1}$$

to parametrize the integral.

After parametrization, the integral is

$$I = \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)} \int dx_1 dx_2 dx_3 dx_4 x_1^{\nu_1-1} x_2^{\nu_2-1} x_3^{\nu_3-1} x_4^{\nu_4-1} \times \int \frac{d^D q_1 d^D q_2}{(i\pi^{\frac{D}{2}})^2} \exp\left(-x_1 q_1^2 - x_2(q_2-p)^2 - x_3 q_2^2 - x_4(q_1-q_2)^2\right) \tag{9.2}$$

`Step 2`: Integrate over the loop momenta using

$$\int \frac{d^Dq_1 \cdots d^Dq_L}{\left(i\pi^{\frac{D}{2}}\right)^L} \exp\left(-\sum_{i=1}^{N} x_iA_i\right) = \frac{(-1)^{\frac{DL}{2}}}{U^{\frac{D}{2}}} \exp\left(-\frac{F}{U}\right). \tag{9.3}$$

The result is

(9.4)

$$I = \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)} \int dx_1dx_2dx_3dx_4 x_1^{\nu_1-1}x_2^{\nu_2-1}x_3^{\nu_3-1}x_4^{\nu_4-1}\frac{1}{U^{\frac{D}{2}}}\exp\left(-\frac{F}{U}\right),$$

where F and U are the Symanzik polynomials determined in Section 4.

`Step 3a`: Factor F and U so as to reduce the number of indices that will be needed in the expansion of the exponential (if possible). The best method is to look for repeated multinomials. The reader can find more details in [6].

The first step in factoring F is to notice that all terms have a common factor of x_2p^2. The remaining factor $x_1x_3 + x_3x_4 + x_1x_4$ is found as a summand in U, so the first step in factoring U is to write it as a sum of $x_1x_3 + x_3x_4 + x_1x_4$ and what is left. Then these terms are factored, noting that $x_1 + x_4$ may appear in both.

$$\begin{aligned} F &= (x_1x_2x_3 + x_1x_2x_4 + x_2x_3x_4)\,p^2 \\ &= x_2p^2\,[x_3(x_1+x_4) + x_1x_4]\,. \\ U &= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_4 + x_3x_4 \\ &= [x_3(x_1+x_4) + x_1x_4] + x_2(x_1+x_4). \end{aligned} \tag{9.5}$$

`Step 3b`: Expand the exponential and resulting multinomials in steps by using the binomial formula, so that repeated multinomials, which appear in subsequent expansions, are factored last.

First, replace F and U by their factored versions, then expand the exponential.

$$\begin{aligned} I = \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)} & \int dx_1dx_2dx_3dx_4 x_1^{\nu_1-1}x_2^{\nu_2-1}x_3^{\nu_3-1}x_4^{\nu_4-1}\times \\ & \sum_{n_1} \frac{(-1)^{n_1}(p^2)^{n_1}}{n_1!} x_2^{n_1}\,[x_3(x_1+x_4)+x_1x_4]^{n_1} \\ & \times [[x_3(x_1+x_4)+x_1x_4] + x_2(x_1+x_4)]^{-\frac{D}{2}-n_1}\,. \end{aligned} \tag{9.6}$$

Begin expansion of the multinomials. In each step, select terms to expand which contain other multinomials of the product as a summand. In the first step, since $[x_3(x_1+x_4)+x_1x_4]$ appears in the product by itself and as a summand of

$[x_3(x_1+x_4)+x_1x_4]+x_2(x_1+x_4)$, the latter is expanded first.

$$(9.7)\quad I = \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)}\int dx_1dx_2dx_3dx_4 x_1^{\nu_1-1}x_2^{\nu_2-1}x_3^{\nu_3-1}x_4^{\nu_4-1} \times \sum_{n_1,n_2,n_3}\frac{(-1)^{n_1}(p^2)^{n_1}\Gamma(1+n_{23})}{n_1!n_2!n_3!}x_2^{n_{13}}\left[x_3(x_1+x_4)+x_1x_4\right]^{n_{12}}(x_1+x_4)^{n_3} \times \delta_{n_{123},-\frac{D}{2}}.$$

Next, expand $x_3(x_1+x_4)+x_1x_4$, noticing that the multinomial x_1+x_4 appears in the product and will be expanded later.

$$(9.8)\quad I = \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)}\int dx_1dx_2dx_3dx_4 x_1^{\nu_1-1}x_2^{\nu_2-1}x_3^{\nu_3-1}x_4^{\nu_4-1} \times \sum_{n_1,\cdots,n_5}\frac{(-1)^{n_1}(p^2)^{n_1}\Gamma(1+n_{23})\Gamma(1+n_{45})}{n_1!n_2!n_3!n_4!n_5!}x_1^{n_5}x_2^{n_{13}}x_3^{n_4}x_4^{n_5}(x_1+x_4)^{n_{34}} \times \delta_{n_{123},-\frac{D}{2}}\delta_{n_{12}-n_{45},0}.$$

Finally, the remaining multinomial x_1+x_4 is expanded and the x-integrals separated.

$$(9.9)\quad I = \sum_{n_1,\cdots,n_7}\frac{(-1)^{n_1}(p^2)^{n_1}\Gamma(1+n_{23})\Gamma(1+n_{45})\Gamma(1+n_{67})}{n_1!n_2!n_3!n_4!n_5!n_6!n_7!}\int\frac{dx_1}{\Gamma(\nu_1)}x_1^{n_{56}+\nu_1-1} \times\int\frac{dx_2}{\Gamma(\nu_2)}x_2^{n_{13}+\nu_2-1}\int\frac{dx_3}{\Gamma(\nu_3)}x_3^{n_4+\nu_3-1}\int\frac{dx_4}{\Gamma(\nu_4)}x_4^{n_{57}+\nu_4-1} \times\delta_{n_{123},-\frac{D}{2}}\delta_{n_{12}-n_{45},0}\delta_{n_{34}-n_{67},0}$$

`Step 4`: Write the integrals in terms of brackets using $\langle\alpha\rangle = \int dx\cdot x^{\alpha-1}$, and write any Kronecker deltas in terms of brackets using the identity

$$(9.10)\qquad \delta_{\alpha,\beta} = \frac{(-1)^{\omega}\langle\alpha-\beta\rangle}{\Gamma(\omega)\Gamma(1-\omega)},$$

where ω is an arbitrary value chosen so that some of the gamma factors in the numerator cancel.

Replacing each integral and Kronecker delta by an expression in terms of brackets gives

$$(9.11)\quad I = \sum_{n_1\cdots n_7}\frac{(-1)^{n_1}(p^2)^{n_1}\Gamma(1+n_{23})\Gamma(1+n_{45})\Gamma(1+n_{67})}{n_1!n_2!n_3!n_4!n_5!n_6!n_7!\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)}\langle n_5+n_6+\nu_1\rangle \times\langle n_1+n_3+\nu_2\rangle\langle n_4+\nu_3\rangle\langle n_5+n_7+\nu_4\rangle \times\frac{(-1)^{n_{23}}\left\langle n_1+n_2+n_3+\frac{D}{2}\right\rangle}{\Gamma(1+n_{23})\Gamma(-n_{23})}\frac{(-1)^{n_{45}}\langle n_1+n_2-n_4-n_5\rangle}{\Gamma(1+n_{45})\Gamma(-n_{45})}\frac{(-1)^{n_{67}}\langle n_3+n_4-n_6-n_7\rangle}{\Gamma(1+n_{67})\Gamma(-n_{67})}.$$

`Step 5a`: Find the values of $n_1^*, n_2^*, \cdots, n_7^*$ and $|\det A|$, where the starred indices are the values that make the system of brackets vanish and A is the matrix of coefficients of the brackets.

The brackets vanish when

$$\begin{aligned} n_5 + n_6 &= -\nu_1 \\ n_1 + n_3 &= -\nu_2 \\ n_4 &= -\nu_3 \\ n_5 + n_7 &= -\nu_4 \\ n_1 + n_2 + n_3 &= -\frac{D}{2} \\ n_1 + n_2 - n_4 - n_5 &= 0 \\ n_3 + n_4 - n_6 - n_7 &= 0, \end{aligned} \tag{9.12}$$

so the matrix A of coefficients of the brackets is

$$A = \begin{vmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 \end{vmatrix} \tag{9.13}$$

and $|\det A| = 1$.

There are 7 equations and 7 indices, so there will be no free index, which allows only one solution. The values are

$$\begin{aligned} n_1^* &= D - \nu_1 - \nu_2 - \nu_3 - \nu_4 \\ n_2^* &= -\frac{D}{2} + \nu_2 \\ n_3^* &= -D + \nu_1 + \nu_3 + \nu_4 \\ n_4^* &= -\nu_3 \\ n_5^* &= \frac{D}{2} - \nu_1 - \nu_4 \\ n_6^* &= -\frac{D}{2} + \nu_4 \\ n_7^* &= -\frac{D}{2} + \nu_1 \end{aligned} \tag{9.14}$$

`Step 5b`: Evaluate the brackets using the formula

$$\sum_{m_1 \cdots m_k} \phi_1 \cdots \phi_k f(m_1 \cdots m_k) \Delta_1 \cdots \Delta_k = \frac{f(m_1^* \cdots m_k^*)}{|\det A|} \Gamma(-m_1^*) \cdots \Gamma(-m_k^*) \tag{9.15}$$

where the Δ_i represent each of the k brackets, and $\phi_j = \dfrac{(-1)^{m_j}}{m_j!}$.

The result is

$$I = \frac{(p^2)^{n_1^*}\Gamma(-n_1^*)\Gamma(-n_2^*)\Gamma(-n_3^*)\Gamma(-n_4^*)\Gamma(-n_5^*)\Gamma(-n_6^*)\Gamma(-n_7^*)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)\Gamma(-n_2^*-n_3^*)\Gamma(-n_4^*-n_5^*)\Gamma(-n_6^*-n_7^*)}. \tag{9.16}$$

Substituting the values of the starred indices gives the final solution.

$$I = \frac{(p^2)^{D-\nu_{1234}}\Gamma\left(\frac{D}{2}-\nu_1\right)\Gamma\left(\frac{D}{2}-\nu_2\right)\Gamma\left(\frac{D}{2}-\nu_4\right)\Gamma\left(-\frac{D}{2}+\nu_{14}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_4)\Gamma\left(\frac{3D}{2}-\nu_{1234}\right)\Gamma\left(-\frac{D}{2}+\nu_{134}\right)\Gamma(D-\nu_{14})} \times \Gamma(-D+\nu_{1234})\Gamma(D-\nu_{134}). \tag{9.17}$$

`Step 6`: Analytic continuation to positive values of D, ν_1, ν_2, ν_3, and ν_4.

Assuming that $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 1$, it is not obvious that any of the gamma factors will require analytic continuation.

10. Conclusions

Ricotta, in [12], gave several methods for evaluating Feynman diagrams using negative dimensional integration. The first method, in Section 5, is referred to here as the Ricotta method without parametrization. It uses the multinomial formula (5.2) to write the integrand as a polynomial, and derives an identity to evaluate the terms. This method gives 9 solutions, each of which is a double sum. Because of the constraints on the indices, it is not obvious that any of the series may be written in closed form; however, under certain conditions on the parameters, they may be written as a product of hypergeometric functions and simplified.

The second method, in Section 6, is referred to here as the Ricotta method with parametrization, and uses the Schwinger parametrization. The Schwinger parametrization introduces a new parameter for each of the internal segments of the diagram. Introducing new parameters allows the integral over the loop momentum q to become a gaussian, which is easily evaluated, leaving integrals over each of the Schwinger parameters. For this method, there are 56 choices for a set of free indices. 20 of these lead to no solution and 36 choices lead to triple hypergeometric series which may be determined by using the Gauss summation formula. Each of these 36 choices lead to the same solution, and analytic continuation of at least three gamma factors is required to obtain the final result.

C. Anastasiou, E. Glover, and C. Oleari, in [1, 2], evaluated the integrals two ways and equated the results, beginning with the integral associated to the Feynman diagram and using the Schwinger parametrization. This method, in Section 7, is referred to here as the Anastasiou method. This method gives the same solutions as the Ricotta method with parametrization.

Instead of beginning with the integral associated to the Feynman diagram, Suzuki and Schmidt began with an associated gaussian. This is the same gaussian obtained by Schwinger parametrization of the integral for the diagram. They changed the algorithm somewhat in that they did not evaluate the integrals over the Schwinger parameters, but instead found two ways to evaluate the integral, equating the results to find the solution. This method, in Section 8, is referred to here as the Suzuki method, and finds the same solutions as in the previous two methods.

Suzuki and Schmidt gave many examples using negative dimensional integration. Among them, they found that negative dimensional integration not only gives the solutions that were able to be calculated using other methods, but that new solutions were simultaneously obtained. These new solutions are analytic continuations of the other solutions, which are convergent in other regions (which depend on external momentum and mass).

I. Gonzalez and I. Schmidt, in [6, 7, 8], further developed negative dimensional integration with the method of brackets algorithm. The method of brackets adds the step of the factorization of F and U when possible, which greatly reduces the number of free indices in the solution for more complicated diagrams (here, 36 solutions were reduced to one). The formulas used for the Schwinger parametrization and the analysis of brackets further simplify the calculations involved to determine the final solution, because the Schwinger parametrization leaves factors of $\Gamma(\nu_i)$, which is analytic for positive ν_i, in the result rather than factors of $\Gamma(1-\nu_i)$, which require continuation, and the expression for $\delta_{\alpha,\beta}$ allows one to choose gamma factors that cancel with existing gamma factors, further simplifying the result.

The method of brackets is an improvement upon the earlier methods of negative dimensional integration. In this example of the flying saucer diagram, the method of brackets gives the same solution as the other methods, and does so without requiring the extra step of analytic continuation. With the factorization of F and U before the expansion into powers of the x parameters, the number of free indices can be greatly reduced. This yields solutions which are sums over fewer indices, which are more likely to be a special type of hypergeometric series, such as an Appell function, for which convergence properties are known. The reader will find more examples in [17].

In this work the one diagram was evaluated considering the two loops simultaneously. The only purpose of the work discussed here was to compare the methods existing in the literature. On the other hand, this diagram belongs to a family that can be evaluated loop by loop in a recursive manner. This produces the solution in a direct and simple manner. The reader can verify this using the strategies described in [14].

References

[1] C. Anastasiou, E. W. N. Glover, and C. Oleari. Application of the negative-dimension approach to massless scalar box integrals. *Nucl. Phys. B*, 565:445–467, 2000.

[2] C. Anastasiou, E. W. N. Glover, and C. Oleari. Scalar one-loop integrals using the negative-dimension approach. *Nucl. Phys. B*, 572:307–360, 2000.

[3] J. Bonnar. *The Gamma Function*. Treasure Trove in Mathemnatics. Online, 2017.

[4] G. Boros and V. Moll. *Irresistible Integrals*. Cambridge University Press, New York, 1st edition, 2004.

[5] I. Gonzalez and V. Moll. Definite integrals by the method of brackets. Part 1. *Adv. Appl. Math.*, 45:50–73, 2010.

[6] I. Gonzalez and I. Schmidt. Optimized negative dimensional integration method (NDIM) and multiloop Feynman diagram calculation. *Nuclear Physics B*, 769:124–173, 2007.

[7] I. Gonzalez and I. Schmidt. Modular application of an integration by fractional expansion (IBFE) method to multiloop Feynman diagrams. *Phys. Rev. D*, 78:086003, 2008.

[8] I. Gonzalez and I. Schmidt. Modular application of an integration by fractional expansion (IBFE) method to multiloop Feynman diagrams II. *Phys. Rev. D*, 79:126014, 2009.

[9] I. G. Halliday and R. M. Ricotta. Negative dimensional integrals. I. Feynman graphs. *Phys. Lett. B*, 193:241–246, 1987.

[10] D. Kreimer. *Knots and Feynman diagrams*. Number 13 in Cambridge Lectures in Physics. Cambridge University Press, 2000.

[11] M. Polyak. Feynman diagrams for pedestrians and mathematicians. In M. Lyubich and L. Takhtajan, editors, *Graphs and Patterns in Mathematics and Theoretical Physics*, volume 73 of *Proc. Symp. Pure Mathematics*. Amer. Math. Soc., 2005.

[12] R. M. Ricotta. *Topics in Field Theory*. PhD thesis, Imperial College London, 1987.

[13] V. A. Smirnov. *Evaluating Feynman integrals*, volume 211 of *Springer Tracts Mod. Phys.* Springer Verlag, Berlin Heildelberg, 2004.

[14] V. A. Smirnov. *Feynman Integral Calculus*. Springer Verlag, Berlin Heildelberg, 2006.

[15] A. T. Suzuki and A. G. M. Schmidt. Negative-dimensional integration revisited. *Journal Physics A*, 31:8023–8039, 1998.

[16] A. T. Suzuki and A. G. M. Schmidt. Two-loop energy diagrams worked out with NDIM. *Eur. Phys. J.*, C-5.1:175–179, 1998.

[17] K. van Deusen. *A comparison of negative-dimensional integration techniques*. PhD thesis, Tulane University, 2021.

[18] M. Veltman. *Diagrammatica. The path to Feynman diagrams*, volume 4 of *Cambridge Lecture Notes in Physics*. Cambridge University Press, 1994.

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