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On some definite integrals connecting with infinite products and infinite series

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Abstract. Some definite integrals involving combinations of powers, exponentials, trigonometric and hyperbolic functions are evaluated through infinite products and infinite series.

1. Introduction.

The classical table of integrals by Gradshteyn and Ryzhik[**2**] contains many entries related to definite integrals in the combinations of powers, algebraic functions of exponentials, trigonometric functions, etc. The solutions of the following integrals are not found in the table of integrals.

$$\int_0^\infty \frac{e^{-zx}}{1-e^{-x}} (\cos ax - \cos bx) \frac{dx}{x} \quad \text{and}$$
$$\int_0^\infty \frac{\cos ax}{x} (\psi(z+ix) - \psi(z-ix))\, dx.$$

Also, they cannot be determined using a symbolic language. The object of this work is to evaluate the above mentioned integrals through the following infinite products [**2**, p. 886] and infinite series[**2**, p. 893]. For real x, y and $x \neq 0, -1, -2, \ldots$

$$\prod_{k=0}^{\infty} \left(1 + \frac{y^2}{(x+k)^2}\right) = \frac{\Gamma(x)^2}{\Gamma(x+iy)\Gamma(x-iy)} = \left|\frac{\Gamma(x)}{\Gamma(x-iy)}\right|^2. \tag{1.1}$$

and

$$\sum_{k=0}^{\infty} \frac{2yi}{y^2 + (x+k)^2} = \psi(x+iy) - \psi(x-iy). \tag{1.2}$$

In addition, some integrals involving gamma and psi functions are evaluated through above mentioned identities.

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2. Definite integrals connecting with infinite products.

In this section, integrals involving various combinations of elementary functions are evaluated by connecting with infinite products. Consider the following definite integral for $a > 0, b > 0$ and $z > 0$ [**2**, p. 494]

$$\int_0^\infty e^{-zx} (\cos ax - \cos bx) \frac{dx}{x} = \frac{1}{2} \log \frac{1 + b^2/z^2}{1 + a^2/z^2}. \tag{2.1}$$

Replacing z by $z + k$

$$\int_0^\infty e^{-(z+k)x} (\cos ax - \cos bx) \frac{dx}{x} = \frac{1}{2} \log \frac{1 + b^2/(z+k)^2}{1 + a^2/(z+k)^2}.$$

Taking summation on both sides for $k = 0, 1, 2, \ldots$ and after simplification, gives

$$\int_0^\infty \frac{e^{-zx}}{1 - e^{-x}} (\cos ax - \cos bx) \frac{dx}{x} = \frac{1}{2} \log \prod_{k=0}^{\infty} \frac{1 + b^2/(z+k)^2}{1 + a^2/(z+k)^2}.$$

Using identity (1.1), then

$$\begin{aligned} \int_0^\infty \frac{e^{-zx}}{1 - e^{-x}} (\cos ax - \cos bx) \frac{dx}{x} &= \frac{1}{2} \log \frac{\Gamma(z - ia)\Gamma(z + ia)}{\Gamma(z - ib)\Gamma(z + ib)} \\ &= \log \left| \frac{\Gamma(z - ia)}{\Gamma(z - ib)} \right|. \end{aligned} \tag{2.2}$$

Differentiating (2.2) n times with respect to z, then for $n \in N$

$$\begin{aligned} \int_0^\infty \frac{x^{n-1} e^{-zx}}{1 - e^{-x}} (\cos ax - \cos bx)\, dx &= \frac{(-1)^n}{2} \Big(\psi^{(n-1)}(z - ia) \\ &+ \psi^{(n-1)}(z + ia) - \psi^{(n-1)}(z - ib) - \psi^{(n-1)}(z + ib) \Big). \end{aligned}$$

Where ψ is psi function. For $n = 1$,

$$\begin{aligned} \int_0^\infty \frac{e^{-zx}}{1 - e^{-x}} (\cos ax - \cos bx)\, dx = -\frac{1}{2} \big(\psi(z - ia) + \\ \psi(z + ia) - \psi(z - ib) - \psi(z + ib)\big). \end{aligned}$$

But for $n > 1$, $\psi^{(n-1)}(x) = (-1)^{n-1}(n-1)!\zeta(n, x)$, where ζ is Riemann zeta function. Hence

$$\begin{aligned} \int_0^\infty \frac{x^{n-1} e^{-zx}}{1 - e^{-x}} (\cos ax - \cos bx)\, dx = -\frac{(n-1)!}{2} [\zeta(n, z - ia) + \\ \zeta(n, z + ia) - \zeta(n, z - ib) - \zeta(n, z + ib)]. \end{aligned} \tag{2.3}$$

Consider the algebraic recursive method from Ref[**3**] for $m \in N$ and $p \in \Re$

$$\sum_{k=1}^{m+1} A_k^{(m+1)} t^{k-1} (1 - t)^{p-k+1} = \sum_{i=0}^{\infty} (-1)^i \binom{p}{i} (z + i)^m t^i. \tag{2.4}$$

where

$$
\begin{aligned}
A_1^{(m+1)} =& zA_1^{(m)}, \\
A_k^{(m+1)} =& -(p-k+2)A_{k-1}^{(m)} + (z+k-1)A_k^{(m)},\ k=2,3,\dots m, \\
A_{m+1}^{(m+1)} =& -(p-m+1)A_m^{(m)}.
\end{aligned}
\tag{2.5}
$$

$A_1^{(1)} = 1$ and other values of A's in (2.4) can be determined through equation (2.5).

Let $t = e^{-x}$, $p = -1$ and multiply by $e^{-zx}(\cos ax - \cos bx)/x$ on both sides of (2.4) and after simplification, gives

$$
\begin{aligned}
&\sum_{k=1}^{m+1} A_k^{(m+1)} \int_0^\infty \frac{e^{-(z+k-1)x}}{(1-e^{-x})^k} (\cos ax - \cos bx)\frac{dx}{x} \\
&\qquad = \sum_{j=0}^{\infty} (z+j)^m \int_0^\infty e^{-(z+j)x} (\cos ax - \cos bx)\frac{dx}{x}.
\end{aligned}
$$

Using (2.1), yields

$$
\begin{aligned}
&\sum_{k=1}^{m+1} A_k^{(m+1)} \int_0^\infty \frac{e^{-(z+k-1)x}}{(1-e^{-x})^k} (\cos ax - \cos bx)\frac{dx}{x} \\
&\qquad = \frac{1}{2}\sum_{j=0}^{\infty} (z+j)^m \log\frac{(z+j)^2+b^2}{(z+j)^2+a^2}.
\end{aligned}
\tag{2.6}
$$

where $m = 0, 1, 2, \dots$. Similarly, consider the following integral [**2**, p. 494] for $m = 1, 2, \dots p$, $a > 0, b > 0$ and $z > 0$

$$
\begin{aligned}
\int_0^\infty e^{-zx}\sin^{2m} ax\frac{dx}{x} =& \frac{(-1)^{m+1}}{2^{2m}}\sum_{j=0}^{m-1}(-1)^j\binom{2m}{j}\times \\
&\log\left(z^2+(2m-2j)a^2\right) - \frac{1}{2^{2m}}\binom{2m}{m}\log z.
\end{aligned}
\tag{2.7}
$$

It can be easily find that,

$$
\begin{aligned}
&\int_0^\infty e^{-zx}\left(\sin^{2m} ax - \sin^{2m} bx\right)\frac{dx}{x} \\
&\qquad = \frac{(-1)^m}{2^{2m}}\sum_{j=0}^{m-1}(-1)^j\binom{2m}{j}\log\frac{1+(2m-2j)^2b^2/z^2}{1+(2m-2j)^2a^2/z^2}.
\end{aligned}
\tag{2.8}
$$

Replacing z by $z+k$ and taking summation on both sides for $k=0,1,2,\dots$ and after simplification, gives

$$\int_0^\infty \frac{e^{-zx}}{1-e^{-x}}\left(\sin^{2m} ax - \sin^{2m} bx\right)\frac{dx}{x} = \frac{(-1)^m}{2^{2m}}\sum_{j=0}^{m-1}(-1)^j\binom{2m}{j}\log\prod_{k=0}^{\infty}\frac{1+(2m-2j)^2b^2/(z+k)^2}{1+(2m-2j)^2a^2/(z+k)^2}. \tag{2.9}$$

Using identity (1.1), then

$$\int_0^\infty \frac{e^{-zx}}{1-e^{-x}}\left(\sin^{2m} ax - \sin^{2m} bx\right)\frac{dx}{x} = \frac{(-1)^m}{2^{2m}}\times \sum_{j=0}^{m-1}(-1)^j\binom{2m}{j}\log\frac{\Gamma(z-ia(2m-2j))\Gamma(z+ia(2m-2j))}{\Gamma(z+ib(2m-2j))\Gamma(z-ib(2m-2j))}. \tag{2.10}$$

Differentiating n times with respect to z and after simplification, for $n>1$

$$\int_0^\infty \frac{x^{n-1}e^{-zx}}{1-e^{-x}}\left(\sin^{2m} ax - \sin^{2m} bx\right)dx = \frac{(-1)^{m+1}}{2^{2m}}\times (n-1)!\sum_{j=0}^{m-1}(-1)^j\binom{2m}{j}\left[\zeta(n,z-ia(2m-2j))+\zeta(n,z+ia(2m-2j)) -\zeta(n,z-ib(2m-2j))-\zeta(n,z+ib(2m-2j))\right]. \tag{2.11}$$

For $n=1$,

$$\int_0^\infty \frac{e^{-zx}}{1-e^{-x}}\left(\sin^{2m} ax - \sin^{2m} bx\right)dx = \frac{(-1)^{m+1}}{2^{2m}}\times \sum_{j=0}^{m-1}(-1)^j\binom{2m}{j}\left[\psi(z-ia(2m-2j))+\psi(z+ia(2m-2j)) -\psi(z-ib(2m-2j))-\psi(z+ib(2m-2j))\right]. \tag{2.12}$$

Let $t=e^{-x}$, $p=-1$ and multiply by $e^{-zx}\left(\sin^{2n} ax - \sin^{2n} bx\right)/x$ on both sides of (2.4) and after simplification, that

$$\sum_{k=1}^{m+1}A_k^{(m+1)}\int_0^\infty \frac{e^{-(z+k-1)x}}{(1-e^{-x})^k}\left(\sin^{2n} ax - \sin^{2n} bx\right)\frac{dx}{x} = \sum_{j=0}^{\infty}(z+j)^m\int_0^\infty e^{-(z+j)x}\left(\sin^{2n} ax - \sin^{2n} bx\right)\frac{dx}{x}.$$

Using (2.8), gives

$$\sum_{k=1}^{m+1} A_k^{(m+1)} \int_0^\infty \frac{e^{-(z+k-1)x}}{(1-e^{-x})^k} \left(\sin^{2n} ax - \sin^{2n} bx\right) \frac{dx}{x}$$
$$= \frac{(-1)^n}{2^{2n}} \sum_{j=0}^{\infty} (z+j)^m \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \log \frac{(z+j)^2 + (2n-2k)^2 b^2}{(z+j)^2 + (2n-2k)^2 a^2}.$$

After simplification, for $m = 1, 2, \ldots$

$$\sum_{k=1}^{m+1} A_k^{(m+1)} \int_0^\infty \frac{e^{-(z+k-1)x}}{(1-e^{-x})^k} \left(\sin^{2n} ax - \sin^{2n} bx\right) \frac{dx}{x} \tag{2.13}$$
$$= \frac{(-1)^n}{2^{2n}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \sum_{j=0}^{\infty} (z+j)^m \log \frac{(z+j)^2 + 4(n-k)^2 b^2}{(z+j)^2 + 4(n-k)^2 a^2}.$$

A variety of similar integrals in the combinations of powers, exponentials and trigonometric functions is found in sections 3.94-3.97 of the table of integrals [**2**].

Now consider the following integral for $b/z > 1$ and $pq > 0$ [**2**, p. 564]

$$\int_0^\infty \log \frac{1 + b^2 e^{-2px}/z^2}{1 + b^2 e^{-2qx}/z^2} dx = -\log\left(1 + b^2/z^2\right) \log\left(p/q\right).$$

It can be easily find that

$$\int_0^\infty \log \left| \frac{\Gamma(z - ibe^{-qx})}{\Gamma(z - ibe^{-px})} \right| dx = -\log \left| \frac{\Gamma(z)}{\Gamma(z - ib)} \right| \log\left(p/q\right).$$

or

$$\int_0^1 \log \left| \frac{\Gamma(z - ix^q)}{\Gamma(z - ix^p)} \right| \frac{dx}{x} = -\log \left| \frac{\Gamma(z)}{\Gamma(z - ib)} \right| \log\left(p/q\right). \tag{2.14}$$

The integrals (2.2), (2.10) and (2.14) cannot be evaluated by using a symbolic language.

3. Evaluation of definite integrals through psi function.

A list of integrals involving psi function is treated through identities of psi function in Ref[**1**]. In this section, integrals that involving psi function are evaluated through following identities of psi function ψ [**2**, p. 894-895].

$$\psi\left(\frac{1}{2} + z\right) = \psi\left(\frac{1}{2} - z\right) + \pi \tan \pi z. \tag{3.1}$$

$$\psi\left(z + n\right) = \psi\left(z\right) + \sum_{k=0}^{n-1} \frac{1}{z+k}. \tag{3.2}$$

Then, it immediately follows that

$$\psi\left(n + \frac{1}{2} + z\right) - \psi\left(n + \frac{1}{2} - z\right) = \pi \tan \pi z - 2 \sum_{k=0}^{n-1} \frac{z}{\left(k + \frac{1}{2}\right)^2 - z^2}. \tag{3.3}$$

Consider the following integral for $a > 0$ and $z > 0$ [**2**, p. 418]

$$\int_0^\infty \frac{\cos ax}{z^2 + x^2} dx = \frac{\pi}{2z} e^{-az} = \frac{\pi}{2} \int_a^\infty e^{-tz} dt. \tag{3.4}$$

Replace z by $z + k$ and taking summation for $k = 0, 1, 2, \ldots$ on both sides, and after simplification, that

$$\int_0^\infty \cos ax \sum_{k=0}^\infty \frac{1}{(z+k)^2 + x^2} dx = \frac{\pi}{2} \int_a^\infty \sum_{k=0}^\infty e^{-t(z+k)} dt.$$

Using the identity (1.2), gives

$$\int_0^\infty \frac{\cos ax}{x} \left(\psi(z+ix) - \psi(z-ix)\right) dx = i\pi \int_a^\infty \frac{e^{-tz}}{1-e^{-t}} dt. \tag{3.5}$$

If $z = n + \frac{1}{2}$ in (3.5) and using (3.3), then

$$-2i \int_0^\infty \sum_{k=0}^{n-1} \frac{\cos ax dx}{(k+\frac{1}{2})^2 + x^2} + i\pi \int_0^\infty \frac{\cos ax}{x} \tanh x dx = i\pi \int_a^\infty \frac{e^{-t\left(n+\frac{1}{2}\right)}}{1-e^{-t}} dt.$$

After simplification, that

$$\int_a^\infty \frac{e^{-t\left(n+\frac{1}{2}\right)}}{1-e^{-t}} dt = -\sum_{k=0}^{n-1} \frac{e^{-a(k+\frac{1}{2})}}{k+\frac{1}{2}} + \log\coth\frac{a}{4}. \tag{3.6}$$

For other values of z,

$$\int_a^\infty \frac{e^{-tz}}{1-e^{-t}} dt = \sum_{k=0}^\infty \frac{e^{-a(z+k)}}{z+k}. \tag{3.7}$$

If $f_c(x)$ can be expressed in finite terms of cosine multiples of x, then the following integral can be evaluated using (3.5) and (3.7).

$$\int_0^\infty \frac{f_c(x)}{x} \left(\psi(z+ix) - \psi(z-ix)\right) dx. \tag{3.8}$$

For instance, using the integral [**2**, p. 425] $z > 0, a \geqslant b \geqslant 0$

$$\int_0^\infty \frac{\sin(ax)\sin(bx)}{z^2 + x^2} dx = \frac{\pi}{2z} e^{-az} \sinh bz.$$

It can be easily find that

$$\begin{aligned} &\int_0^\infty \frac{\sin(ax)\sin(bx)}{x} \left(\psi(z+ix) - \psi(z-ix)\right) dx \\ &= \frac{i\pi}{2} \left[\int_{a-b}^\infty \frac{e^{-zt}}{1-e^{-t}} dt - \int_{a+b}^\infty \frac{e^{-zt}}{1-e^{-t}} dt\right]. \end{aligned}$$

If $z = n + 1/2$, then the RHS of the above equation can be easily evaluated through (3.5) and (3.6). For other values of z, one can use (3.5) and (3.7). Consider the following integral for $z > 0$ and $a > 0$ [**2**, p. 418]

$$\int_0^\infty \frac{x \sin ax}{z^2 + x^2} dx = \frac{\pi}{2} e^{-az}.$$

Replace z by $z+k$, taking summation $k=0,1,2,\ldots$ on both sides, using the identity (1.2) and after simplification that

$$\int_0^\infty \sin ax\,(\psi(z+ix)-\psi(z-ix))\,dx = i\pi\frac{e^{-az}}{1-e^{-a}}. \tag{3.9}$$

The solution of the integral (3.9) is found in Ref[**2**, p. 652]. If $f_s(x)$ is expressed in finite terms of sine multiples of x, then the following type of integrals can be evaluated using (3.9).

$$\int_0^\infty f_s(x)\,(\psi(z+ix)-\psi(z-ix))\,dx.$$

For instance, using the integral [**2**, p. 455],

$$\begin{aligned}\int_0^\infty \frac{x\sin 2a\cos^2 bx}{x^2+z^2}dx &= \frac{\pi}{8}\left[2e^{-2az}+e^{-2(a+b)z}-e^{-2(a-b)z}\right], \quad a<b\\ &= \frac{\pi}{8}\left[e^{-4az}+2e^{-2az}\right], \quad a=b.\end{aligned}$$

It is easy to find that

$$\begin{aligned}&\int_0^\infty \sin 2ax\cos^2 bx\,(\psi(z+ix)-\psi(z-ix))\,dx\\ &\quad= \frac{i\pi}{4}\left[\frac{2e^{-2az}}{1-e^{-2a}}+\frac{e^{-2(a+b)z}}{1-e^{-2(a+b)}}-\frac{e^{-2(a-b)z}}{1-e^{-2(a-b)}}\right], \qquad a<b\\ &\quad= \frac{i\pi}{4}\left[\frac{e^{-4az}}{1-e^{-4a}}+2\frac{e^{-2az}}{1-e^{-2a}}\right], \qquad a=b.\end{aligned}$$

If $z=1/2$, then

$$\begin{aligned}&\int_0^\infty \sin 2ax\cos^2 bx\tanh x dx\\ &\quad= \frac{1}{4}\left[\frac{2e^{-a}}{1-e^{-2a}}+\frac{e^{-(a+b)}}{1-e^{-2(a+b)}}-\frac{e^{-(a-b)}}{1-e^{-2(a-b)}}\right], \qquad a<b\\ &\quad= \frac{1}{4}\left[\frac{e^{-2a}}{1-e^{-4a}}+2\frac{e^{-a}}{1-e^{-2a}}\right], \qquad a=b.\end{aligned}$$

The similar integrals involving powers, trigonometric functions and $\tanh x$ are found in sections 3.98-3.99 and integrals involving psi functions are listed in sections 6.46-6.47 of Ref [**2**].

4. Conclusion.

The integrals involving combinations of powers, exponentials, hyperbolic and trigonometric functions are evaluated by connecting with infinite products and infinite series. Some integrals are derived through identities of psi function. Most of integrals given here are not available in the classical tables by Gradshteyn and Ryzhik [**2**]. Some of them cannot be expressed in closed form using a symbolic language.

References

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