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# Evaluation of integrals through psi function

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Abstract. Some integrals involving various combinations of elementary functions with psi function are evaluated through elementary methods on properties of psi function. In addition, evaluation of integrals involving various combinations of trigonometric and hyperbolic functions with elementary functions are shown.

## 1. Introduction

The classical table of integrals by Gradshteyn and Ryzhik contains list of integrals involving psi function (logarithmic derivative of gamma function[**2**, p. 892]) in the combinations of powers, algebraic functions of exponentials, trigonometric functions, etc[**2**, p. 651-652]. It is usually denoted by

$$\psi(x) = \frac{d}{dx}\left(\log \Gamma(x)\right). \tag{1.1}$$

Where $\psi$ and $\Gamma$ are psi and gamma functions respectively. It satisfies following functional relations for $x \neq 0, -1, -2, \dots$ [**2**, p. 894-895]

$$\psi(x+1) = \psi(x) + \frac{1}{x}. \tag{1.2}$$

$$\psi(1-x) = \psi(x) + \pi \cot \pi x. \tag{1.3}$$

$$\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right) = 2\beta(x). \tag{1.4}$$

Using these three properties, Glasser[**3**] evaluated some integrals involving psi function in closed form. The properties and integral representations of $\psi$ and $\beta$ functions are collected in Ref[**4**] and Ref[**1**] respectively. In Ref[**5**], integrals involving psi function are evaluated through following infinite series [**2**, p. 893]. For real $x$ and $y$,

$$\sum_{k=0}^{\infty} \frac{2yi}{y^2 + (x+k)^2} = \psi(x+iy) - \psi(x-iy). \tag{1.5}$$

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In the present study, evaluation of following integrals involving psi function are given either by finite or infinite number of terms using the identities (1.2)-(1.4) and infinite series given in (1.5).

$$\text{(A)} \qquad \begin{aligned} &\textstyle\int_\alpha^\beta f(x)\left(\psi(z+ax) \pm \psi(z-ax)\right) dx, \\ &\textstyle\int_\alpha^\beta \dfrac{f(x)}{\cos 2ax \pm \cos 2z} dx, \\ &\textstyle\int_\alpha^\beta \dfrac{f(x)\sin 2ax}{\cos 2ax \pm \cos 2z} dx. \end{aligned}$$

The solutions of integrals given in this paper are not found in the classical table of integrals[**2**]. Also, they cannot be expressed in closed form using symbolic language such as Mathematica.

## 2. Some identities between integrals

Some relationships between integrals in (A) are derived in this section.

Definition 2.1. Let $f$ be a continuous function on $(\alpha, \beta)$, and let $a$ and $z$ are any numbers. Then, define

$$A(z,a) = \int_\alpha^\beta f(x)\left[\psi(z+ax) - \psi(z-ax)\right] dx. \tag{2.1}$$

and

$$B(z,a) = \int_\alpha^\beta f(x)\left[\psi(z+ax) + \psi(z-ax)\right] dx. \tag{2.2}$$

Theorem 2.2. *Let $f$ be a continuous function on $(\alpha, \beta)$, and let $z$ and $a$ are any numbers. If $z \notin Z$, then*

$$\int_\alpha^\beta \frac{f(x)\sin 2\pi ax}{\cos 2\pi z - \cos 2\pi ax} dx = \frac{1}{2\pi}\left[A(z,a) + A(1-z,a)\right]. \tag{2.3}$$

$$\int_\alpha^\beta \frac{f(x)}{\cos 2\pi z - \cos 2\pi ax} dx = \frac{1}{2\pi}\frac{B(z,a) - B(1-z,a)}{\sin 2\pi z}, \quad 2z \notin Z. \tag{2.4}$$

$$\int_\alpha^\beta \frac{f(x)\sinh 2\pi ax}{\cos 2\pi z - \cosh 2\pi ax} dx = \frac{1}{2\pi i}\left[A(z,ia) + A(1-z,ia)\right]. \tag{2.5}$$

$$\int_\alpha^\beta \frac{f(x)}{\cos 2\pi z - \cosh 2\pi ax} dx = \frac{1}{2\pi}\frac{B(z,ia) - B(1-z,ia)}{\sin 2\pi z}, \quad 2z \notin Z. \tag{2.6}$$

Proof. Replacing $z$ by $1-z$ in (2.1), then

$$A(1-z,a) = \int_\alpha^\beta f(x)\left(\psi(1-z+ax) - \psi(1-z-ax)\right) dx. \tag{2.7}$$

Adding (2.1) and (2.7), gives

$$\int_{\alpha}^{\beta} f(x)\left(\psi(z+ax)-\psi(1-z-ax)-\psi(z-ax)+\psi(1-z+ax)\right)dx = A(z,a)+A(1-z,a).$$

Using the identity (1.3), then

$$\int_{\alpha}^{\beta} f(x)\left(\cot\pi(z+ax)-\cot\pi(z-ax)\right)dx = \frac{1}{\pi}\left(A(z,a)+A(1-z,a)\right).$$

After simplification, gives (2.3). Similarly, (2.4) can be easily found. Replacing $a$ by $ai$ in (2.3) and (2.4) and after simplification, gives (2.5) and (2.6). This completes the theorem. □

THEOREM 2.3. *Let $f$ be a continuous function on $(\alpha,\beta)$ and, let $z, 2z \notin Z$ and $a$ be any number. Then*

$$\int_{\alpha}^{\beta} \frac{f(x)\sin 2\pi ax dx}{\cos 2\pi ax+\cos 2\pi z} = \frac{1}{2\pi}\left[A(z,a)+A(1-z,a)-2A(2z,2a) -2A(1-2z,2a)\right]. \tag{2.8}$$

$$\int_{\alpha}^{\beta} \frac{f(x)}{\cos 2\pi ax+\cos 2\pi z}dx = \frac{1}{2\pi}\frac{1}{\sin 2\pi z}\left[2B(2z,2a)-2B(1-2z,2a) -B(z,a)+B(1-z,a)\right]. \tag{2.9}$$

$$\int_{\alpha}^{\beta} \frac{f(x)\sinh 2\pi ax dx}{\cosh 2\pi ax+\cos 2\pi z} = \frac{1}{2\pi i}\left[A(z,ia)+A(1-z,ia)-2A(2z,2ia) -2A(1-2z,2ia)\right]. \tag{2.10}$$

$$\int_{\alpha}^{\beta} \frac{f(x)}{\cosh 2\pi ax+\cos 2\pi z}dx = \frac{1}{2\pi}\frac{1}{\sin 2\pi z}\left[2B(2z,ia)-2B(1-2z,ia) -B(z,ai)+B(1-z,ai)\right]. \tag{2.11}$$

PROOF. Replacing $a$ by $2a$ and $z$ by $2z$ in (2.3), then

$$\int_{\alpha}^{\beta} \frac{f(x)\sin 4\pi ax}{\cos 4\pi z-\cos 4\pi ax}dx = \frac{1}{2\pi}\left(A(2z,2a)+A(1-2z,2a)\right).$$

Since $\cos 2\pi ax = 2\cos^2\pi ax - 1$, gives

$$\int_{\alpha}^{\beta} \frac{f(x)\sin 2\pi ax\cos 2\pi ax}{\cos^2 2\pi z-\cos^2 2\pi ax}dx = \frac{1}{2\pi}\left(A(2z,2a)+A(1-2z,2a)\right).$$

Splitting the denominator by partial fraction, gives

$$\int_{\alpha}^{\beta} \frac{f(x)\sin 2\pi ax}{\cos 2\pi z - \cos 2\pi ax}dx - \int_{\alpha}^{\beta} \frac{f(x)\sin 2\pi ax}{\cos 2\pi z + \cos 2\pi ax}dx$$
$$= \frac{1}{\pi}\left[A(2z, 2a) + A(1-2z, 2a)\right].$$

After simplification gives (2.8). Similarly, (2.9) can be easily found . Replacing $a$ by $ia$ in (2.8) and (2.9), gives (2.9) and (2.10). This completes the theorem. □

Example 2.4. Let $f(x) = \sin xy$ and $a = 1$ in (2.5). Then

$$\int_{0}^{\infty} \frac{\sin xy \sinh 2x}{\cos 2\pi z - \cosh 2\pi x}dx = \frac{1}{2\pi}\left[A(z, i) + A(1-z, i)\right].$$

But entry **6.466** from Ref [**2**, p. 652] states that for $z > 0$ and $y > 0$

$$\int_{0}^{\infty} \left[\psi(z+ix) - \psi(z-ix)\right]\sin xy dx = i\pi e^{-zy}(1-e^{-y})^{-1} = A(z, i).$$

So that

$$\int_{0}^{\infty} \frac{\sin xy \sinh 2x}{\cos 2\pi z - \cosh 2\pi x}dx = \frac{1}{2}\frac{e^{-zy} + e^{-z(1-y)}}{1-e^{-y}}.$$

Similarly, using (2.10), gives

$$\int_{0}^{\infty} \frac{\sin xy \sinh 2x}{\cosh 2\pi x + \cos 2\pi z}dx = \frac{1}{2}\frac{1}{1-e^{-y}}\left[\frac{e^{-2zy} + e^{-(1-2z)y}}{\cos 2\pi z} - e^{-zy} - e^{-(1-z)y}\right].$$

## 3. Evaluation of integrals involving psi function in closed form

In this section, the integrals involving various combinations of psi function are evaluated through integral representation of psi function.

Theorem 3.1. *Let $f$ be a continuous function on $(\alpha, \beta)$, and let $z$ and $a$ are any numbers. Then*

$$A(z, ia) = 2i\int_{0}^{\infty} \frac{e^{-zt}}{1-e^{-t}} \int_{\alpha}^{\beta} f(x)\sin axt dt dx. \tag{3.1}$$

Proof. Entry **3.427.1** from Ref [**2**, p. 355 ] states that for $\mu > 0$

$$\int_{0}^{\infty} \left(\frac{e^{-x}}{x} + \frac{e^{-\mu x}}{e^{-x}-1}\right)dx = \psi(\mu).$$

For $a$ and $z$, it can be written as

$$\psi(z+ax) - \psi(z-ax) = 2\int_{0}^{\infty} \frac{e^{-zt}\sinh axt}{1-e^{-t}}dt.$$

Using definition 2.1, gives

$$A(z, a) = \int_{\alpha}^{\beta} f(x)\left(\psi(z+ax) - \psi(z-ax)\right)dx dt.$$
$$= 2\int_{\alpha}^{\beta} f(x)\int_{0}^{\infty} \frac{e^{-zt}\sinh axt}{1-e^{-t}}dt dx.$$

Rearranging above integrals, gives

$$A(z,a) = 2\int_0^\infty \frac{e^{-zt}}{1-e^{-t}} \int_\alpha^\beta f(x)\sinh axtdtdx. \tag{3.2}$$

Replacing $a$ by $ia$ gives (3.1). This completes the theorem. □

REMARK 3.2. The following results are immediately obtained from (3.1) and (3.2)

$$A(z,ia) - A(y,ia) = 2i\int_0^\infty \frac{e^{-zt}-e^{-yt}}{1-e^{-t}} \int_\alpha^\beta f(x)\sin axtdtdx. \tag{3.3}$$

$$A(z,a) - A(y,a) = 2\int_0^\infty \frac{e^{-zt}-e^{-yt}}{1-e^{-t}} \int_\alpha^\beta f(x)\sinh axtdtdx. \tag{3.4}$$

EXAMPLE 3.3. Many integrals can be evaluated through Theorem 3.1. The following are few of them.

(1) Let $f(x) = (\beta+ix)^{-v} - (\beta-ix)^{-v}$ in (3.1) on $(0,\infty)$. Then

$$\begin{aligned} &\int_0^\infty \left[(\beta+ix)^{-v} - (\beta-ix)^{-v}\right][\psi(z+iax) - \psi(z-iax)]\,dx \\ &= 2i\int_0^\infty \frac{e^{-zt}}{1-e^{-t}} \int_0^\infty \left[(\beta+ix)^{-v} - (\beta-ix)^{-v}\right]\sin axtdxdt. \end{aligned} \tag{3.5}$$

The formula **3.769.1** in Ref[**2**, p.435] states that for $a>0$, $\mathrm{Re}(\beta)>0$ and $\mathrm{Re}(v)>0$

$$\int_0^\infty \left[(\beta+ix)^{-v} - (\beta-ix)^{-v}\right]\sin axdx = -i\pi\frac{a^{v-1}e^{-a\beta}}{\Gamma(v)}. \tag{3.6}$$

Using (3.6) in (3.5), gives

$$\begin{aligned} &\int_0^\infty \left[(\beta+ix)^{-v} - (\beta-ix)^{-v}\right][\psi(z+iax) - \psi(z-iax)]\,dx \\ &= 2\pi\frac{a^{v-1}}{\Gamma(v)}\int_0^\infty \frac{e^{-(z+a\beta)t}}{1-e^{-t}}t^{v-1}dt. \end{aligned} \tag{3.7}$$

The formula **3.411.7** from Ref[**3**, p. 349] states that

$$\int_0^\infty \frac{x^{v-1}e^{-\mu x}}{1-e^{-\beta x}}dx = \frac{1}{\beta^v}\Gamma(v)\zeta\left(v,\mu/\beta\right). \tag{3.8}$$

Using (3.8) in (3.7), yields

$$\begin{aligned} &\int_0^\infty \left[(\beta+ix)^{-v} - (\beta-ix)^{-v}\right][\psi(z+iax) - \psi(z-iax)]\,dx \\ &= 2\pi a^{v-1}\zeta(v,z+a\beta). \end{aligned} \tag{3.9}$$

Let $z=1/2$ in (3.9), gives

$$\int_0^\infty \left[(\beta+ix)^{-v} - (\beta-ix)^{-v}\right]\tanh axdx = -2ia^{v-1}\zeta(v,1/2+a\beta).$$

(2) Let $f(x) = \frac{1}{\beta^2+(\gamma-x)^2} - \frac{1}{\beta^2+(\gamma+x)^2}$ in (3.1) on $(0,\infty)$. Using the integral **3.732.1** in Ref [**2**, p. 422]

$$\int_0^\infty \left[\frac{1}{\beta^2+(\gamma-x)^2} - \frac{1}{\beta^2+(\gamma+x)^2}\right] \sin ax dx = \frac{\pi}{\beta} e^{-a\beta} \sin(a\gamma).$$

Then, (3.1) gives

$$\int_0^\infty \left[\frac{1}{\beta^2+(\gamma-x)^2} - \frac{1}{\beta^2+(\gamma+x)^2}\right] [\psi(z+ix) - \psi(z-iax)]\, dx = 2i\frac{\pi}{\beta}\int_0^\infty \frac{e^{-(z+a\beta)t}}{1-e^{-t}} \sin at\gamma dt. \tag{3.10}$$

Further, the integral **3.911.6** from Ref[**2**, p. 484] states that

$$\int_0^\infty \frac{\sin ax}{e^{\beta x}(e^{-x}-1)} dx = \frac{i}{2}\left[\psi(\beta+ia) - \psi(\beta-ai)\right].$$

Using above entry in (3.10), gives

$$\int_0^\infty \left[\frac{1}{\beta^2+(\gamma-x)^2} - \frac{1}{\beta^2+(\gamma+x)^2}\right] [\psi(z+ix) - \psi(z-iax)]\, dx = \frac{\pi}{\beta}\left[\psi(z+a\beta+i\gamma) - \psi(z+a\beta-\gamma i)\right]. \tag{3.11}$$

Let $z = 1/2$ in (3.11). Then

$$\int_0^\infty \left[\frac{1}{\beta^2+(\gamma-x)^2} - \frac{1}{\beta^2+(\gamma+x)^2}\right] \tanh a\pi x dx = -\frac{i}{\beta}\left[\psi(1/2+a\beta+i\gamma) - \psi(1/2+a\beta-\gamma i)\right].$$

(3) Let $f(x) = x/(x^2+\beta^2)^{n+1}$ in (3.1). Formula **3.737.2** in Ref[**3**, p. 423] states that

$$\int_0^\infty \frac{x \sin ax}{(x^2+\beta^2)^{n+1}} dx = \frac{\pi e^{-a\beta}}{2^{2n} n! \beta^{2n-1}} \sum_{k=0}^{n-1} \frac{2^k (2n-k-2)! \beta^k a^{k+1}}{k!(n-k-1)!}. \tag{3.12}$$

Using (3.12) in (3.1), gives

$$\int_0^\infty \frac{x\left[\psi(z+iax) - \psi(z-iax)\right]}{(x^2+\beta^2)^{n+1}} dx = \frac{2i\pi}{2^{2n} n! \beta^{2n-1}} \sum_{k=0}^{n-1} \frac{2^k (2n-k-2)! \beta^k a^{k+1}(k+1)}{(n-k-1)!} \zeta(k+2, z+a\beta). \tag{3.13}$$

(4) Similarly, using the entry **3.776.1** from Ref[**3**, p. 440], gives

$$\int_0^\infty \frac{a^2(b+x)^2 + p(p+1)}{(b+x)^{p+2}} \left[\psi(z+iax) - \psi(z-iax)\right] dx = 2i\frac{a}{b^p}\zeta(2,z).$$

EXAMPLE 3.4. Let $f(x) = 1/x$ in (3.3) on $(0, \infty)$ and using the integral $\int_0^\infty \frac{\sin ax}{x} dx = \pi/2$. Then

$$\int_0^\infty [\psi(z+iax) - \psi(z-iax) - \psi(y+iax) + \psi(y-iax)] \frac{dx}{x} = i\pi [\psi(y) - \psi(z)].$$

THEOREM 3.5. *Let $f$ be a continuous function on $(\alpha, \beta)$ and let $z$ and $a$ are any numbers. Then*

$$B(z, ia) - B(z, ib) = 2\int_0^\infty \frac{e^{-zt}}{1-e^{-t}} \int_\alpha^\beta f(x)(\cos bxt - \cos axt) dt dx. \tag{3.14}$$

PROOF. Using entry **3.427.1** from Ref[**2**, p. 355] for $a > 0$ and $z > 0$

$$\psi(z+ax) + \psi(z-ax) - \psi(z+bx) - \psi(z-bx) = 2\int_0^\infty \frac{e^{-zt}(\cosh axt - \cosh bxt)}{e^{-t}-1} dt.$$

Using definition (2.1), gives

$$B(z, a) - B(z, b) = \int_\alpha^\beta f(x) \left(\psi(z+ax) + \psi(z-ax) - \psi(z+bx) - \psi(z-bx)\right) dx dt.$$
$$= 2\int_\alpha^\beta f(x) \int_0^\infty \frac{e^{-zt}(\cosh axt - \cosh bxt)}{e^{-t}-1} dt dx.$$

Rearranging above integrals, gives

$$B(z, a) - B(z, b) = 2\int_0^\infty \frac{e^{-zt}}{e^{-t}-1} \int_\alpha^\beta f(x)(\cosh axt - \cosh bxt) dx dt.$$

Replacing $a$ by $ia$ and $b$ and $ib$, after simplification gives (3.14). This completes the theorem. ☐

EXAMPLE 3.6. The following are some examples of evaluation of integrals through Theorem 3.5.

(1) Let $f(x) = 1/x^2$ in (3.14). Then

$$\int_0^\infty \frac{\psi(z+ax) + \psi(z-ax) - \psi(z+bx) - \psi(z-bx)}{x^2} dx = 2\int_0^\infty \frac{e^{-zt}}{1-e^{-t}} \int_0^\infty \frac{\cos axt - \cos bxt}{x^2} dx dt. \tag{3.15}$$

Entry **3.784** from Ref[**2**, p. 441] states that

$$\int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx = \frac{\pi}{2}(b-a). \tag{3.16}$$

Using (3.16) in (3.15), gives

$$\int_0^\infty \frac{\psi(z+ax) + \psi(z-ax) - \psi(z+bx) - \psi(z-bx)}{x^2} dx = \pi(b-a)\zeta(2, z).$$

(2) Let $f(x) = 1/(x^2(x^2+y^2))$ in (3.14) and use entry **3.784.8** from Ref[**2**, p. 441]. Then

$$\int_0^\infty \frac{\psi(z+ax)+\psi(z-ax)-\psi(z+bx)-\psi(z-bx)}{x^2(x^2+y^2)}dx$$
$$= \frac{\pi(b-a)}{y^2}\zeta(2,z) + \frac{\pi}{y^3}\left(\psi(z+ay)-\psi(z+by)\right).$$

(3) Let $f(x) = \frac{1}{(x^2+y^2)^n}$ in (3.14). Then

$$\int_0^\infty \frac{\psi(z+ax)+\psi(z-ax)-\psi(z+bx)-\psi(z-bx)}{(x^2+y^2)^n}dx \tag{3.17}$$
$$= 2\int_0^\infty \frac{e^{-zt}}{1-e^{-t}}\int_0^\infty \frac{(\cos axt - \cos bxt)}{(x^2+y^2)^n}dxdt.$$

Entry **3.737.1** from Ref[**2**, p. 423] states that

$$\int_0^\infty \frac{\cos ax}{(x^2+y^2)^n}dx = \frac{\pi e^{-ay}}{(2y)^{2n-1}(n-1)!}\sum_{k=0}^{n-1}\frac{(2n-k-2)!(2ay)^k}{k!(n-k-1)!}. \tag{3.18}$$

Using (3.18) in (3.17), gives

$$\int_0^\infty \frac{\psi(z+ax)+\psi(z-ax)-\psi(z+bx)-\psi(z-bx)}{(x^2+y^2)^n}dx = \frac{\pi}{(2y)^{2n-1}(n-1)!}\times$$
$$\sum_{k=0}^{n-1}\frac{(2n-k-2)!2^{k+1}y^k}{(n-k-1)!}\left(a^k\zeta(z+ay,k)-b^k\zeta(z+by,k)\right).$$

## 4. Integrals through infinite series

THEOREM 4.1. *Let $f$ be a continuous function on $(\alpha,\beta)$, and let $z$ be any number and $|a|<1$,*

$$A(z,a) = 2\sum_{k=1}^\infty a^{2k-1}\zeta(2k,z)\int_\alpha^\beta x^{2k-1}f(x)dx. \tag{4.1}$$

$$B(z,a) = 2\psi(z) - 2\sum_{k=1}^\infty a^{2k}\zeta(2k+1,z)\int_\alpha^\beta x^{2k}f(x)dx. \tag{4.2}$$

PROOF. Expanding psi function $\psi(z+ax)$ and $\psi(z-ax)$ by Taylor series at $x=0$, gives

$$\psi(z+ax) = \psi(z) + \sum_{k=1}^\infty (-1)^{k-1}a^k x^k \zeta(k+1,z). \tag{4.3}$$

$$\psi(z-ax) = \psi(z) - \sum_{k=1}^\infty a^k x^k \zeta(k+1,z). \tag{4.4}$$

Subtracting (4.3) from (4.4), gives (4.1). Adding (4.3) and (4.4), gives (4.2). ☐

EXAMPLE 4.2. The following are some examples of evaluation of integrals using (4.1) and (4.2).

(1) Let $f(x) = x^{r-1}(1-x)^{s-1}$ on $(0,1)$ in (4.1) and (4.2). Then

$$\int_0^1 x^{r-1}(1-x)^{s-1}\left(\psi(z+ax)-\psi(z-ax)\right)dx = 2K\sum_{k=1}^{\infty}\frac{a^{2k-1}(r)_k}{(r+s)_k}\zeta(2k,z).$$

$$\int_0^1 \frac{x^{r-1}(1-x)^{s-1}\sin 2\pi ax}{\cos 2\pi ax-\cos 2\pi z}dx = -\frac{K}{\pi}\sum_{k=1}^{\infty}\frac{a^{2k-1}(r)_{2k-1}}{(r+s)_{2k-1}}\left[\zeta(2k,z)+\zeta(2k,1-z)\right].$$

$$\int_0^1 \frac{x^{r-1}(1-x)^{s-1}}{\cos 2\pi ax-\cos 2\pi z}dx = -\frac{1}{\pi}\frac{\psi(z)-\psi(1-z)}{\sin 2\pi z} + \frac{K}{\pi}\sum_{k=1}^{\infty}\frac{a^{2k-1}(r)_{2k-1}}{(r+s)_{2k-1}}\left[\frac{\zeta(2k+1,z)-\zeta(2k+1,1-z)}{\sin 2\pi z}\right].$$

Where $K = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$ and $(a)_k$ is Pochhammer symbol.

(2) Let $f(x) = \mathrm{li}(x)x^{p-1}$ on $(0,1)$ in (4.1) and (4.2). Use the solution of the integral $\int_0^1 \mathrm{li}(x)x^{p-1}dx = -\frac{1}{p}\log(p+1)$ for $p > -1$ to get following results.

$$\int_0^1 \mathrm{li}(x)x^{p-1}\left(\psi(z+ax)-\psi(z-ax)\right)dx = -2\sum_{k=1}^{\infty}\frac{a^{2k-1}\log(p+2k)}{p+2k-1}\zeta(2k,z).$$

$$\int_0^1 \frac{\mathrm{li}(x)x^{p-1}\sin 2axdx}{\cos 2\pi ax-\cos 2\pi z} = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{a^{2k-1}\log(p+2k)}{p+2k-1}\left[\zeta(2k,z)+\zeta(2k,1-z)\right].$$

$$\int_0^1 \frac{\mathrm{li}(x)x^{p-1}dx}{\cos 2\pi ax-\cos 2\pi z} = \frac{1}{\pi}\frac{\psi(z)-\psi(1-z)}{\sin 2\pi z} - \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{a^{2k-1}\log(p+2k+1)}{p+2k}\left[\frac{\zeta(2k+1,z)-\zeta(2k+1,1-z)}{\sin 2\pi z}\right].$$

## 5. Evaluation of integrals though infinite series

In this section, the integrals involving psi and Hurwitz zeta functions are evaluated through infinite series.

Entry **3.232** from Ref[**2**, p. 318] states that for $\mathrm{Re}\mu > -1$, $a > 0$, $b > 0$, $c > 0$

$$\int_0^{\infty}\frac{(c+ax)^{-\mu}-(c+bx)^{-\mu}}{x}dx = c^{-\mu}\log(b/a).$$

Replacing $c$ by $c+k$ and taking summation for $k = 0,1,2,\ldots$, gives

$$\int_0^{\infty}\sum_{k=0}^{\infty}(c+k+ax)^{-\mu}-(c+k+bx)^{-\mu}\frac{dx}{x} = \log(b/a)\sum_{k=0}^{\infty}(c+k)^{-\mu}.$$

Using definition of Hurwitz zeta function[**2**, p. 1027], yields

$$\int_0^\infty (\zeta(\mu, c+ax) - \zeta(\mu, c+bx))\, \frac{dx}{x} = \zeta(\mu, c)\log(b/a). \tag{5.1}$$

Entry **3.194.3** from Ref[**2**, p. 313] states that for $|\arg\beta| < \pi$, Re $v >$ Re $\mu > 0$

$$\int_0^\infty \frac{x^{\mu-1}}{(1+\beta x)^v} dx = \beta^{-v} B(\mu, v-\mu).$$

It is equivalent to

$$\int_0^\infty \frac{x^{\mu-v-1}}{(1/x+\beta)^v} dx = \beta^{-v} B(\mu, v-\mu).$$

Then it is easy to find that

$$\int_0^\infty x^{u-v-1}\zeta(v, 1/x+\beta)dx = B(u, v-u)\zeta(v,\beta).$$

Also, it can be written as

$$\int_0^\infty \zeta(v, x+\beta)\frac{dx}{x^{u-v+1}} dx = B(u, v-u)\zeta(v,\beta). \tag{5.2}$$

Entry **3.249.1** from Ref[**2**, p. 321] states that for $n \in N$

$$\int_0^\infty \frac{dx}{(x^2+a^2)^n} = \frac{(2n-3)!!}{2.(2n-2)!!}\frac{\pi}{a^{2n-1}}.$$

It is easy to find that

$$\int_0^\infty \zeta(n, t+a)\frac{dt}{\sqrt{t}} = \frac{(2n-3)!!}{(2n-2)!!}\pi\zeta(n-1/2, a). \tag{5.3}$$

Entry **3.749.1** from Ref[**3**, p. 428] states that for $a > 0$, $b > 0$

$$\int_0^\infty \frac{x\tan ax}{x^2+b^2} dx = \frac{\pi}{e^{2ab}+1}. \tag{5.4}$$

Replace $b$ by $b+k$ and multiply by $2i$ and taking summation on both sides $k = 1, 2, \ldots$, gives the following identity

$$\int_0^\infty \tan ax(\psi(b+ix) - \psi(b-ix))dx = 2i\pi\sum_{k=0}^\infty \frac{1}{e^{2a(b+k)}+1}. \tag{5.5}$$

Let $b = 1/2$ in (5.5). Then

$$\int_0^\infty \tan ax \tanh x dx = \sum_{k=0}^\infty \frac{1}{e^{a(2k+1)}+1}.$$

Similarly, using formula **3.749.2** from Ref[**2**, p. 428], gives

$$\int_0^\infty \cot ax(\psi(b+ix) - \psi(b-ix))dx = 2i\pi\sum_{k=0}^\infty \frac{1}{e^{2a(b+k)}-1}. \tag{5.6}$$

Let $b = 1/2$ in (5.6). Then

$$\int_0^\infty \cot ax \tanh x dx = 2\sum_{k=0}^\infty \frac{1}{e^{a(2k+1)} - 1}.$$

Using (5.5) and (5.6), gives

$$\int_0^\infty \frac{\psi(b+ix) - \psi(b-ix)}{\sin ax} dx = 2i\pi \sum_{k=0}^\infty \operatorname{csch}\, 2a(b+k). \tag{5.7}$$

Let $b = 1/2$ in (5.7). Then

$$\int_0^\infty \frac{\tanh x}{\sin 2ax} dx = \sum_{k=0}^\infty \operatorname{csch}\, a(2k+1).$$

## 6. Conclusion

The integrals involving various combinations of elementary functions with psi functions are evaluated through elementary methods using properties of psi function and infinite series. Most of integrals given here are not available in the classical table of integrals by Gradshteyn and Ryzhik [**2**]. Also, they cannot be expressed in closed form using a symbolic language.

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