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The integrals in Gradshteyn and Ryzhik. Part 25: Evaluation by series

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Abstract. The table of Gradshteyn and Rhyzik contains many integrals that can be evaluated by expanding the integrand in series. Some examples are discussed.

1. Introduction

The table of integrals [**2**] contains a large variety of definite integrals that can be evaluated by expanding the integrand. The idea is remarkably simple: to evaluate

$$I = \int_a^b f(x)\,dx \tag{1.1}$$

one chooses a set of functions $\{f_n : n \in \mathbb{N}\}$ for which it is possible to expand

$$f(x) = \sum_{n=1}^{\infty} a_n f_n, \tag{1.2}$$

uniformly on $[a,b]$. Then, with

$$b_n = \int_a^b f_n(x)\,dx \tag{1.3}$$

it follows that

$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} a_n b_n. \tag{1.4}$$

In order to obtain a simpler form of the integral I, it is required to identify the series in (1.4).

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2. A hypergeometric example

The first example is entry **3.311.4** in [**2**]:

$$\int_0^\infty \frac{e^{-qx}\,dx}{1-ae^{-px}} = \sum_{k=0}^\infty \frac{a^k}{q+kp}. \tag{2.1}$$

Expanding the integrand as a geometric series produces

$$\frac{1}{1-ae^{-px}} = \sum_{k=0}^\infty a^k e^{-kpx}, \tag{2.2}$$

and integrating over $[0,\infty)$ gives

$$I = \sum_{k=0}^\infty a^k \int_0^\infty e^{-(q+kp)x}\,dx = \sum_{k=0}^\infty \frac{a^k}{q+kp}. \tag{2.3}$$

The resulting series may be identified as a hypergeometric sum. Recall that the **hypergeometric function** is defined by

$$ {}_pF_q\left(a_1,\,\cdots\, a_p; b_1, \cdots, b_q; x\right) := \sum_{k=0}^\infty \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\,\frac{x^k}{k!}, \tag{2.4}$$

where the Pochhammer symbol $(a)_k$ is

$$(a)_k = \begin{cases} a(a+1)(a+2)\cdots(a+k-1), & \text{if } k>0 \\ 1 & \text{if } k=0. \end{cases} \tag{2.5}$$

The reader will find in [**3**] a selection of entries in [**2**] that are evaluated in terms of these functions.

To identify the series in (2.3), write it as

$$\sum_{k=0}^\infty \frac{a^k}{q+kp} = \frac{1}{p}\sum_{k=0}^\infty \frac{a^k}{k+c}, \tag{2.6}$$

where $c=q/p$. Now use $k! = (1)_k$ and

$$k+c = \frac{c\,(c+1)_k}{(c)_k} \tag{2.7}$$

to write

$$\sum_{k=0}^\infty \frac{a^k}{q+kp} = \frac{1}{q}\sum_{k=0}^\infty \frac{(c)_k\,(1)_k}{(1+c)_k}\,\frac{a^k}{k!}. \tag{2.8}$$

It follows that

$$\int_0^\infty \frac{e^{-qx}\,dx}{1-ae^{-px}} = \frac{1}{q}\,{}_2F_1\left(\frac{q}{p},\,1;\,1+\frac{q}{p};\,a\right). \tag{2.9}$$

3. An integral involving the binomial theorem

Entry **3.194.8**:

$$\int_0^1 \frac{x^{n-1}\,dx}{(1+x)^m} = 2^{-n}\sum_{k=0}^{\infty}\binom{m-n-1}{k}\frac{(-2)^{-k}}{n+k} \tag{3.1}$$

is now evaluated using the binomial theorem

$$(1-t)^{-a} = \sum_{k=0}^{\infty}\frac{(a)_k}{k!}t^k. \tag{3.2}$$

Indeed, the change of variables $t = x/(1+x)$ produces

$$\int_0^1 \frac{x^{n-1}\,dx}{(1+x)^m} = \int_0^{1/2}\frac{t^{n-1}\,dt}{(1-t)^{n-m+1}}. \tag{3.3}$$

The integrand is expanded by the binomial theorem (3.2) in the form

$$(1-t)^{m-n-1} = \sum_{k=0}^{\infty}\binom{m-n-1}{k}(-t)^k \tag{3.4}$$

and replacing in (3.3) produces the stated result.

4. A product of logarithms

This section considers the evaluation of entries in [**2**] where the integrand is the product of two logarithmic functions. The entries are evaluated by expanding the integrand in series. Alternative proofs are sometimes offered.

Example 4.1. The value of entry **4.221.1**:

$$\int_0^1 \ln x\,\ln(1-x)\,dx = 2 - \frac{\pi^2}{6} \tag{4.1}$$

can be obtained from the expansion

$$\ln(1-x) = -\sum_{k=1}^{\infty}\frac{x^k}{k}. \tag{4.2}$$

It follows that

$$\int_0^1 \ln x\,\ln(1-x)\,dx = -\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 x^k\,\ln x\,dx \tag{4.3}$$

and the integral can be evaluated by integration by parts to produce

$$\int_0^1 x^k\,\ln x\,dx = -\frac{1}{(k+1)^2}. \tag{4.4}$$

Therefore

$$\int_0^1 \ln x\,\ln(1-x)\,dx = \sum_{k=1}^{\infty}\frac{1}{k(k+1)^2} \tag{4.5}$$

and the partial fraction decomposition

$$\frac{1}{k(k+1)^2} = \frac{1}{k} - \frac{1}{k+1} - \frac{1}{(k+1)^2} \tag{4.6}$$

gives the result.

EXAMPLE 4.2. The evaluation of entry **4.221.2**:

$$\int_0^1 \ln x \, \ln(1+x)\, dx = 2 - \frac{\pi^2}{12} - 2\ln 2 \tag{4.7}$$

can be obtained by using the expansion

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \tag{4.8}$$

and replacing in the integral to obtain

$$\int_0^1 \ln x \, \ln(1+x)\, dx = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \int_0^1 x^k \, \ln x \, dx. \tag{4.9}$$

Integration by parts produces

$$\int_0^1 x^k \, \ln x \, dx = -\frac{1}{(k+1)^2}, \tag{4.10}$$

and this leads to

$$\int_0^1 \ln x \, \ln(1+x)\, dx = \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)^2}. \tag{4.11}$$

Expanding

$$\frac{1}{k(k+1)^2} = \frac{1}{k} - \frac{1}{k+1} - \frac{1}{(k+1)^2} \tag{4.12}$$

and using the values

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -\ln 2 \text{ and } \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}, \tag{4.13}$$

produces the result.

EXAMPLE 4.3. The evaluation of entry **4.221.3**:

$$\int_0^1 \ln\left(\frac{1-ae^{-t}}{1-a} \right) \frac{dx}{\ln x} = -\sum_{k=1}^{\infty} \frac{a^k}{k} \, \ln(1+k) \tag{4.14}$$

is obtained via the change of variables $x = e^{-t}$ to produce

$$\int_0^1 \ln\left(\frac{1-ae^{-t}}{1-a} \right) \frac{dx}{\ln x} = -\int_0^{\infty} \ln\left(\frac{1-ae^{-t}}{1-a} \right) \frac{e^{-t}}{t} \, dt. \tag{4.15}$$

The expansions

$$\ln(1 - ae^{-t}) = -\sum_{k=1}^{\infty} \frac{a^k}{k} e^{-kt} \text{ and } \ln(1-a) = -\sum_{k=1}^{\infty} \frac{a^k}{k} \tag{4.16}$$

produce

$$\int_0^1 \ln\left(\frac{1 - ae^{-t}}{1-a}\right) \frac{dx}{\ln x} = \sum_{k=1}^{\infty} \frac{a^k}{k} \int_0^{\infty} \frac{e^{-kt}-1}{t} e^{-t} dt. \tag{4.17}$$

The integral

$$g(k) = \int_0^{\infty} \frac{e^{-kt}-1}{t} e^{-t}\, dt \tag{4.18}$$

appearing above, satisfies $g(0) = 0$ and $g'(k) = -1/(k+1)$. Thus $g(k) = -\ln(1+k)$ as required.

The series

$$h(a) = -\sum_{k=1}^{\infty} \frac{a^k}{k} \ln(1+k) \tag{4.19}$$

in the formula (4.14) is related to the **polylogarithm function**

$$\mathrm{Li}_b(x) := \sum_{k=1}^{\infty} \frac{x^k}{k^b}. \tag{4.20}$$

Indeed, $h(a)$ satisfies

$$h'(a) = -\sum_{k=1}^{\infty} a^{k-1} \ln(1+k) = -\frac{1}{a^2} \sum_{k=2}^{\infty} a^k \ln k = -\frac{1}{a^2} \frac{\partial}{\partial b} \mathrm{Li}_{-b}(a)\Big|_{b=0}. \tag{4.21}$$

5. Some integrals involving the exponential function

This section presents some examples involving the exponential function.

EXAMPLE 5.1. The evaluation of entry **3.342**:

$$\int_0^1 \exp(-px \ln x)\, dx = \int_0^1 x^{-px}\, dx = \frac{1}{p} \sum_{k=1}^{\infty} \left(\frac{p}{k}\right)^k \tag{5.1}$$

can be established by expanding the integrand in series. Indeed,

$$\int_0^1 \exp(-px \ln x)\, dx \quad = \quad \sum_{k=0}^{\infty} \frac{(-1)^k p^k}{k!} \int_0^1 x^k \ln^k dx. \tag{5.2}$$

The change of variables $x = e^{-t}$ gives

$$\begin{aligned} \int_0^1 x^k \ln^k x \, dx &= (-1)^k \int_0^\infty t^k e^{-(k+1)t} \, dt \\ &= \frac{(-1)^k}{(k+1)^{k+1}} \int_0^\infty s^k e^{-s} \, ds \\ &= \frac{(-1)^k \, k!}{(k+1)^{k+1}}. \end{aligned}$$

Replacing in (5.2) gives the result.

EXAMPLE 5.2. A similar procedure provides the evaluation of entry **3.466.3**:

$$\int_0^1 \frac{e^{x^2}-1}{x^2} \, dx = \sum_{k=1}^\infty \frac{1}{k! \, (2k-1)}. \tag{5.3}$$

Expand the exponential in the integrand and integrate term by term. The resulting series can be identified as

$$\sum_{k=1}^\infty \frac{1}{k! \, (2k-1)} = 1 - e + \sqrt{\pi} \, \text{erfi}(1), \tag{5.4}$$

where

$$\text{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{t^2} \, dt. \tag{5.5}$$

The generalization

$$d_n := \int_0^1 \frac{1}{x^{2n+2}} \left(e^{x^2} - \sum_{k=0}^n \frac{x^{2k}}{k!} \right) dx \tag{5.6}$$

is evaluated as

$$d_n = \sum_{k=n+1}^\infty \frac{1}{k! \, (2k-2n-1)}. \tag{5.7}$$

The first few values are

$$\begin{aligned} d_0 &= 1 - e + \sqrt{\pi} \, \text{erfi}(1) \\ d_1 &= \frac{4}{3} - e + \frac{2}{3} \sqrt{\pi} \, \text{erfi}(1) \\ d_2 &= \frac{31}{30} - \frac{3e}{5} + \frac{4}{15} \sqrt{\pi} \, \text{erfi}(1) \\ d_3 &= \frac{71}{105} - \frac{11e}{35} + \frac{8}{105} \sqrt{\pi} \, \text{erfi}(1) \\ d_4 &= \frac{379}{840} - \frac{19e}{105} + \frac{16}{945} \sqrt{\pi} \, \text{erfi}(1). \end{aligned}$$

The reader will check that the coefficient of $\sqrt{\pi} \, \text{erfi}(1)$ in d_n is $2^n/(2n+1)!!$. The remaining coefficients will be explored in a future article.

6. Some combinations of powers and algebraic functions

This section considers entries of the form

$$\int_0^\infty x^n A\left(e^{-x}\right)\,dx \tag{6.1}$$

where A is an algebraic function; that is, it satisfies $P(x, A(x)) = 0$, for some polynomial P.

Theorem 6.1. Let $a > 0$. Then

$$\int_0^\infty xe^{-x}\sqrt{1-e^{-ax}}\,dx = 1 - \sum_{k=0}^\infty \frac{(2k)!}{2^{2k+1}\,(k+1)!\,k!\,(b+ak)^2} \tag{6.2}$$

with $b = 1 + a$.

Proof. The binomial theorem shows that

$$\begin{aligned}\sqrt{1-e^{-ax}} &= \sum_{k=0}^\infty \binom{\frac{1}{2}}{k}(-1)^k e^{-akx} \\ &= -\sum_{k=0}^\infty \frac{(2k)!}{(2k-1)\,2^{2k}k!^2}e^{-akx}.\end{aligned}$$

Integration yields

$$\begin{aligned}\int_0^\infty xe^{-x}\sqrt{1-e^{-ax}}\,dx &= -\sum_{k=0}^\infty \frac{(2k)!}{(2k-1)\,2^{2k}k!^2}\int_0^\infty xe^{-(1+ak)x}\,dx \\ &= -\sum_{k=0}^\infty \frac{(2k)!}{(2k-1)\,2^{2k}k!^2}\,\frac{1}{(1+ak)^2}.\end{aligned}$$

Now shift the index of summation to obtain the stated form. $\square$

In the examples below, the notation

$$I(a) = \int_0^\infty xe^{-x}\sqrt{1-e^{-ax}}\,dx = 1 - S(a) \tag{6.3}$$

where

$$S(a) := \sum_{k=0}^\infty \frac{(2k)!}{2^{2k+1}\,(k+1)!\,k!\,(b+ak)^2}, \tag{6.4}$$

is employed.

Example 6.1. Entry **3.451.1** states that

$$\int_0^\infty xe^{-x}\sqrt{1-e^{-x}}\,dx = \frac{4}{3}\left(\frac{4}{3} - \ln 2\right). \tag{6.5}$$

This entry corresponds to $I(1) = 1 - S(1)$, where

$$S(1) = \sum_{k=0}^\infty \frac{(2k)!}{2^{2k+1}(k+2)^2(k+1)k!^2}. \tag{6.6}$$

To evaluate this sum by elementary means, start with

$$f(x) = \sum_{k=0}^{\infty} \frac{(2k)!}{k!^2} x^k = \frac{1}{\sqrt{1-4x}}, \tag{6.7}$$

where the last evaluation comes from the binomial theorem. Define

$$g(x) = \int_0^x f(t)\,dt = \sum_{k=0}^{\infty} \frac{(2k)!}{k!^2} \frac{x^{k+1}}{k+1}. \tag{6.8}$$

An elementary calculation shows that

$$g(x) = \frac{1}{2}(1 - \sqrt{1-4x}). \tag{6.9}$$

Now define

$$h(x) = \int_0^x g(t)\,dt = \sum_{k=0}^{\infty} \frac{(2k)!}{k!^2} \frac{x^{k+2}}{(k+1)(k+2)} = \frac{x}{2} - \frac{1}{12} + \frac{1}{12}(1-4x)^{3/2}, \tag{6.10}$$

and

$$w(x) = \int_0^x \frac{h(t)}{t}\,dt = \sum_{k=0}^{\infty} \frac{(2k)!}{k!^2} \frac{x^{k+2}}{(k+1)(k+2)^2}. \tag{6.11}$$

The relation $S(1) = 8w(1/4)$ comes by comparing this last series to the one defining $S(1)$ in (6.6). Now observe that

$$w(x) = \int_0^x \frac{h(t)}{t}\,dt = \frac{x}{2} + \frac{1}{12}J(x), \tag{6.12}$$

where

$$J(x) = \int_0^x \frac{(1-4t)^{3/2}-1}{t}\,dt. \tag{6.13}$$

The change of variables $u = \sqrt{1-4t}$ gives

$$J(x) = -2\int_{\sqrt{1-4x}}^1 \frac{u(1+u+u^2)}{1+u}\,du, \tag{6.14}$$

and the further change of variables $v = 1+u$ gives

$$J(x) = -2\int_{\sigma}^2 (v^2 - 2v + 2 - 1/v)\,dv, \tag{6.15}$$

where $\sigma = 1 + \sqrt{1-4x}$. This last integral can be evaluated in elementary terms to produce

$$w(x) = \frac{1}{18}\left(-4 + 4\sqrt{1-4x} + x(9 - 4\sqrt{1-4x}) + 3\ln 2 - 3\ln(1+\sqrt{1-4x})\right).$$

In particular $w\left(\frac{1}{4}\right) = \frac{1}{18}\left(3\ln 2 - \frac{7}{4}\right)$ and then

$$I(1) = 1 - S(1) = 1 - 8w\left(\tfrac{1}{4}\right) = \frac{4}{9}(4 - 3\ln 2) \tag{6.16}$$

as required.

EXAMPLE 6.2. The second entry in this family is **3.451.2** that states

$$\int_0^\infty xe^{-x}\sqrt{1-e^{-2x}}\,dx = \frac{\pi}{4}\left(\frac{1}{2}+\ln 2\right). \tag{6.17}$$

The same technique used in the previous example is now used to evaluate $I(2) = 1 - S(2)$, where

$$S(2) = \sum_{k=0}^\infty \frac{(2k)!}{2^{2k+1}\,(k+1)!\,k!\,(2k+3)^2}. \tag{6.18}$$

Start with

$$f(x) = \sum_{k=0}^\infty \frac{(2k)!}{k!^2}x^k = \frac{1}{\sqrt{1-4x}}, \tag{6.19}$$

and then evaluate

$$g(x) := \int_0^x f(t)dt = \sum_{k=0}^\infty \frac{(2k)!}{k!\,(k+1)!}x^{k+1}, \tag{6.20}$$

and as before

$$g(x) = \frac{1-\sqrt{1-4x}}{2}. \tag{6.21}$$

The next step is to form

$$h(x) = \int_0^x g(t^2)\,dt = \sum_{k=0}^\infty \frac{(2k)!}{(k+1)!\,k!}\,\frac{x^{2k+3}}{2k+3}. \tag{6.22}$$

Now

$$\begin{aligned} h(x) &= \frac{1}{2}\int_0^x (1-\sqrt{1-4t^2})\,dt \\ &= \frac{x}{2} - \frac{1}{2}\int_0^x \sqrt{1-4t^2}\,dt. \end{aligned}$$

Elementary changes of variables yield

$$h(x) = \frac{x}{2} - \frac{1}{8}\sin^{-1}(2x) - \frac{x}{4}x\sqrt{1-4x^2}. \tag{6.23}$$

Define

$$w(x) = \int_0^x \frac{h(t)}{t}\,dt = \sum_{k=0}^\infty \frac{(2k)!}{(k+1)!\,k!}\,\frac{x^{2k+3}}{(2k+3)^2}, \tag{6.24}$$

so that $S_2 = 4w(1/2)$. Now,

$$\begin{aligned} w(x) &= \int_0^x \left(\frac{1}{2} - \frac{1}{8}\frac{\sin^{-1}(2t)}{t} - \frac{1}{4}\sqrt{1-4t^2}\right)dt \\ &= \frac{x}{2} - \frac{1}{8}x\sqrt{1-4x^2} - \frac{1}{16}\sin^{-1}(2x) - \frac{1}{8}\int_0^x \frac{\sin^{-1}(2t)}{t}\,dt \end{aligned}$$

The change of variables $\varphi = \sin^{-1}(2t)$ yields

$$\int_0^x \frac{\sin^{-1}(2t)}{t}\,dt = \int_0^{\sin^{-1}(2x)} \varphi \cot\varphi\, d\varphi. \tag{6.25}$$

It follows that

$$w(\tfrac{1}{2}) = \frac{1}{4} - \frac{\pi}{12} - \frac{1}{8}\int_0^{\pi/2} \frac{\varphi\, d\varphi}{\tan\varphi}. \tag{6.26}$$

The evaluation

$$\int_0^{\pi/2} \frac{\varphi\, d\varphi}{\tan\varphi} = \frac{\pi}{2}\ln 2, \tag{6.27}$$

appears as entry **3.747.7** and it has been presented in [**1**]. Therefore

$$I(2) = \frac{\pi}{8}(1 + 2\ln 2), \tag{6.28}$$

as claimed.

The integral in Theorem 6.1 is evaluated in an alternative form. The answer involves the digamma function

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = \frac{d}{dx}\log\Gamma(x). \tag{6.29}$$

The reader will find in [**4**] a variety of entries from [**2**] evaluated in terms of this function.

Theorem 6.2. Let $a > 0$. Then

$$\int_0^\infty xe^{-x}\sqrt{1-e^{-ax}}\,dx = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{a}\right)}{2a^2\Gamma\left(\frac{3}{2}+\frac{1}{a}\right)}\left[\psi\left(\tfrac{3}{2}+\tfrac{1}{a}\right) - \psi\left(\tfrac{1}{a}\right)\right]. \tag{6.30}$$

Proof. The change of variables $t = e^{-x}$ gives

$$\int_0^\infty xe^{-x}\sqrt{1-e^{-ax}}\,dx = -\int_0^1 \ln t\,\sqrt{1-t^a}\,dt. \tag{6.31}$$

This last form of the integral is evaluated by differentiating the identity

$$\int_0^1 (1-t^a)^{1/2}t^b\,dt = \frac{\sqrt{\pi}\Gamma\left(\frac{1+b}{a}\right)}{2a\Gamma\left(\frac{3}{2}+\frac{1+b}{a}\right)} \tag{6.32}$$

at $b = 0$. □

The special values required for the evaluations of the entries discussed in this section are

$$\Gamma(n) = (n-1)! \text{ and } \Gamma\left(n+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}\,(2n)!}{2^{2n}\,n!}, \tag{6.33}$$

which appear as entries **8.339.1** and **8.339.2**, respectively, and also

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1}\frac{1}{k} \text{ and } \psi\left(n+\tfrac{1}{2}\right) = -\gamma + 2\left(\sum_{k=1}^{n}\frac{1}{2k-1} - \ln 2\right), \tag{6.34}$$

which are found as entries **8.365.4** and **8.366.3**, respectively.

7. Some examples related to geometric series

The paper [**5**] contains a variety of entries in [**2**] that are obtained by manipulating the geometric series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \tag{7.1}$$

and the alternating version

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k \tag{7.2}$$

A couple of examples are presented here.

Integrating term by term yields

$$\int_0^1 \frac{x^m}{1+x} \ln x \, dx = \sum_{k=0}^{\infty} (-1)^k \int_0^1 x^{k+m} \ln x \, dx. \tag{7.3}$$

Integration by parts gives

$$\int_0^1 x^{k+m} \ln x \, dx = -\frac{1}{(k+m+1)^2}, \tag{7.4}$$

therefore

$$\begin{aligned}
\int_0^1 \frac{x^m}{1+x} \ln x \, dx &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(k+m+1)^2} \\
&= (-1)^m \sum_{k=m+1}^{\infty} \frac{(-1)^k}{k^2} \\
&= (-1)^{m+1} \left[\frac{\pi^2}{12} + \sum_{k=1}^{m} \frac{(-1)^k}{k^2} \right],
\end{aligned}$$

using

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}. \tag{7.5}$$

This establishes

$$\int_0^1 \frac{x^m}{1+x} \ln x \, dx = (-1)^{m+1} \left[\frac{\pi^2}{12} + \sum_{k=1}^{m} \frac{(-1)^k}{k^2} \right]. \tag{7.6}$$

EXAMPLE 7.1. Entry **4.251.5** states that

$$\int_0^1 \frac{x^{2n}}{1+x} \ln x \, dx = -\frac{\pi^2}{12} - \sum_{k=1}^{2n} \frac{(-1)^k}{k^2}. \tag{7.7}$$

This is the case $m = 2n$ an even integer of (7.6).

EXAMPLE 7.2. Entry **4.251.6** states that

$$\int_0^1 \frac{x^{2n-1}}{1+x} \ln x \, dx = \frac{\pi^2}{12} + \sum_{k=1}^{2n-1} \frac{(-1)^k}{k^2}. \tag{7.8}$$

This is the case $m = 2n-1$ an odd integer of (7.6).

EXAMPLE 7.3. The integral

$$I(\alpha) = \int_0^1 \left(\ln \frac{1}{x} \right)^{\alpha} \frac{dx}{1+x^2} \tag{7.9}$$

is evaluated by expanding $1/(1+x^2)$ as a geometric series to obtain

$$I(\alpha) = \sum_{j=0}^{\infty} (-1)^j \int_0^1 x^{2j} (-\ln x)^{\alpha} \, dx. \tag{7.10}$$

The changes of variables $u = -\ln x$ and $v = (2j+1)u$ give

$$\begin{aligned} I(\alpha) &= \sum_{j=0}^{\infty} (-1)^j \int_0^{\infty} u^{\alpha} e^{-(2j+1)u} \, du \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^{\alpha+1}} \int_0^{\infty} v^{\alpha} e^{-v} \, dv \\ &= \Gamma(\alpha+1) \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^{\alpha+1}}. \end{aligned}$$

Entry **4.269.1** is the special case $\alpha = \frac{1}{2}$ that produces

$$\int_0^1 \sqrt{\ln \frac{1}{x}} \frac{dx}{1+x^2} = \frac{\sqrt{\pi}}{2} \sum_{k=0}^{\infty} \frac{(-1)^j}{\sqrt{(2j+1)^3}} \tag{7.11}$$

and entry **4.269.2**

$$\int_0^1 \frac{dx}{\sqrt{\ln \frac{1}{x}} \, (1+x^2)} = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^j}{\sqrt{2j+1}} \tag{7.12}$$

corresponds to $\alpha = -\frac{1}{2}$.

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