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On certain generalized polynomial system associated with Humbert polynomials

Ritu Agarwal[a] and Hari Singh Parihar[b]

Abstract. The object of this paper is to present a unification and generalization of a class of Humbert polynomials which generalizes the well known class of Gegenbauer, Legendre, Pincherle, Horadam, Kinney, Horadam-Pethe, Gould, Milovanovic-Dordevic, Pathan-Khan and many not so well known polynomials. We shall give some basic relations involving the generalized Humbert polynomials and then take up several generating functions, hypergeometric representations and expansions in series of some relatively more familiar polynomials of Legendre, Gegenbauer, Hermite and Laguerre. The results obtained are of general character and include the investigations carried out by several authors including Dilcher, Horadam, Sinha, Shreshtha, Milovanovic-Dordevic and Pathan-Khan.

1. Introduction

A systematic study of an interesting generalization of Humbert, Gegenbauer and several other polynomial systems is presented and defined by Gould [**3**]

$$(c - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, c)t^n \tag{1.1}$$

where m is a positive integer and other parameters are unrestricted in general. For the special case of (1.1), including Gegenbauer, Legendre, Techebycheff, Princherle, Kinney and Humbert polynomials, see Gould [**3**].

Milovanovic and Dordevic [**10**] considered the polynomials $\{p^{\lambda}_{n,m}\}_{n=0}^{\infty}$ defined by the generating function

$$G^{\lambda}_m(x, t) = (1 - 2xt + t^m)^{-\lambda} = \sum_{n=0}^{\infty} p^{\lambda}_{n,m} t^n \tag{1.2}$$

where $m \in N$ and $\lambda > -\frac{1}{2}$.

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The explicit form of the polynomial $P^{\lambda}_{n,m}(x)$ is

$$p^{\lambda}_{n,m}(x) = \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^k (\lambda)_{n-(m-1)k} (2x)^{n-mk}}{k!(n-mk)!} \tag{1.3}$$

Sinha [**14**] considered another set of polynomials denoted by $S^{\nu}_n(x)$ and which is defined by the following generating function:

$$[1 - 2xt + t^2(2x-1)]^{-\nu} = \sum_{n=0}^{\infty} S^{\nu}_n(x) t^n \tag{1.4}$$

which is precisely a generalization of $S_n(x)$ defined and studied by Shreshtha [**13**].

Recently, Pathan and Khan [**11**, p.54, Eq. (1.5)], have defined and studied the following polynomial system

$$[c - axt + bt^m(2x-1)^d]^{-\nu} = \sum_{n=0}^{\infty} P^{\nu}_{n,m,a,b,c,d}(x) t^n = \sum_{n=0}^{\infty} \Theta_n(x) t^n \tag{1.5}$$

Here we introduce and study a new polynomial system which provides a generalization (and unification) of various polynomials mentioned above. This set of polynomials is defined by the following generating function:

$$\begin{aligned} &[c - ax^{\mu}t + bt^m(px^{\nu} + qx + s)^{\rho}]^{-\omega} \\ &= \sum_{n=0}^{\infty} P^{\omega}_n(m, a, b, c, \rho, p, q, r, \mu, \nu)(x) t^n = \sum_{n=0}^{\infty} \Phi_n(x) t^n \end{aligned} \tag{1.6}$$

where $m, \mu, \nu \in N$(the set of natural numbers), $\rho \in N \cup \{0\}$ and other parameters are unrestricted in general.

In this paper, we shall give some basic relations involving the generalized Humbert polynomials $\Phi_n(x)$ and then take up several operational results, series representation, hypergeometric representations and expansions of $\Phi_n(x)$ in series of other polynomials which are best stated in terms of the generalized polynomials. The relationship with other polynomial systems are also developed. Definition (1.6) of $\Phi_n(x)$ is general enough to account for many of polynomials involved in generalized potential problems [**6**], [**7**], [**8**]. The particular cases of the generalized Humbert polynomials $\Phi_n(x)$ are also discussed.

2. Relations with other polynomial systems

On comparing the new polynomial system (1.6) with the other polynomial systems , we find the following relationships hold:

Liouville(1722)

$$P^{1/2}_n(2, q, -1, p^2, 1, 0, 0, 1, 1, 0)(x) = f_n(p, q) \tag{2.1}$$

Legendre(1784)

$$P_n^{1/2}(2,x,1,1,1,0,0,1,1,0)(x) = P_n(x) \tag{2.2}$$

Tchebycheff(1859)

$$P_n^{1}(2,x,1,1,1,0,0,1,1,0)(x) = U_n(x) \tag{2.3}$$

Gegenbauer(1874)

$$P_n^{\delta}(2,x,1,1,1,0,0,1,1,0)(x) = C_n^{\delta}(x) \tag{2.4}$$

Pincherle(1890)

$$P_n^{1/2}(3,x,1,1,1,0,0,1,1,0)(x) = P_n(x) \tag{2.5}$$

Humbert(1921)

$$P_n^{\delta}(m,x,1,1,1,0,0,1,1,0)(x) = \Pi_{n,m}^{\delta}(x) \tag{2.6}$$

Kinney(1963)

$$P_n^{1/m}(m,x,1,1,1,0,0,1,1,0)(x) = P_n(m,x) \tag{2.7}$$

Gould(1965)

$$P_n^{-p}(m,a,-y,c,1,0,0,1,1,0)(x) = P_n(m,x,y,p,c) \tag{2.8}$$

Horadam and Pethe(1981)

$$P_n^{\omega}(3,2,1,1,1,0,0,1,1,0)(x) = P_{n,3}^{\omega}(x) \tag{2.9}$$

Horadam(1985)

$$P_n^{\omega}(1,2,1,1,1,0,0,1,1,0)(x) = P_{n,1}^{\omega}(x) \tag{2.10}$$

Milovanovic and Dordevic(1987)

$$P_n^{\omega}(m,2,1,1,1,0,0,1,1,0)(x) = P_{n,m}^{\omega}(x) \tag{2.11}$$

Sinha(1989)

$$P_n^{\omega}(2,2,1,1,1,2,0,-1,1,1)(x) = S_n^{\omega}(x) \tag{2.12}$$

Pathan and Khan(1997)

$$P_n^{\omega}(m,a,b,c,\rho,2,0,1,1,1)(x) = P_n^{\omega}(m,a,b,c,\rho)(x) \tag{2.13}$$

3. Finite series representations for $\Phi_n(x)$

In this section, we obtain the following two finite series representations for $\Phi_n(x)$, viz. (i)

$$\Phi_n(x) = \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-b)^k(\omega)_{n-(m-1)k}(ax^{\mu})^{n-mk}(px^{\nu}+qx+s)^{\rho k}c^{-\omega-n+(m-1)k}}{k!(n-mk)!} \tag{3.1}$$

and

(ii)

$$\Phi_n(x) = \sum_{k=0}^{\left[\frac{(n-(m-2)r)}{2}\right]} \sum_{r=0}^{k} \frac{(\omega)_k c^{-\omega-n+(m-2)r}(2\omega+2k)_{n-2k-(m-2)r}}{k!r!(n-2k-(m-2)r)!} (-k)_r \left(\frac{ax^\mu}{2}\right)^{n-(m-2)r} \left\{\frac{4bc(px^\nu+qx+s)^\rho}{a^2x^{2\mu}}\right\}^r \tag{3.2}$$

Proof. of (3.1):

By using binomial expansion in (1.6), we have

$$\sum_{n=0}^{\infty} \Phi_n(x)t^n = c^{-\omega} \sum_{n=0}^{\infty} \frac{(\omega)_n}{n!} \left(\frac{ax^\mu t - bt^m(px^\nu+qx+s)^\rho}{c}\right)^n. \tag{3.3}$$

Also, we know that

$$(t+v)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} t^k v^{n-k}. \tag{3.4}$$

By using (3.4) in (3.3), we get

$$\sum_{n=0}^{\infty} \Phi_n(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{c^{-\omega-n}(\omega)_n}{k!(n-k)!}(-b)^k(ax^\mu)^{n-k}(px^\nu+qx+s)^{\rho k}t^{n+(m-1)k}$$

which on applying series manipulation [**12**, p.57, Eq.(2)]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k) \tag{3.5}$$

gives

$$\sum_{n=0}^{\infty} \Phi_n(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{c^{-\omega-n-k}(\omega)_{n+k}}{k!n!}(-b)^k(ax^\mu)^n(px^\nu+qx+s)^{\rho k}t^{n+mk}. \tag{3.6}$$

Again, using series manipulation

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{m}\right]} A(k,n-mk) \tag{3.7}$$

in (3.6), we have

$$\sum_{n=0}^{\infty} \Phi_n(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{c^{-\omega-n+(m-1)k}(\omega)_{n-(m-1)k}}{k!(n-mk)!} (-b)^k(ax^\mu)^{n-mk}(px^\nu+qx+s)^{\rho k}t^n. \tag{3.8}$$

On comparing the coefficients of t^n, on both sides of (3.8), we get the finite series representation of (3.1) for $\Phi_n(x)$.

Proof of (3.2): From (1.6), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\Phi_n(x)t^n &= c^{-\omega}\left[\left(1-\frac{ax^{\mu}t}{2c}\right)^2-\left(\frac{ax^{\mu}t}{2c}\right)^2+\frac{bt^m}{c}(px^{\nu}+qx+s)^{\rho}\right]^{-\omega}\\ &= c^{-\omega}\left(1-\frac{ax^{\mu}t}{2c}\right)^{-2\omega}\left[1-\frac{\left(\frac{ax^{\mu}t}{2c}\right)^2-\frac{bt^m}{c}(px^{\nu}+qx+s)^{\rho}}{\left(1-\frac{ax^{\mu}t}{2c}\right)^2}\right]^{-\omega}\end{aligned} \tag{3.9}$$

Using Binomial expansion in (3.9), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\Phi_n(x)t^n &= c^{-\omega}\sum_{k=0}^{\infty}\frac{(\omega)_k}{k!}\left(1-\frac{ax^{\mu}t}{2c}\right)^{-2\omega-2k}\left(\frac{ax^{\mu}t}{2c}\right)^{2k}\\ &\quad .\left[1-\frac{4bct^m(px^{\nu}+qx+s)^{\rho}}{a^2t^2x^{2\mu}}\right]^k\\ &= c^{-\omega}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\omega)_k(2\omega+2k)_n}{k!n!}\left(\frac{ax^{\mu}t}{2c}\right)^{n+2k}\\ &\quad .\left[1-\frac{4bct^{m-2}(px^{\nu}+qx+s)^{\rho}}{a^2x^{2\mu}}\right]^k\\ &= c^{-\omega}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{k}\frac{(\omega)_k(2\omega+2k)_n(-k)_r}{k!n!r!}\left(\frac{ax^{\mu}}{2c}\right)^{n+2k}\\ &\quad .\left(\frac{4bc(px^{\nu}+qx+s)^{\rho}}{a^2x^{2\mu}}\right)^r t^{(n+2k+(m-2)r)}\end{aligned} \tag{3.10}$$

Replacing n by $n-2k-(m-2)r$ in (3.10), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\Phi_n(x)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{(n-(m-2)r)}{2}\right]}\sum_{r=0}^{k}\frac{c^{-\omega-n+(m-2)r}(\omega)_k}{k!r!(n-2k-(m-2)r)!}\\ &\quad .(2\omega+2k)_{n-2k-(m-2)r}(-k)_r\left(\frac{ax^{\mu}}{2}\right)^{n-(m-2)r}\\ &\quad .\left(\frac{4bc(px^{\nu}+qx+s)^{\rho}}{a^2x^{2\mu}}\right)^r t^n\end{aligned} \tag{3.11}$$

On comparing the coefficients of t^n on both sides of (3.11), we get (3.2).

4. Hypergeometric representations for $\Phi_n(x)$

(4.1)
$$\Phi_n(x) = \frac{(\omega)_n c^{-\omega-n}(ax^\mu)^n}{n!}$$
$$\cdot {}_mF_{m-1}\left[\begin{array}{c} \frac{-n}{m}, \frac{-n+1}{m}, ..., \frac{-n+m-1}{m}; \\ \frac{-\omega-n+1}{m-1}, \frac{-\omega-n+2}{m-1}, ..., \frac{-\omega-n+m-1}{m-1}; \end{array} \frac{m^m bc^{m-1}(px^\nu+qx+s)^\rho}{(m-1)^{(m-1)}(ax^\mu)^m}\right]$$

for $m \geqslant 2$.

Proof. Since we know that [**12**, p.58, Eq.(2)]

$$(\alpha)_{n-k} = \frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k}, 0 \leqslant k \leqslant n. \tag{4.2}$$

Using (4.2) in (3.1), we have

$$\Phi_n(x) = \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^{(m-1)k}(-b)^k(\omega)_n(ax^\mu)^{n-mk}(px^\nu+qx+s)^{\rho k}c^{-\omega-n+(m-1)k}}{(1-\omega-n)_{(m-1)k}k!(n-mk)!} \tag{4.3}$$

Now, using the well-known results

$$(n-mk)! = \frac{(-1)^{mk}n!}{(-n)_{mk}}, 0 \leqslant mk \leqslant n, \tag{4.4}$$

$$(-n)_{mk} = m^{mk}\prod_{s=1}^{m}\left(\frac{-n+s-1}{m}\right)_k \tag{4.5}$$

and

$$(1-\nu-n)_{(m-1)k} = (m-1)^{(m-1)k}\prod_{p=1}^{(m-1)}\left(\frac{-\nu-n+p}{m-1}\right)_k, k = 0, 1, 2, ... \tag{4.6}$$

in (4.3), we find that

(4.7)
$$\Phi_n(x) = \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(m)^{mk}(b)^k c^{-\omega-n+(m-1)k}(\omega)_n \prod_{r=1}^{m}\left(\frac{-n+r-1}{m}\right)_k}{\prod_{r=1}^{m-1}\left(\frac{-\omega-n+r}{m-1}\right)_k (m-1)^{(m-1)k}k!n!}$$
$$.(ax^\mu)^{n-mk}(px^\nu+qx+s)^{\rho k}$$

After a little simplification in the right hand side of (4.7), we get (4.1). □

5. Generating functions for $\Phi_n(x)$

(i)

$$\sum_{n=0}^{\infty}\frac{\Phi_n(x)t^n}{(\omega)_n}=\sum_{n=0}^{\infty}\frac{c^{-\omega-n}(ax^{\mu}t)^n}{n!} \cdot {}_1F_m\left[\begin{matrix}\omega+n; \\ \frac{\omega+n}{m},\frac{\omega+n+1}{m},...,\frac{\omega+n+m-1}{m};\end{matrix}\ \frac{-bt^m(px^{\nu}+qx+s)^{\rho}}{cm^m}\right] \tag{5.1}$$

(ii)

$$\sum_{n=0}^{\infty}\frac{\Phi_n(x)t^n(e)_n}{(\omega)_n}=\sum_{n=0}^{\infty}\frac{c^{-\omega-n}(ax^{\mu}t)^n(e)_n}{n!} \cdot {}_{m+1}F_m\left[\begin{matrix}\omega+n,\frac{e+n}{m},\frac{e+n+1}{m},...,\frac{e+n+m-1}{m}; \\ \frac{\omega+n}{m},\frac{\omega+n+1}{m},...,\frac{\omega+n+m-1}{m};\end{matrix}\ \frac{-bt^m(px^{\nu}+qx+s)^{\rho}}{cm^m}\right] \tag{5.2}$$

(iii)

$$\sum_{n=0}^{\infty}\frac{\Phi_n(x)t^n}{(2\omega)_n}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{c^{-\omega-n-2k}(-k)_r\left(\frac{ax^{\mu}t}{2}\right)^{n+2k}}{n!k!r!2^{2k}(\omega+\frac{1}{2})_k(2\omega+n+2k)_{(m-2)r}} \cdot\left(\frac{4bct^{m-2}(px^{\nu}+qx+s)^{\rho}}{a^2x^{2\mu}}\right)^r \tag{5.3}$$

and

(iv)

$$\sum_{n=0}^{\infty}\frac{\Phi_n(x)t^n(e)_n}{(2\omega)_n}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{c^{-\omega-n-2k}(-k)_r\left(\frac{ax^{\mu}t}{2}\right)^{n+2k}}{n!k!r!2^{2k}(2\omega+n+2k)_{(m-2)r}} .(e)_{n+2k}\frac{(e+n+2k)_{(m-2)r}}{(\omega+\frac{1}{2})_k}\left(\frac{4bct^{m-2}(px^{\nu}+qx+s)^{\rho}}{a^2x^{2\mu}}\right)^r \tag{5.4}$$

where e is an arbitrary number, may be a complex number.

Proof. of (5.1):

$$\sum_{n=0}^{\infty}\frac{\Phi_n(x)t^n}{(\omega)_n}=\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{m}\right]}\frac{(-b)^k(\omega)_{n-(m-1)k}c^{-\omega-n+(m-1)k}}{(\omega)_nk!(n-mk)!} .(ax^{\mu})^{n-mk}(px^{\nu}+qx+s)^{\rho k}t^n \tag{5.5}$$

By using series manipulation [**15**, p.101, Eq.(6)], we have

$$\sum_{n=0}^{\infty}\frac{\Phi_n(x)t^n}{(\omega)_n}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^k(\omega)_{n+k}c^{-\omega-n-k}}{(\omega)_{n+mk}k!n!} .(ax^{\mu})^n(px^{\nu}+qx+s)^{\rho k}t^{n+mk} \tag{5.6}$$

Using the identity

$$(\lambda)_{m+n}=(\lambda)_m(\lambda+m)_n$$

and the well-known Gauss's multiplication theorem in (5.6), we find that

$$\sum_{n=0}^{\infty}\frac{\Phi_n(x)t^n}{(\omega)_n}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^k(\omega+n)_k c^{-\omega-n-k}}{k!n!(m)^{mk}\prod_{p=1}^{m}\left(\frac{\omega+n+p-1}{m}\right)_k} .(ax^{\mu})^n(px^{\nu}+qx+s)^{\rho k}t^{n+mk} \tag{5.7}$$

By summing the k^{th} series, we easily arrive at the right hand side of (5.1). □

The proof of (5.2) is similar to the proof of (5.1). (5.3) and (5.4) can be made on similar lines of (5.1) using (3.2).

6. Expansion of $\Phi_n(x)$ in series of polynomials

Expansion of $\Phi_n(x)$ in series of Legendre, Geganbauer, Hermite and Laguerre polynomials relevant to our present investigation are given by

(i)

$$\Phi_n(x)=\sum_{k=0}^{\left[\frac{(n-(m-2)r)}{2}\right]}\sum_{r=0}^{k}\frac{(-1)^k(\omega)_{n+(m-1)(r-k)}\ c^{-\omega-n+(m-1)(k-r)}}{k!r!(3/2)_{(n-mk+(m-1)r)}} .(-k)_r\left\{b(px^{\nu}+qx+s)^{\rho}\right\}^{k-r}\left\{2n+2r(m-2)-2mk+1\right\} .P_{n+(m-2)r-mk}\left(\frac{ax^{\mu}}{2}\right), \tag{6.1}$$

where $P_n(x)$ is Legendre Polynomial.

(ii)

$$\Phi_n(x)=\sum_{k=0}^{\left[\frac{(n-(m-2)r)}{2}\right]}\sum_{r=0}^{k}\frac{(-1)^k(\omega)_{n+(m-1)(k-r)}c^{-\omega-n+(m-1)(k-r)}}{k!r!(\omega)_{(n+1-mk+(m-1)r)}} .(-k)_r\left\{b(px^{\nu}+qx+s)^{\rho}\right\}^{k-r}\left\{\omega+n-2r-m(k-r)\right\} .C^{\omega}_{n-2r-m(k-r)}\left(\frac{ax^{\mu}}{2}\right), \tag{6.2}$$

where $C^{\omega}_n(x)$ stands for Gegenbauer polynomial.

(iii)

$$\Phi_n(x) = \sum_{k=0}^{\left[\frac{(n-(m-2)r)}{2}\right]} \sum_{r=0}^{k} \frac{(-1)^k(\omega)_{n-(m-1)(k-r)}c^{-\omega-n+(m-1)(k-r)}}{k!r!(n-2r-m(k-1))!} \tag{6.3}$$
$$.(-k)_r \left\{b(px^\nu+qx+s)^\rho\right\}^{k-r} H_{n-2r-m(k-r)}\left(\frac{ax^\mu}{2}\right),$$

where $H_n(x)$ stands for Hermite polynomial.

(iv)

$$\Phi_n(x) = \sum_{r=0}^{\left[\frac{(n-(m-2)r)}{2}\right]} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^{k+r}(\omega)_{n-(m-1)k}c^{-\omega-n+(m-1)k}}{k!(n-r-mk)!(1+\alpha)_r} \tag{6.4}$$
$$.(1+\alpha)_n 2^{n-mk} \left\{b(px^\nu+qx+s)^\rho\right\}^k L_n^{(\alpha)}\left(\frac{ax^\mu}{2}\right),$$

where $L_n^{(\alpha)}(x)$ stands for Laguerre polynomial.

Proof. of (6.1)
From (3.1), we have

$$\sum_{n=0}^{\infty}\Phi_n(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-b)^k(\omega)_{n-(m-1)k}c^{-\omega-n+(m-1)k}}{k!(n-mk)!} \tag{6.5}$$
$$.(ax^\mu)^{n-mk}(px^\nu+qx+s)^{\rho k}t^n$$

Using a known result [**15**, p.101, Eq.(6)] in (6.5), we get

$$\sum_{n=0}^{\infty}\Phi_n(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-b)^k(\omega)_{n+k}c^{-\omega-n-k}}{k!n!}(ax^\mu)^n \tag{6.6}$$
$$(px^\nu+qx+s)^{\rho k}t^{n+mk}.$$

Again on using the result [**12**, p.181, Eq.(4)] in (6.6), we get

$$\sum_{n=0}^{\infty}\Phi_n(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-b)^k(\omega)_{n+k}c^{-\omega-n-k}}{k!r!(3/2)_{n-r}} \tag{6.7}$$
$$.(px^\nu+qx+s)^{\rho k}(2n-4r+1)P_{n-2r}\left(\frac{ax^\mu}{2}\right)t^{n+mk}.$$

Using the results [**12**, p.57, Eq.(8) and p.56, Eq.(1)] in (6.7), we have

$$\sum_{n=0}^{\infty}\Phi_n(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{k} \frac{(-b)^{k-r}(\omega)_{n+r+k}c^{-\omega-n-r-k}}{(k-r)!r!(3/2)_{n+r}} \tag{6.8}$$
$$.(px^\nu+qx+s)^{\rho(k-r)}(2n+1)P_n\left(\frac{ax^\mu}{2}\right)t^{n-(m-2)r+mk}.$$

Now replacing n by $n+(m-2)r-mk$, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\Phi_n(x)t^n=&\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n+(m-2)r}{m}\right]}\sum_{r=0}^{k}\frac{(-b)^{k-r}(\omega)_{n-(m-1)(k-r)}}{(k-r)!r!(3/2)_{n+(m-1)r-mk}}\\&.(px^{\nu}+qx+s)^{\rho(k-r)}(1-2mk+2n+2(m-2)r)\\&.c^{-\omega-n+(m-1)(k-r)}P_{n+(m-2)r-mk}\left(\frac{ax^{\mu}}{2}\right)t^n.\end{aligned}\tag{6.9}$$

On using (4.4) in (6.9), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\Phi_n(x)t^n=&\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n+(m-2)r}{m}\right]}\sum_{r=0}^{k}\frac{(-1)^{k}(\omega)_{n-(m-1)(k-r)}}{(k-r)!r!(3/2)_{n+(m-1)r-mk}}\\&.(-k)_r\{b(px^{\nu}+qx+s)^{\rho}\}^{(k-r)}(1-2mk+2n+2(m-2)r)\\&.c^{-\omega-n+(m-1)(k-r)}P_{n+(m-2)r-mk}\left(\frac{ax^{\mu}}{2}\right)t^n.\end{aligned}\tag{6.10}$$

On comparing the coefficients of t^n, we obtain (6.1). □

In a similar manner, results (6.2) to (6.4) can be proved by using [**12**, p.283, Eq.(36), p.194, Eq.(4) and p.207, Eq.(2)] respectively.

7. Particular Cases

(1) For $a=m=p=2$, $b=c=\rho=\mu=\nu=1$, $q=0$, $r=-1$, (3.1) gives

$$S_n^{\omega}(x)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(\omega)_{n-k}(2x)^{n-2k}(2x-1)^k}{k!(n-2k)!}\tag{7.1}$$

where $S_n^{\omega}(x)$ stands for Sinha's polynomial defined by (1.4).

(2) Making same substitutions in (3.2), we get

$$\begin{aligned}S_n^{\omega}(x)&=\sum_{k=0}^{\left[\frac{n}{2}\right]}\sum_{r=0}^{k}\frac{(\omega)_k(2\omega+2k)_{n-2k}(-k)_r}{k!r!(n-2k)!}x^n\left(\frac{2x-1}{x^2}\right)^r\\&=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(\omega)_k(2\omega+2k)_{n-2k}}{k!(n-2k)!}x^n\left(1-\frac{2x-1}{x^2}\right)^k\\&=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{\Gamma(2\omega)\Gamma(\omega+k)\Gamma(2\omega+n)}{\Gamma(2\omega+2k)\Gamma(\omega)\Gamma(2\omega)k!(n-2k)!}x^{n-2k}\left(x-1\right)^{2k}\end{aligned}$$

Now, using the well-known Legendre's duplication formula, we finally get

$$S_n^{\omega}(x)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(2\omega)_n}{2^{2k}(\omega+\frac{1}{2})_k k!(n-2k)!}x^{n-2k}\left(x-1\right)^{2k}\tag{7.2}$$

The results (7.1)and (7.2) are due to Sinha [**14**, p.439, Eqs. (3) and (4)].

(3) Setting $a = m$, $b = c = \mu = 1$, $\rho = 0$ in (3.1) and (3.2), we get

$$h_{n,m}^{\omega}(x) = \sum_{k=0}^{[\frac{n}{m}]} \frac{(-1)^k (\omega)_{n-(m-1)k} (mx)^{n-mk}}{k!(n-mk)!} \tag{7.3}$$

and

$$h_{n,m}^{\omega}(x) = \sum_{k=0}^{[\frac{(n-(m-2)r)}{2}]} \sum_{r=0}^{k} \frac{(\omega)_k (2\omega+2k)_{n-2k-(m-2)r}}{k!r!(n-2k-(m-2)r)!} (-k)_r \left(\frac{mx}{2}\right)^{n-mr} \tag{7.4}$$

where $h_{n,m}^{\omega}(x)$ stands for Humbert polynomials.

(4) Substituting $m = 3$, $\omega = \frac{1}{2}$ in (7.3)and (7.4), we get

$$P_n(x) = \sum_{k=0}^{[\frac{n}{3}]} \frac{(-1)^k (\frac{1}{2})_{n-2k} (3x)^{n-3k}}{k!(n-3k)!} \tag{7.5}$$

and

$$P_n(x) = \sum_{k=0}^{[\frac{(n-r)}{2}]} \sum_{r=0}^{k} \frac{(\frac{1}{2})_k (1+2k)_{n-2k-r}}{k!r!(n-2k-r)!} (-k)_r \left(\frac{3x}{2}\right)^{n-3r} \tag{7.6}$$

where $P_n(x)$ is Pincherle polynomials [**6**].

(5) For $a = m = 2$, $\omega = \frac{1}{2}$, (7.3)and (7.4) give finite series representation of Legendre polynomials [**12**, p.164, Eq.(1)].

(6) Putting $m = 2$ in (7.3), we get

$$C_n^{\omega}(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (\omega)_{n-k} (2x)^{n-2k}}{k!(n-2k)!} \tag{7.7}$$

(7) Putting $m = 2$ in (7.4), we get

$$\begin{aligned} C_n^{\omega}(x) &= \sum_{k=0}^{[\frac{n}{2}]} \sum_{r=0}^{k} \frac{(\omega)_k (2\omega+2k)_{n-2k}}{k!r!(n-2k)!} (-k)_r \, (x)^{n-2r} \\ &= \sum_{k=0}^{[\frac{n}{2}]} \frac{(\omega)_k (2\omega+2k)_{n-2k}}{k!(n-2k)!} x^n \left(1 - \frac{1}{x^2}\right)^k \\ &= \sum_{k=0}^{[\frac{n}{2}]} \frac{\Gamma(2\omega)\Gamma(\omega+k)\Gamma(2\omega+n)}{\Gamma(2\omega+2k)\Gamma(\omega)\Gamma(2\omega)k!(n-2k)!} x^{n-2k} \left(x^2-1\right)^k \end{aligned} \tag{7.8}$$

Now, using the well-known Legendre's duplication formula [**12**, p.23, Eq. (19)], we finally get

$$C_n^{\omega}(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(2\omega)_n}{(\omega+\frac{1}{2})_k k!(n-2k)!} x^{n-2k} \left(x^2-1\right)^k \tag{7.9}$$

where $C_n^\omega(x)$ is the well-known Gegenbauer polynomial.

(8) In (3.1), setting $a=c=1$, $\rho=0$, $m=2$ and replacing b and x by λz^2 and $1+z+z^2$ respectively, we get

$$f_n^{\lambda,\omega}(z)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(\omega)_{n-k}}{k!(n-2k)!}(1+z+z^2)^{n-2k}\left(-\lambda z^2\right)^k \tag{7.10}$$

Note that $f_n^{\lambda,\omega}(z)$ is related to $C_n^\omega(z)$ by the relation (see, e.g. [**11**, p.57])

$$f_n^{\lambda,\omega}(z)=\lambda^{n/2}z^nC_n^\omega\left(\frac{1+z+z^2}{2\sqrt{\lambda}z}\right)$$

(9) Making the same substitutions in (3.2) as mentioned above, we get after a little simplification

$$f_n^{\lambda,\omega}(z)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(2\omega)_n\left(\frac{(1+z+z^2)^\mu}{2}\right)^n}{2^{2k}(\omega+\frac{1}{2})_k k!(n-2k)!}\left(1-\frac{4\lambda z^2}{(1+z+z^2)^{2\mu}}\right)^k \tag{7.11}$$

(10) If we set $a=m=p=2$, $b=c=\rho=\nu=1$ and $q=0$, $r=-1$ in (4.1), we arrive at a known result [**14**, p.442, Eq. (12)].

(11) By setting $a=m$, $b=c=1$, $\rho=0$ and $\mu=1$ in (4.1), we get

$$\begin{aligned}h_{n,m}^\omega(x)&=\frac{(\omega)_n(mx)^n}{n!}\\&\cdot {}_mF_{m-1}\left[\begin{matrix}\frac{-n}{m},\frac{-n+1}{m},...,\frac{-n+m-1}{m};\\ \frac{-\omega-n+1}{m-1},\frac{-\omega-n+2}{m-1},...,\frac{-\omega-n+m-1}{m-1};\end{matrix}\ \frac{1}{(m-1)^{(m-1)}x^m}\right]\end{aligned} \tag{7.12}$$

which is hypergeometric representation of Humbert polynomials.

(12) For $m=2$, (7.12) gives hypergeometric representation of Gegenbauer polynomial

$$C_n^\omega(x)=\frac{(\omega)_n(2x)^n}{n!}\cdot{}_2F_1\left[\frac{-n}{2},\frac{-n+1}{2};-\omega-n+1;\frac{1}{x^2}\right] \tag{7.13}$$

(13) Setting $a=c=\mu=1$, $m=2$, $\rho=0$ in (4.1) and replacing b and x by λz^2 and $1+z+z^2$ respectively, we get

$$\begin{aligned}f_n^{\lambda,\omega}(z)&=\frac{(\omega)_n(1+z+z^2)^n}{n!}\\&\cdot{}_2F_1\left[\frac{-n}{2},\frac{-n+1}{2};-\omega-n+1;\frac{4\lambda z^2}{(1+z+z^2)^2}\right]\end{aligned} \tag{7.14}$$

(14) For $a=2$, $b=c=1$, $\rho=0$ and $\mu=1$, (5.1) gives the generating function for $P_{n,m}^\omega(x)$ defined by (1.2):

$$\sum_{n=0}^\infty\frac{P_{n,m}^\omega(x)t^n}{(\omega)_n}=\sum_{n=0}^\infty\frac{(2xt)^n}{n!}\cdot{}_1F_m\left[\begin{matrix}\omega+n;\\ \frac{\omega+n}{m},\frac{\omega+n+1}{m},...,\frac{\omega+n+m-1}{m};\end{matrix}\ \frac{-t^m}{m^m}\right] \tag{7.15}$$

(15) In (5.2), setting $a = m = p = 2$, $b = c = \rho = \mu = \nu = 1$, $q = 0$, $r = -1$, we get the generating function for $S_n^\omega(x)$:

$$\sum_{n=0}^{\infty} \frac{S_n^\omega(x) t^n (e)_n}{(\omega)_n} = \sum_{n=0}^{\infty} \frac{(2xt)^n (e)_n}{n!} \cdot {}_3F_2 \left[\begin{array}{l} \omega + n, \frac{e+n}{2}, \frac{e+n+1}{2}; \\ \frac{\omega+n}{2}, \frac{\omega+n+1}{2}; \end{array} - t^2(2x-1) \right] \tag{7.16}$$

For $e = \omega$, (7.16) reduces to a known result of Sinha [**14**, p.439, Eq.(2)].

(16) For $a = m$, $b = c = \mu = 1$, $\rho = 0$, (5.2) gives the generating function for Humbert polynomials:

$$\sum_{n=0}^{\infty} \frac{h_{n,m}^\omega(x) t^n (e)_n}{(\omega)_n} = \sum_{n=0}^{\infty} \frac{(mxt)^n (e)_n}{n!} \cdot {}_1F_m \left[\begin{array}{l} \omega + n, \frac{e+n}{m}, \frac{e+n+1}{m}, \ldots, \frac{e+n+m-1}{m}; \\ \frac{\omega+n}{m}, \frac{\omega+n+1}{m}, \ldots, \frac{\omega+n+m-1}{m}; \end{array} - t^m \right] \tag{7.17}$$

(17) For $m = 3$ and $\omega = 1/2$, (7.17) gives generating function for Pincherle polynomials $P_n(x)$

$$\sum_{n=0}^{\infty} \frac{P_n(x) t^n (e)_n}{(1/2)_n} = \sum_{n=0}^{\infty} \frac{(3xt)^n (e)_n}{n!} \cdot {}_4F_3 \left[\begin{array}{l} \frac{1}{2} + n, \frac{e+n}{3}, \frac{e+n+1}{3}, \frac{e+n+2}{3}; \\ \frac{\frac{1}{2}+n}{3}, \frac{\frac{3}{2}+n}{3}, \frac{\frac{5}{2}+n}{3}; \end{array} - t^3 \right] \tag{7.18}$$

If we set $\mu = \nu = 1$, $p = 2$, $q = 0$, $r = -1$ in (3.1), (3.2), (4.1), (5.1) to (5.4) (6.1) to (6.4), we get the results established by Pathan and Khan [**11**].

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[a]Institute of Engineering and Technology,
JK Lakshmipat University,
Laliya Ka Vaas, Mahapura,
Jaipur, Rajasthan,
India.
E-mail address: rituagarwal@jklu.edu.in

[b]Central University of Rajasthan,
Bandarsindri, Kishangarh,
Ajmer, Rajasthan,
India.
E-mail address: harisingh.p@rediffmail.com