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The integrals in Gradshteyn and Ryzhik. Part 34: Bessel functions

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Abstract. This work explores the application of the method of brackets to evaluate definite integrals. We show the effectiveness of this method through numerous examples involving Bessel function taken from the classical table of integrals by I. S. Gradshteyn and I. M. Ryzhik

1. Introduction

The table of integrals by Gradshetyn and Ryzhik [**18**] is one the most used compendiums of integrals. A project dedicated to producing proofs of these entries began in [**20**] was motivated by an incorrect entry appearing in this table. Indeed, the 6^{th} edition [**16**] contains, as entry 3.248.5, the beautiful evaluation

$$\int_0^\infty \frac{dx}{(1+x^2)^{3/2}\left[\varphi(x)+\sqrt{\varphi(x)}\right]^{1/2}} = \frac{\pi}{2\sqrt{6}}, \tag{1.1}$$

where $\varphi(x) = 1 + \dfrac{4x^2}{3(1+x^2)^2}$. Unfortunately this evaluation is incorrect. A direct numerical evaluation of the left-hand side gives 0.666377 for the left-hand side is approximately 0.641275. The initial solution to this problem was to `exclude` this entry from the next two editions [**17, 18**]. At this point one of the authors of the current work had become the scientific editor of the table.

The correct value for this entry was found by J. Arias de Reyna [**6**], expressed as the difference of two elliptic integrals. The correct question was found by P. Blaschke

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[**8**] in the form

$$\int_0^\infty \frac{dx}{(1+x^2)^{3/2}\left[\varphi(x)+\sqrt{\varphi(x)^3}\right]^{1/2}} = \frac{\pi}{2\sqrt{6}}, \tag{1.2}$$

The difference between (1.1) and (1.2) is a single number (the extra 3 in the exponent). Transcribing formulas is a delicate subject. A generalization of (1.2) has been discussed by L. Glasser [**11**].

The goal of the present work is to present some proofs of entries in [**18**] containing the Bessel function in the integrand. Some examples have appeared in [**12**], [**14**]. The evaluation of entries containing products of Bessel functions is in preparation [**2**].

Bessel functions are an important family of special functions known for their crucial role in the analysis of wave propagation phenomena. Among the expressions for these functions one finds power series and integral representations. These formulas make them particularly well-suited for the application of the method of brackets. This is a relatively new method of integration, created by I. Gonzalez [**15**] in his Ph. D. Thesis. This method is described in Section 3. The reader is referred to [**7, 22**] for general information about these functions and to [**21**] for a complete treatise on them.

2. Fundamentals of Bessel, Gamma and Beta Functions

This section presents the definitions and fundamental properties of the special functions that are considered in this work.

`Bessel functions` are solutions of the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-\nu^2)y = 0. \tag{2.1}$$

These are divided into two types:

- `Bessel functions of the first kind`, denoted by J_ν, are defined by the power series representation

$$J_\nu(x) = \sum_{k=0}^\infty \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k+\nu}. \tag{2.2}$$

- `Bessel functions of the second kind`, denoted by Y_ν, is defined by

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\pi\nu) - J_{-\nu}(z)}{\sin(\pi\nu)}, \quad \nu \notin \mathbb{Z}. \tag{2.3}$$

The case when ν is an integer is treated by a limiting procedure. This is a second solution of (2.1), linearly independent of J_ν.

In addition to these two main types, one finds the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$. These are variations of the Bessel functions of the first and second kinds, respectively, defined by

$$I_\nu(x) = i^{-\nu}J_\nu(ix) = \sum_{k=0}^\infty \frac{1}{k!\,\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k+\nu} \tag{2.4}$$

and

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu x)}. \tag{2.5}$$

Again, the limit must be used for the case when ν is an integer. The function K_ν admits the integral representation,

$$K_\nu(x) = \int_0^\infty e^{-x \cosh t} \cosh(\nu t) dt. \tag{2.6}$$

appearing as entry **8.432.1** in [**18**].

Through this work we will frequently encounter with the `gamma function`, defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}.\, dt \tag{2.7}$$

This functions is a generalization of factorials to the complex numbers, except for negative integers. Indeed, the value $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$ is an elementary consequence of the functional equation $\Gamma(x+1) = x\Gamma(x)$.

The well-known Gaussian integral

$$\int_{-\infty}^\infty \exp(-x^2)\, dx = \sqrt{\pi}, \tag{2.8}$$

yields the special value $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Some properties of the Gamma function that will be frequently used along this paper are given next:

- The *Euler's reflection formula*:

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)}, \text{ for } x \notin \mathbb{Z}, \tag{2.9}$$

which implies

$$\Gamma(-n+x) = (-1)^{n-1} \frac{\Gamma(-x)\Gamma(x+1)}{\Gamma(n+1-x)}. \tag{2.10}$$

- `Legendre's duplication formula`:

$$\Gamma\left(x + \frac{1}{2}\right) = \frac{\sqrt{\pi}\, \Gamma(2z)}{2^{2x-1}\Gamma(x)}, \tag{2.11}$$

which produces for $x = n \in \mathbb{N}$ the values

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}\,(2n)!}{2^{2n} n!}. \tag{2.12}$$

A companion function is the `beta function`, defined by

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx. \tag{2.13}$$

It also admits the integral representation (see entry 8.380.3 in [**18**])

$$B(a,b) = \int_0^\infty \frac{t^{a-1}\, dt}{(1+t)^{a+b}}, \tag{2.14}$$

which is preferred for the application of the method of brackets since the integral is taken over the interval $(0, \infty]$. Also, it is related with the gamma function via the relation

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{2.15}$$

In particular, this gives the identity

$$B(a, 1-a) = \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi\, a)}. \tag{2.16}$$

More properties, as well as analytic information about the gamma and beta functions can be found at [**1, 5, 9, 22**].

3. The method of brackets

This section presents the main steps in the application of the method of brackets. This is a technique used to compute definite integrals over the half line $[0, \infty)$, with relatively few computations. The primary object in this method is the so-called bracket series, which is produced and evaluated according to a small number of rules, initially derived in a heuristic manner, some of which are placed on solid ground [**4**]. A complete analysis of this method has been given in [**10**]

The key idea is the assignment of the formal symbol

$$\langle a \rangle = \int_0^\infty x^{a-1}\, dx, \tag{3.1}$$

called the `bracket` associated to the divergent integral on the right. The expansion of the integrand as a power series will be combined with the use of this symbol to construct the previously mentioned bracket series.

An important notation while operating with brackets is the use of the symbol

$$\phi_n = \frac{(-1)^n}{\Gamma(n+1)}, \tag{3.2}$$

called the `indicator` of n. The symbol $\phi_{i_1, i_2, \ldots, i_r}$, denotes the product $\phi_{i_1}\phi_{i_2}\cdots\phi_{i_r}$.

The evaluation of the iintegral

$$I(f) = \int_0^\infty f(x)\, dx, \tag{3.3}$$

by the method of brackets consists of the application of a small number of rules. These are of two types: production and evaluation of a `bracket series` (see [**3**] for details). First we present the rules for the production of brackets.

Rule $\mathbf{P_1}$. Assume f has the expansion

$$f(x) = \sum_{n=0}^{\infty} \phi_n a_n x^{\alpha n + \beta - 1}. \tag{3.4}$$

Then $I(f)$ is assigned the bracket series

$$I(f) = \sum_{n=0}^{\infty} \phi_n a_n \langle \alpha n + \beta \rangle. \tag{3.5}$$

Rule $\mathbf{P_2}$. For $\alpha \in \mathbb{R}$, the multinomial power $(a_1 + a_2 + \cdots + a_r)^\alpha$ is assigned the r-dimensional bracket series

$$\sum_{n_1 \geqslant 0} \sum_{n_2 \geqslant 0} \cdots \sum_{n_r \geqslant 0} \phi_{n_1, n_2, \ldots, n_r} a_1^{n_1} \cdots a_r^{n_r} \frac{\langle n_1 + \cdots + n_r - \alpha \rangle}{\Gamma(-\alpha)}. \tag{3.6}$$

Rule $\mathbf{P_3}$. Each representation of an integral by a bracket series has associated an `index of the representation` via

$$\text{index} = \text{ number of sums -number of brackets.} \tag{3.7}$$

REMARK 3.1. It is important to note that the index is attached to each specific representation of the integral and not just to integral itself. The level of difficulty in the analysis of the resulting bracket series increases with the index. Hence, among all representations of an integral as a bracket series, the one with minimal index should be chosen.

Now we introduce the rules for the evaluation of brackets.

Rule $\mathbf{E_1}$. Let a, $b \in \mathbb{R}$. The one-dimensional bracket series (3.5) is assigned the value

$$\sum_{n=0}^{\infty} \phi_n a_n \langle \alpha n + \beta \rangle = \frac{1}{|\alpha|} f(n^*) \Gamma(-n^*), \tag{3.8}$$

where n^* is obtained from the vanishing of the bracket; that is n^* solves $\alpha n + b = 0$. This is precisely the Ramanujan's Master Theorem.

The next rule evaluates a multi-dimensional bracket series of index 0, that is, the number of sums is equal to the number of brackets.

Rule $\mathbf{E_2}$. Assume the matrix $A = (a_{ij})$, with $a_{ij} \in \mathbb{R}$, is non-singular. Then we have the assignment

(3.9)
$$\sum_{n_1 \geqslant 1} \sum_{n_2 \geqslant 1} \cdots \sum_{n_r \geqslant 1} \phi_{n_1 \cdots n_r} f(n_1, \ldots, n_r) \langle a_{11} n_1 + \cdots + a_{1r} n_r + c_1 \rangle$$
$$\cdots \langle a_{r1} n_1 + \cdots + a_{rr} n_r + c_r \rangle = \frac{1}{|\det(A)|} f({n_1}^*, \ldots, {n_r}^*) \Gamma(-{n_1}^*) \cdots \Gamma(-{n_r}^*),$$

where $\{n_i^*\}$ is the (unique) solution of the linear system obtained from the vanishing of the brackets. There is no assignment if A is singular.

Rule $\mathbf{E_3}$. The value of a multi-dimensional bracket series of positive index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded. There is no assignment to a bracket series of negative index. If all the resulting series are discarded, then the method is nor applicable.

The next section begins the evaluation of entries in [**18**].

4. Integrals in section 6.561 from Gradshteyn and Ryzhik

Sections 6.56 − 6.58 contain entries under the title `Combinations of Bessel functions and powers`. Some examples are evaluated next. Two examples from the Subsection 6.561 are given.

4.1. Entry 6.561.14.

$$\int_0^\infty x^\mu J_\nu(\alpha x)dx = 2^\mu \alpha^{-\mu-1} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)}. \tag{4.1}$$

Proof. Let

$$I = \int_0^\infty x^\mu J_\nu(\alpha x)dx, \tag{4.2}$$

the series representation of the Bessel function (2.2) gives

$$\begin{aligned} I &= \int_0^\infty x^\mu \left[\sum_{n=0}^\infty \phi_n \frac{(\frac{\alpha x}{2})^{\nu+2n}}{\Gamma(\nu+n+1)} \right] \\ &= \int_0^\infty \sum_{n=0}^\infty \phi_n \left(\frac{\alpha}{2}\right)^{\nu+2n} \frac{1}{\Gamma(\nu+n+1)} x^{\nu+2n+\mu} \\ &= \sum_{n=0}^\infty \phi_n \left(\frac{\alpha}{2}\right)^{\nu+2n} \frac{1}{\Gamma(\nu+n+1)} \langle 2n+\nu+\mu+1 \rangle. \end{aligned} \tag{4.3}$$

The vanishing of the bracket gives $n^* = -\frac{1}{2}(\nu+\mu+1)$ and rule E_1 yields

$$I = \frac{1}{2}\left(\frac{\alpha}{2}\right)^{-\mu-1} \frac{1}{\Gamma(\frac{\nu-\mu+1}{2})} \Gamma\left(\frac{\nu+\mu+1}{2}\right), \tag{4.4}$$

which is (4.1). □

4.2. Entry 6.561.16.

$$\int_0^\infty x^\mu K_\nu(ax)dx = 2^{\mu-1} a^{-\mu-1} \Gamma\left(\frac{1+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\mu-\nu}{2}\right). \tag{4.5}$$

Proof. The substitution $z = ax$ produces

$$I = \int_0^\infty x^\mu K_\nu(ax)dx = a^{-\mu-1} \int_0^\infty z^\mu K_\nu(z)dz. \tag{4.6}$$

The integral representation of K_ν in (2.6) now yields

$$\begin{aligned} I &= a^{-\mu-1} \int_0^\infty z^\mu \left(\int_0^\infty e^{-z\cosh t} \cosh(\nu t)\, dt \right) dz \\ &= a^{-\mu-1} \int_0^\infty \cosh(\nu t) \left(\int_0^\infty z^\mu e^{-z\cosh t}\, dz \right) dt. \end{aligned} \tag{4.7}$$

The substitution $s = z\cosh t$ for the interior integral then gives

$$(4.8)\qquad \begin{aligned} I &= a^{-\mu-1}\int_0^\infty \cosh(\nu t)\left[\int_0^\infty \left(\frac{s}{\cosh t}\right)^\mu e^{-s}\frac{ds}{\cosh t}\right]dt \\ &= a^{-\mu-1}\Gamma(\mu+1)\int_0^\infty \frac{\cosh(\nu t)}{\cosh^{\mu+1}(t)}dt. \end{aligned}$$

Since the integrand is even, then

$$(4.9)\qquad I = \frac{a^{-\mu-1}\Gamma(\mu+1)}{2}\int_{-\infty}^\infty \frac{\cosh(\nu t)}{\cosh^{\mu+1}(t)}dt,$$

and the change of variables $w = e^t$ now produces

$$(4.10)\qquad \begin{aligned} I &= \frac{a^{-\mu-1}\Gamma(\mu+1)}{2}\int_0^\infty \frac{\frac{1}{2}(w^\nu + w^{-\nu})}{(\frac{1}{2})^{\mu+1}(w+w^{-1})^{\mu+1}}\frac{dw}{w} \\ &= 2^{\mu-1}a^{-\mu-1}\Gamma(\mu+1)\left[\int_0^\infty \frac{w^{\nu-1}}{(w+w^{-1})^{\mu+1}}dw + \int_0^\infty \frac{w^{-\nu-1}}{(w+w^{-1})^{\mu+1}}dw\right] \\ &= 2^{\mu-1}a^{-\mu-1}\Gamma(\mu+1)\left[\int_0^\infty \frac{w^{\nu+\mu}}{(1+w^2)^{\mu+1}}dw + \int_0^\infty \frac{w^{-\nu+\mu}}{(1+w^2)^{\mu+1}}dw\right]. \end{aligned}$$

Finally, the substitution $x = w^2$ gives

(4.11)

$$\begin{aligned} I &= 2^{\mu-2}a^{-\mu-1}\Gamma(\mu+1)\left[\int_0^\infty \frac{x^{\frac{\nu+\mu-1}{2}}}{(1+x)^{\mu+1}}dx + \int_0^\infty \frac{x^{\frac{-\nu+\mu-1}{2}}}{(1+x)^{\mu+1}}dx\right] \\ &= 2^{\mu-2}a^{-\mu-1}\Gamma(\mu+1)\left[B\left(\frac{\nu+\mu+1}{2}, \frac{-\nu+\mu+1}{2}\right) + B\left(\frac{-\nu+\mu+1}{2}, \frac{\nu+\mu+1}{2}\right)\right] \\ &= 2^{\mu-1}a^{-\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right), \end{aligned}$$

where the last two steps follow from the definition of the beta function (2.14) and its representation in terms of the gamma function (2.15).

□

5. Integrals in section 6.623 from Gradshteyn and Ryzhik

Sections 6.62 – 6.63 contains entries under the title `Combinations of Bessel functions, exponentials, and powers`. Two examples are evaluated.

5.1. Entry 6.623.2.

$$(5.1)\qquad I = \int_0^\infty e^{-\alpha x}J_\nu(\beta x)x^{\nu+1}\,dx = \frac{2\alpha(2\beta)^\nu\Gamma(\nu+\frac{3}{2})}{\sqrt{\pi}(\alpha^2+\beta^2)^{\nu+1/2}}.$$

Two proofs are presented. The first one is a classical one, using elementary properties of Bessel functions. The second proof uses the method of brackets.

First we establish a result giving an integral representation of $e^{-\alpha x}$. This will be used to give a proof of (5.1).

Lemma 5.1. *Let $r > 0$, then*

$$e^{-r} = \frac{2r}{\sqrt{\pi}} \int_0^\infty e^{-r^2w^2 - \frac{1}{4w^2}}\, dw \tag{5.2}$$

Proof. Recall the identity

$$I(r) = \int_0^\infty e^{-\frac{r}{2}t^2}\, dt = \frac{\sqrt{\pi}}{\sqrt{2r}}. \tag{5.3}$$

Now, use the substitution $t = y - \frac{1}{y}$, to obtain

$$\begin{aligned} I(r) = \int_0^\infty e^{-\frac{r}{2}t^2}\, dt &= \int_1^\infty \left(1 + \frac{1}{y^2}\right) e^{-\frac{r}{2}\left(y-\frac{1}{y}\right)^2}\, dy \\ &= \int_1^\infty e^{-\frac{r}{2}\left(y-\frac{1}{y}\right)^2}\, dy + \int_1^\infty \frac{1}{y^2} e^{-\frac{r}{2}\left(y-\frac{1}{y}\right)^2}\, dy. \end{aligned} \tag{5.4}$$

The change of variables $t = \frac{1}{y}$ gives

$$\int_1^\infty \frac{1}{y^2} e^{-\frac{r}{2}(y-\frac{1}{y})^2}\, dy = \int_0^1 e^{-\frac{r}{2}(t-\frac{1}{t})^2}\, dt, \tag{5.5}$$

and (5.4) becomes

$$I(r) = \int_0^1 e^{-\frac{r}{2}\left(y-\frac{1}{y}\right)^2}\, dy + \int_1^\infty e^{-\frac{r}{2}\left(y-\frac{1}{y}\right)^2}\, dy = \int_0^\infty e^{-\frac{r}{2}(y-\frac{1}{y})^2}\, dy. \tag{5.6}$$

Finally, the substitution $y = \sqrt{2r}w$ yields

$$I(r) = \sqrt{2r} \int_0^\infty e^{-r^2w^2 - \frac{1}{4w^2}}\, e^r\, dw, \tag{5.7}$$

which using (5.3) produces the desired result. □

Proof. (of entry 6.623.2). Start with the slightly different integral

$$\tilde{I}(\alpha) = \int_0^\infty e^{-\alpha x} J_\nu(\beta x)\, x^\nu\, dx. \tag{5.8}$$

Lemma 5.1 yields

$$\begin{aligned} \tilde{I}(\alpha) &= \int_0^\infty \left(\frac{2\alpha x}{\sqrt{\pi}} \int_0^\infty e^{-(\alpha x t)^2 - \frac{1}{4t^2}}\, dt \right) J_\nu(\beta x) x^\nu\, dx \\ &= \int_0^\infty \frac{2\alpha}{\sqrt{\pi}} e^{-\frac{1}{4t^2}} \int_0^\infty e^{-(\alpha x t)^2} J_\nu(\beta x) x^{\nu+1}\, dx\, dt. \end{aligned} \tag{5.9}$$

The integral with respect to x can be evaluated using entry 6.631.4 (see (6.1) for the proof). Therefore

$$\begin{aligned}
\tilde{I}(\alpha) &= \frac{2\alpha}{\sqrt{\pi}} \int_0^\infty e^{-\frac{1}{4t^2}} \frac{\beta^\nu e^{-\frac{\beta^2}{4\alpha^2 t^2}}}{(2\alpha^2 t^2)^{\nu+1}} \, dt \\
&= \frac{2\alpha\beta^\nu}{\sqrt{\pi}(2\alpha^2)^{\nu+1}} \int_0^\infty t^{2\nu} e^{-\frac{t^2}{4}\left(1+\frac{\beta^2}{\alpha^2}\right)} \, dt \\
&= \frac{\alpha\beta^\nu}{\sqrt{\pi}(2\alpha^2)^{\nu+1}} \frac{\Gamma(\nu+\frac{1}{2})}{(\frac{1}{4}(1+\frac{\beta^2}{\alpha^2}))^{\nu+\frac{1}{2}}} \\
&= \frac{(2\beta)^\nu \Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}(\alpha^2+\beta^2)^{\nu+\frac{1}{2}}}.
\end{aligned} \tag{5.10}$$

Finally, (5.1) follows by differentiation of $\tilde{I}(\alpha)$, using its integral representation given by (5.8) and the explicit representation found in (5.10). Indeed, from (5.8)

$$\frac{d}{d\alpha}(\tilde{I}(\alpha)) = \frac{d}{d\alpha}\left(\int_0^\infty e^{-\alpha x} J_\nu(\beta x) x^\nu \, dx \right) = -\int_0^\infty e^{-\alpha x} J_\nu(\beta x) x^{\nu+1} \, dx = -I, \tag{5.11}$$

while from (5.10)

$$\frac{d}{d\alpha}(\tilde{I}(\alpha)) = -\frac{2\alpha(2\beta)^\nu \Gamma(\nu+\frac{3}{2})}{\sqrt{\pi}(\alpha^2+\beta^2)^{\nu+\frac{3}{2}}}. \tag{5.12}$$

This completes the proof.

An evaluation of the entry above is now given using the method of brackets. This should serve as an example illustrating the simplicity of this method.

Start with the original integral and use the standard series for the exponential function and the Bessel function J_ν given in (2.2). Then

$$\begin{aligned}
I &= \int_0^\infty e^{-\alpha x} J_\nu(\beta x) x^{\nu+1} \, dx \\
&= \int_0^\infty \left[\sum_{n=0}^\infty \frac{(-\alpha x)^n}{n!} \right] \left[\sum_{m=0}^\infty \frac{(-1)^m (\frac{\beta}{2})^{\nu+2m}}{m! \, \Gamma(\nu+m+1)} x^{\nu+2m} \right] x^{\nu+1} \, dx \\
&= \int_0^\infty \sum_{n=0}^\infty \sum_{m=0}^\infty \phi_{n,m} \frac{\alpha^n (\frac{\beta}{2})^{\nu+2m}}{\Gamma(\nu+m+1)} x^{2\nu+n+2m+1} \, dx \\
&= \sum_{n=0}^\infty \sum_{m=0}^\infty \phi_{n,m} \frac{\alpha^n (\frac{\beta}{2})^{\nu+2m}}{\Gamma(\nu+m+1)} \langle n+2m+2\nu+2 \rangle.
\end{aligned} \tag{5.13}$$

This is a 2-dimensional bracket series of index one (one bracket and two sums). Then according to rule E_3 we must consider the two contributions coming from taking, separately, n and m as free parameters:

n `as a free parameter`. In this case $m^* = -\nu - 1 - \frac{n}{2}$ and the coefficient of m is $a = 2$, so the contribution to the value of the integral is

$$I_1 = \frac{1}{2}\sum_{n=0}^{\infty} \phi_n \frac{\alpha^n (\frac{\beta}{2})^{-\nu-n-2}\,\Gamma(\nu+1+\frac{n}{2})}{\Gamma(-\frac{n}{2})}. \tag{5.14}$$

Note that due to the appearance of $\Gamma(-\frac{n}{2})$ in the denominator, only the odd indexes contribute to the sum. Therefore

$$\begin{aligned} I_1 &= \frac{1}{2}\sum_{n=0}^{\infty} \phi_{2n+1} \frac{\alpha^{2n+1} (\frac{\beta}{2})^{-\nu-2n-3}\,\Gamma(\nu+n+\frac{3}{2})}{\Gamma(-n-\frac{1}{2})} \\ &= \frac{2^{\nu+2}\alpha}{\beta^{\nu+3}} \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{(2n+1)!} \frac{2^{2n}(\frac{\alpha}{\beta})^{2n}\,\Gamma(\nu+n+\frac{3}{2})}{\Gamma(-n-\frac{1}{2})}. \end{aligned} \tag{5.15}$$

This expression can be simplified using the identities

$$\frac{1}{\Gamma(-n-\frac{1}{2})} = \frac{(-1)^{n-1}(2n+1)!}{\sqrt{\pi}\,2^{2n+1}n!}, \tag{5.16}$$

and the relation of the Pochammer symbol with the gamma function

$$\Gamma\left(\nu+n+\tfrac{3}{2}\right) = \Gamma\left(\nu+\tfrac{3}{2}\right)\left(\nu+\tfrac{3}{2}\right)_n. \tag{5.17}$$

The last two identities imply that, for $|\alpha| < |\beta|$,

$$\begin{aligned} I_1 &= \frac{2^{\nu+1}\alpha\,\Gamma\left(\nu+\frac{3}{2}\right)}{\sqrt{\pi}\,\beta^{\nu+3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\nu+\frac{3}{2}\right)_n \left(\frac{\alpha}{\beta}\right)^{2n} \\ &= \frac{2^{\nu+1}\alpha\,\Gamma\left(\nu+\frac{3}{2}\right)}{\sqrt{\pi}\,\beta^{\nu+3}} \left(1+\frac{\alpha^2}{\beta^2}\right)^{-\nu-\frac{3}{2}} \\ &= \frac{2\alpha(2\beta)^{\nu}\Gamma(\nu+\frac{3}{2})}{\sqrt{\pi}(\alpha^2+\beta^2)^{\nu+\frac{3}{2}}}. \end{aligned} \tag{5.18}$$

m `as a free parameter`. In this case $n^* = -2\nu - 2m - 2$ and $a = 1$. Then, the contribution to the integral, for $|\beta| < |\alpha|$, is

$$
\begin{aligned}
I_2 &= \sum_{m=0}^{\infty} \phi_m \frac{\alpha^{-2\nu-2m-2}(\frac{\beta}{2})^{\nu+2m}\,\Gamma(2\nu+2m+2)}{\Gamma(\nu+m+1)} \\
&= \frac{\beta^\nu}{2^\nu \alpha^{2\nu+2}} \sum_{m=0}^{\infty} \phi_m \frac{\Gamma(2\nu+2m+2)}{\Gamma(\nu+m+1)} \left(\frac{\beta}{2\alpha}\right)^{2m} \\
&= \frac{2^{\nu+1}\beta^\nu}{\sqrt{\pi}\,\alpha^{2\nu+2}} \sum_{m=0}^{\infty} \phi_m \,\Gamma(\nu+m+3/2) \left(\frac{\beta}{\alpha}\right)^{2m} \\
&= \frac{2^{\nu+1}\beta^\nu\,\Gamma(\nu+3/2)}{\sqrt{\pi}\,\alpha^{2\nu+2}} \sum_{m=0}^{\infty} \phi_m \left(\nu+\frac{3}{2}\right)_m \left(\frac{\beta}{\alpha}\right)^{2m} \\
&= \frac{2^{\nu+1}\beta^\nu\,\Gamma(\nu+3/2)}{\sqrt{\pi}\,\alpha^{2\nu+2}} \left(1+\frac{\beta^2}{\alpha^2}\right)^{-\nu-\frac{3}{2}} \\
&= \frac{2\alpha(2b)^\nu\Gamma(\nu+\frac{3}{2})}{\sqrt{\pi}(\alpha^2+\beta^2)^{\nu+\frac{3}{2}}}.
\end{aligned} \tag{5.19}
$$

Finally, since both contributions yield the same result ($I_1 = I_2$), rule E_3 states that the integral is given by this common value. This concludes the proof of (5.1). □

6. Entry 6.631

This section uses the method of brackets to evaluate an integral containing the gaussian function (quadratic exponential) and a Bessel J_ν term.

6.1. Entry 6.631.4.

$$
I = \int_0^\infty e^{-\alpha x^2} J_\nu(\beta x) x^{\nu+1}\,dx = \frac{\beta^\nu e^{-\frac{\beta^2}{4\alpha}}}{(2\alpha)^{\nu+1}}. \tag{6.1}
$$

PROOF. The first proof is in classical style. The series representation of the Bessel function (2.2) gives

$$
\begin{aligned}
I &= \int_0^\infty e^{-\alpha x^2} \sum_{n=0}^{\infty} \left[\frac{(-1)^n (\frac{\beta}{2})^{\nu+2n}}{n!\,\Gamma(\nu+n+1)} x^{\nu+2n} \right] x^{\nu+1}\,dx \\
&= \int_0^\infty e^{-\alpha x^2} \sum_{n=0}^{\infty} \left[\frac{(-1)^n (\frac{\beta}{2})^{\nu+2n}}{n!\,\Gamma(\nu+n+1)} x^{2\nu+2n+1} \right] dx.
\end{aligned} \tag{6.2}
$$

Using the substitution $x = \left(\frac{t}{\alpha}\right)^{1/2}$, this becomes

$$\begin{aligned} I &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n(\frac{\beta}{2})^{\nu+2n}}{n!\,\Gamma(\nu+n+1)\alpha^{n+\nu+1}} \int_0^{\infty} t^{\nu+n}e^{-t}\,dt \\ &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n\left(\frac{\beta}{2}\right)^{\nu+2n}}{n!\alpha^{n+\nu+1}} \\ &= \frac{\beta^{\nu}e^{-\frac{\beta^2}{4\alpha}}}{(2\alpha)^{\nu+1}}, \end{aligned} \tag{6.3}$$

which is the desired result.

On the other hand, to evaluate this integral using the method of brackets, start with (6.2) and use the series expansion of the exponential to obtain a bracket series as follows:

$$\begin{aligned} I &= \int_0^{\infty}\left[\sum_{m=0}^{\infty}\frac{(-\alpha x^2)^m}{m!}\right]\left[\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\beta}{2})^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}x^{2\nu+2n+1}\right]dx \\ &= \int_0^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\phi_{n,m}\frac{\alpha^m(\frac{\beta}{2})^{\nu+2n}}{\Gamma(\nu+n+1)}x^{2n+2m+2\nu+1}\,dx \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\phi_{n,m}\frac{\alpha^m(\frac{\beta}{2})^{\nu+2n}}{\Gamma(\nu+n+1)}\langle 2n+2m+2\nu+2\rangle. \end{aligned} \tag{6.4}$$

This is a 2-dimensional bracket series of index one. According to rule E_2 we consider the two contributions coming from considering n and m as free parameters.

First take m as the free parameter. Then $n^* = -\nu-m-1$ and $a = 2$, and according to rule E_1 the contribution in this case is

$$I_1 = \frac{1}{2}\sum_{n=0}^{\infty}\phi_m\frac{\alpha^m(\frac{\beta}{2})^{-\nu-2m-2}\Gamma(\nu+m+1)}{\Gamma(-m)} \tag{6.5}$$

which diverges due to the appearance of the term $\Gamma(-m)$. Therefore this is discarded.

Now, when n as the free parameter, $m^* = -\nu-n-1$ and $a = 2$. Thus the contribution of this case is given by

$$\begin{aligned} I_2 &= \frac{1}{2}\sum_{n=0}^{\infty}\phi_n\frac{\alpha^{-\nu-n-1}(\frac{\beta}{2})^{\nu+2n}\Gamma(\nu+n+1)}{\Gamma(\nu+n+1)} \\ &= \frac{\beta^{\nu}}{(2\alpha)^{\nu+1}}\sum_{n=0}^{\infty}\phi_n\left(\frac{\beta^2}{4\alpha}\right)^n \\ &= \frac{\beta^{\nu}e^{-\frac{\beta^2}{4\alpha}}}{(2\alpha)^{\nu+1}}. \end{aligned} \tag{6.6}$$

This completes the proof. □

7. Section 6.512

This section contains 10 entries where the integrand is a product of Bessel functions. One of them, entry 6.512.1 has been evaluated in [**19**] in order to illustrate methods of automatic computation. In this section the method of brackets is used to evaluate half of the entries in this section.

7.1. Entry 6.512.1.

$$\int_0^\infty J_\mu(ax)J_\nu(bx)\,dx = \frac{b^\nu}{a^{\nu+1}}\frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\nu+1)\Gamma\left(\frac{\mu-\nu+1}{2}\right)}\,{}_2F_1\left(\left.\begin{matrix}\frac{\mu+\nu+1}{2} & \frac{\nu-\mu+1}{2}\\ \nu+1 & \end{matrix}\right|\frac{b^2}{a^2}\right) \tag{7.1}$$

for a, $b>0$ and $b<a$. For $a<b$, the positions of μ, ν and a, b should be reversed.

Proof. Denote the integral by I and assume $b<a$. Using the power series of the Bessel function gives the representation of I as an index 1 bracket series

$$I = \sum_{n_1,n_2}\phi_{1,2}\frac{a^{2n_1+\mu}b^{2n_2+\nu}}{2^{2n_1+2n_2+\mu+\nu}}\frac{\langle 2n_1+2n_2+\mu+\nu+1\rangle}{\Gamma(\mu+n_1+1)\Gamma(\nu+n_2+1)}. \tag{7.2}$$

The evaluation of this series, using Rule E_3, is now divided into two cases, according to which index is chosen as the free one.

`Case 1`: n_1 `is a free index`. The vanishing of the bracket gives $n_2^* = -n_1-\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{2}$. Rule E_3 now gives

$$I_1 = \frac{a^\mu}{b^{\mu+1}}\sum_{n_1=0}^\infty\frac{(-1)^{n_1}}{n_1!}\frac{\Gamma(n_1+\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2})}{\Gamma(n_1+\mu+1)\Gamma(\frac{1}{2}\nu+\frac{1}{2}-\frac{1}{2}\mu-n_1)}\left(\frac{a^2}{b^2}\right)^{n_1}. \tag{7.3}$$

The gamma factors are now converted to Pochhammer symbols using

$$\Gamma(x+m)=\Gamma(x)(x)_m \quad\text{and}\quad (x)_{-m}=\frac{(-1)^m}{(1-x)_m}\quad\text{for } x\in\mathbb{R}\text{ and } m\in\mathbb{N}. \tag{7.4}$$

This produces

$$I_1 = \frac{a^\mu}{b^{\mu+1}}\frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\mu+1)\Gamma\left(\frac{\nu+1-\mu}{2}\right)}\sum_{n_1=0}^\infty\frac{\left(\frac{\mu+\nu+1}{2}\right)_{n_1}\left(\frac{\mu-\nu+1}{2}\right)_{n_1}}{n_1!(\mu+1)_{n_1}}\left(\frac{a^2}{b^2}\right)^{n_1}. \tag{7.5}$$

Since $b<a$, the series diverges and it does not contribute to the value of the integral.

`Case 2`: n_2 `is a free index`. The vanishing of the brackets gives $n_1^* = -n_2-\frac{1}{2}(\mu+\nu+1)$. Rule E_3 now gives the integral as

$$I_2 = \sum_{n_2=0}^\infty\frac{(-1)^{n_2}}{n_2!}\frac{\Gamma(-n_1^*)}{\Gamma(\mu+n_1^*+1)\Gamma(\nu+n_2+1)}a^{2n_1^*+\mu}b^{2n_2+\nu}. \tag{7.6}$$

Now replace the value of n_1^* and use the identities

$$\begin{aligned}\Gamma(-n_1^*) &= \Gamma(n_2+\tfrac{1}{2}(\mu+\nu+1)) = \left(\tfrac{1}{2}(\mu+\nu+1)\right)_{n_2}\Gamma\left(\tfrac{1}{2}(\mu+\nu+1)\right)\\ \Gamma(\mu+n_1^*+1) &= \Gamma(\tfrac{1}{2}(\mu-\nu+1)-n_2) = (-1)^{n_2}/\Gamma(\tfrac{\nu-\mu+1}{2})\\ \Gamma(\nu+n_2+1) &= \Gamma(\nu+1)(\nu+1)_{n_2}.\end{aligned} \tag{7.7}$$

in (7.7) to obtain the expression (7.1). □

7.2. Entry 6.512.2. Let

$$I = \int_0^\infty J_{\nu+k}(\alpha t) J_{\nu-k-1}(\beta t)\, dt. \tag{7.8}$$

Then

$$I = \frac{\beta^{\nu-k-1}\Gamma(\nu)}{\alpha^{\nu-k} k! \Gamma(\nu-k)} {}_2F_1\left(\begin{matrix} \nu \quad -k \\ \nu - k \end{matrix} \middle| \frac{\beta^2}{\alpha^2} \right) \quad \text{if } 0 < \beta < \alpha \tag{7.9}$$

and the special cases

$$I = (-1)^k \frac{1}{2\alpha} \quad \text{if} \quad 0 < \beta = \alpha \tag{7.10}$$

and

$$I = 0 \quad \text{if} \quad 0 < \alpha < \beta. \tag{7.11}$$

Proof. In order to apply the method of brackets, expand both Bessel functions in series to obtain

$$I = \int_0^\infty \sum_{n_1, n_2} \phi_{1,2} \frac{(\alpha/2)^{2n_1+\nu+k}(\beta/2)^{2n_2+\nu-k-1}}{\Gamma(n_1+\nu+k+1)\Gamma(n_2+\nu-k)} t^{2n_1+2n_2+2\nu-1}\, dt. \tag{7.12}$$

Integrating converts this into a bracket series of index 1:

$$I = \sum_{n_1, n_2} \phi_{1,2} \frac{(\alpha/2)^{2n_1+\nu+k}(\beta/2)^{2n_2+\nu-k-1}}{\Gamma(n_1+\nu+k+1)\Gamma(n_2+\nu-k)} \langle 2n_1 + 2n_2 + 2\nu \rangle. \tag{7.13}$$

This is now expressed as a single series by choosing a free parameter.
n_2 `is free`. Using Rule E_2 with $n_1^* = -\nu - n_2$ gives

$$I = \frac{\beta^{\nu-k-1}}{2\alpha^{\nu-k}} \sum_{n_2=0}^\infty \frac{(-1)^{n_2}}{\Gamma(n_2+1)} \frac{\Gamma(\nu+n_2)}{\Gamma(k+1-n_2)\Gamma(n_2+\nu-k)} \left(\frac{\beta^2}{\alpha^2} \right)^{n_2} \tag{7.14}$$

`Case 1`: $0 < \beta < \alpha$. Then the series converges. In order to simplify this expression, use $\Gamma(u+1) = u\Gamma(u)$ and $(\alpha - m)_m = (-1)^m (1-\alpha)_m$ to prove the relation

$$\Gamma(\alpha - n) = \frac{\Gamma(\alpha)}{(-1)^n (1-\alpha)_n}. \tag{7.15}$$

Now use this to simplify the term $\Gamma(k+1-n_2)$ in (7.14) (the only gamma factor where the index n_2 appears with a negative sign). This gives the result (7.9), for the case $0 < \beta < \alpha$.
`Case 2`: $0 < \alpha = \beta$. Then (7.14) becomes

$$I_1 = \frac{1}{\alpha} \sum_{n_2=0}^\infty \frac{(-1)^{n_2}}{\Gamma(n_2+1)} \frac{\Gamma(\nu+n_2)}{\Gamma(k+1-n_2)\Gamma(n_2+\nu-k)}. \tag{7.16}$$

The term $1/\Gamma(k+1-n_2)$ vanishes for $n_2 > k$ and expression I reduces to

$$I_1 = \frac{1}{\alpha} \sum_{n_2=0}^{k} \frac{(-1)^{n_2}}{\Gamma(n_2+1)} \frac{\Gamma(\nu+n_2)}{\Gamma(k+1-n_2)\Gamma(n_2+\nu-k)}. \tag{7.17}$$

Using (7.4) the desired result is seen to be equivalent to the identity

$$\sum_{n_2=0}^{k} \binom{k}{n_2} (\nu)_{n_2} (1-\nu)_{k-n_2} = k! \tag{7.18}$$

PROOF. Assume X_ν be a gamma distributed random variable with parameter ν. Then

$$\mathbb{E}[X_\nu^m] = (\nu)_m. \tag{7.19}$$

Recall that the sum of independent gamma distributed random variables is again gamma distributed and the corresponding parameters are added; that is, if $X_i \sim \Gamma(\nu_i)$ are independent, then $X_1 + X_2 \sim \Gamma(\nu_1 + \nu_2)$.

Now

$$\begin{aligned} \sum_{n=0}^{k} \binom{k}{n} (\nu)_n (1-\nu)_{k-n} &= \sum_{n=0}^{k} \binom{k}{n} \mathbb{E}[X_\nu^n] \mathbb{E}[X_{1-\nu}^{k-n}] \\ &= \mathbb{E}[(X_\nu + X_{1-\nu})^k] \\ &= \mathbb{E}[X(1)^k] \\ &= (1)_k \\ &= k!, \end{aligned} \tag{7.20}$$

and the proof is complete. □

□

7.3. Entry 6.512.3.

$$\int_0^\infty J_\nu(\alpha x) J_{\nu-1}(\beta x)\, dx = \begin{cases} \beta^{\nu-1}/\alpha^\nu & \text{for } \beta < \alpha \\ \frac{1}{2\beta} & \text{for } \beta = \alpha \\ 0 & \text{for } \beta > \alpha. \end{cases} \tag{7.21}$$

PROOF. Using the power series of the Bessel function, it follows that

$$\begin{aligned} &\int_0^\infty J_\nu(\alpha x) J_{\nu-1}(\beta x) dx \\ &\int_0^\infty \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}}{n_1! \Gamma(n_1+\nu+1)} \left(\frac{\alpha x}{2} \right)^{2n_1+\nu} \sum_{n_2=0}^{\infty} \frac{(-1)^{n_2}}{n_2! \Gamma(n_2+\nu)} \left(\frac{\beta x}{2} \right)^{2n_2+\nu-1} dx \\ &= \sum_{n_1,n_2=0}^{\infty} \frac{(-1)^{n_1+n_2} \alpha^{2n_1+v} \beta^{2n_2+\nu-1}}{n_1! n_2! \Gamma(n_2+\nu) \Gamma(n_1+\nu+1) 2^{2(n_1+n_2+\nu)-1}} \int_0^\infty x^{2(n_1+n_2+\nu)-1} dx \\ &= \sum_{n_1,n_2=0}^{\infty} \frac{(-1)^{n_1+n_2} \alpha^{2n_1+\nu} \beta^{2n_2+\nu-1}}{n_1! n_2! \Gamma(n_2+\nu) \Gamma(n_1+\nu+1) 2^{2(n_1+n_2+\nu)-1}} \langle 2(n_1+n_2+\nu) \rangle \end{aligned} \tag{7.22}$$

This is a bracket series of index is 1. In its evaluation we use rule E_3:

n_1 `is a free parameter`: The corresponding equation is $2n_2 = -2\nu - 2n_1$ with determinant 2. Therefore, the contribution with n_1 free is

$$I_1 = \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}\alpha^{2n_1+v}}{\beta^{2n_1+v+1}(n_1+v)\Gamma(-n_1)\Gamma(n_1+1)}.$$

This series vanishes identically in view of the presence of $\Gamma(-n_1)$.

n_2 `is a free parameter`: The corresponding equation is now $2n_1 = -2\nu - 2n_2$, with determinant 2. As in the previous case, the contribution of n_2 free is

$$I_2 = \frac{1}{\beta}\sum_{n_2=0}^{\infty} \frac{(-1)^{n_2}}{\Gamma(n_2+1)\Gamma(1-n_2)}\left(\frac{\beta}{\alpha}\right)^{2n_2+\nu}. \tag{7.23}$$

The presence of $\Gamma(1-n_2)$ shows that every term, except $n_2 = 0$, vanishes. This confirms the evaluation.

□

7.4. Entry 6.512.9.

$$\int_0^\infty K_0(ax)J_1(bx)\,dx = \frac{1}{2b}\ln\left(1+\frac{b^2}{a^2}\right), \quad a>0,\, b>0. \tag{7.24}$$

Proof. The creation of a bracket series for this problem starts with the integral representation

$$K_0(x) = \int_0^\infty \frac{\cos(xt)}{(t^2+1)^{1/2}}\,dt \tag{7.25}$$

discussed in [**13**]. Therefore

$$I = \int_0^\infty \int_0^\infty \frac{\cos(axt)}{\sqrt{t^2+1}} J_1(bx)dtdx \tag{7.26}$$

Now use the Taylor series for $\cos(axt)$ and $J_1(bx)$ and rule P_2 to expand $(t^2+1)^{-1/2}$ into a bracket series. two sums and a bracket. The value of the integral is therefore given by

(7.27)

$$I = \sum_{n_1,n_2,n_3,n_4} \phi_{1234}\frac{n_1!}{(2n_1)!}\frac{a^{2n_1}}{\Gamma(n_2+2)}\frac{(b/2)^{n_2+1}}{\Gamma(1/2)}\langle n_3+n_4+1/2\rangle\langle 2n_1+2n_2+2\rangle\langle 2n_1+2n_4+1\rangle.$$

This is a bracket series representation of index 1. The value of the integral is now obtained by taking each of the four indices free, one at the time.

n_1 `is a free index`. The vanishing of the brackets leads to the system of equations

$$\begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} n_2 \\ n_3 \\ n_4 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -2n_1-2 \\ -2n_1-1 \end{pmatrix} \tag{7.28}$$

with a matrix A of determinant 4 and solutions

$$n_2 = -n_1 - 1, \quad n_3 = -n_1 - \tfrac{1}{2}, \quad n_4 = n_1. \tag{7.29}$$

Therefore the contribution of the free index n_1 to the value of the integral is

$$I_1 = \frac{1}{4} \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}}{n_1!} \frac{\Gamma^2(n_1+1)\Gamma(n_1+\tfrac{1}{2})\Gamma(-n_1)}{\Gamma(2n_1+1)\Gamma(-n_1+1)} \frac{a^{2n_1}b^{-2n_1-1}}{\sqrt{\pi}2^{-2n_1-1}}. \tag{7.30}$$

Using the identity $\Gamma(n+\frac{1}{2}) = \dfrac{(2n)!\sqrt{\pi}}{2^{2n}n!}$, the series above can be written as

$$I_1 = -\frac{1}{2b} \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}}{n_1} \left(\frac{a}{b} \right)^{n_1}. \tag{7.31}$$

The first term diverges, so this contribution is discarded.

n_2 `is a free index`. Now the solution to the corresponding linear system is

$$n_1 = -n_2 - 1, \quad n_3 = n_2 + \tfrac{1}{2}, \quad n_4 = -n_2 - 1. \tag{7.32}$$

Using elementary properties of the gamma function the corresponding series converges for $b < a$ and it reduces to

$$I_2 = \frac{b}{2a^2} \sum_{n_2=0}^{\infty} \frac{(-1)^{n_2}}{n_2+1} \left(\frac{b^2}{a^2} \right)^{n_2}. \tag{7.33}$$

This series can be summed to obtain

$$I_2 = \frac{1}{2b} \log \left(1 + \frac{b^2}{a^2} \right). \tag{7.34}$$

n_3 `is a free index`. The solutions are now

$$n_1 = -n_3 - \tfrac{1}{2}, \quad n_2 = n_3 - \tfrac{1}{2}, \quad n_4 = -n_3 - \tfrac{1}{2}. \tag{7.35}$$

The standard procedure gives as contribution a series where every terms containing the vanishing factor $1/\Gamma(-2n_3)$. Therefore $I_3 = 0$.

n_4 `is a free index`. In the final case, the computation is similar to the case when n_1 is a free index, leading to no contribution.

This completes the proof. □

7.5. Entry 6.512.10.

$$\int_0^\infty K_0(ax)I_1(bx)\,dx = -\frac{1}{2b} \ln \left(1 - \frac{b^2}{a^2} \right), \quad a > 0,\, b > 0. \tag{7.36}$$

Proof. Denote the integral by I and use the representations

$$K_0(ax) = \int_0^\infty \frac{\cos(axt)}{\sqrt{t^2+1}}\,dt \tag{7.37}$$

and

$$I_1(bx) = \sum_{n_1=0}^{\infty} \frac{1}{n_1!\Gamma(n_1+2)} \left(\frac{bx}{2} \right)^{2n_1+1} \tag{7.38}$$

and using them in the integral I. The usual procedure for the representation of this integral as a bracket series (following similar steps as the ones described in the previous example) yields

(7.39)
$$I = \frac{b}{2\sqrt{\pi}} \sum_{n_1,n_2,n_3,n_4} \phi_{1234} \frac{\Gamma(n_2+1)}{\Gamma(2n_2+1)} \frac{(-1)^{n_1}}{\Gamma(n_1+2)} \langle 2n_1+2n_2+2\rangle \langle 2n_2+2n_3+1\rangle \langle n_3+n_4+\tfrac{1}{2}\rangle$$

This is a bracket series of index 1. The evaluation is obtained by treating each index as a free one.

n_1 `is a free index`. The usual linear system has a matrix of determinant 4 and gives the solutions

$$n_2 = -n_1 - 1, \quad n_3 = n_1 + \tfrac{1}{2}, \quad n_4 = -n_1 - 1. \tag{7.40}$$

Therefore, the contribution of this index to the value of the integral I is

$$I_1 = \frac{b}{8a^2\sqrt{\pi}} \sum_{n_1=0}^{\infty} \frac{\Gamma(-n_1-\frac{1}{2})\Gamma(n_1+1)\Gamma(-n_1)}{\Gamma(-2n_1-1)\Gamma(n_1+2)} \frac{b^{2n_1}}{2^{2n_1}a^{2n_1}}. \tag{7.41}$$

Using the duplication formula for the gamma function

$$\Gamma(2u) = \frac{1}{\sqrt{\pi}} 2^{2u-1}\Gamma(u)\Gamma(u+\tfrac{1}{2}) \tag{7.42}$$

gives the simplified form

$$I_1 = \frac{b}{2a^2} \sum_{n_1=0}^{\infty} \frac{1}{(n_1+1)} \left(\frac{b^2}{a^2}\right)^{n_1} \tag{7.43}$$

and this series can be summed to produce

$$I_1 = -\frac{1}{2b} \log\left(1 - \frac{b^2}{a^2}\right). \tag{7.44}$$

n_2 `is a free index`. The corresponding linear system has a matrix of determinant 4 and solutions

$$n_1 = -n_2 - 1, \quad n_3 = n_2 - \tfrac{1}{2}, \quad n_4 = n_2. \tag{7.45}$$

Therefore the contribution to the integral is

$$I_2 = \frac{1}{8\sqrt{\pi}} \sum_{n_2=0}^{\infty} \frac{(-1)^{n_2}}{n_2!} \times \frac{\Gamma(n_2+1)}{\Gamma(2n_2+1)} \frac{(-1)^{-n_2-1}}{\Gamma(-n_2+1)} \Gamma(1+n_2)\Gamma(n_2+\tfrac{1}{2})\Gamma(-n_2) \frac{a^{2n_2}b^{-2-2n_2}}{2^{-2-2n_2}}. \tag{7.46}$$

The term corresponding to $n_2 = 0$ diverges. Therefore this series does not contribute to the integral.

n_3 `is a free index`. In this case the case the contribution is seen to be a series with $\Gamma(-2n_3)$ in the denominator. This series vanishes, so $I_3 = 0$.

n_4 `is a free index`. In this case the case the contribution is seen to be a series with $\Gamma(-n_4)$ in the numerator. This series diverges, so it is discarded.

In summary, only n_1 as a free index gives a contribution to the integral. The final result is

$$I = -\frac{1}{2b} \log \left(1 - \frac{b^2}{a^2} \right), \tag{7.47}$$

as claimed in (7.36). The evaluation is complete. □

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