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One modulo N gracefulness of H-class of graphs

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Abstract. A function f is called a graceful labelling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, \ldots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. A graph G is said to be one modulo N graceful (where N is a positive integer) if there is a function ϕ from the vertex set of G to $\{0, 1, N, (N + 1), 2N, (2N + 1), \ldots, N(q - 1), N(q - 1) + 1\}$in such a way that (i) ϕ is $1 - 1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N + 1, 2N + 1, \ldots, N(q - 1) + 1\}$where $\phi^*(uv)$=$|\phi(u) - \phi(v)|$. In this paper we prove that the H-graph, $H \bigodot mK_1$ and $H^{(a)}_{(n)}$ are one modulo N graceful for all positive integers N.

1. Introduction

Golomb [2] introduced graceful labelling. Rosa [6] has shown that if every vertex has even degree and the number of edges is congruent to 1 or 2 $(mod 4)$ then the graph is not graceful. Odd gracefulness was introduced by Gnanajothi [3]. Sekar [8] introduced one modulo three graceful labelling. Ramachandran and Sekar [7] introduced the concept of one modulo N graceful where N is any positive integer. In the case $N = 2$, the labelling is odd graceful and in the case $N = 1$ the labelling is graceful. We also introduce the concept $H^{(a)}_{(n)}$. Gallian [4] surveyed numerous graph labelling methods. We prove that the H-graph, $H \bigodot mK_1$ and $H^{(a)}_{(n)}$ are one modulo N graceful for all positive integers N.

2. Main Results

Definition 2.1. A graph G with q edges is said to be **one modulo N graceful** (where N is a positive integer) if there is a function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$ in such a way that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \ldots, N(q-1) + 1\}$where $\phi^*(uv)$=$|\phi(u) - \phi(v)|$.

Definition 2.2. Let $P_1(u_1, u_2, \ldots, u_n)$ and $P_2(v_1, v_2, \ldots, v_n)$ be two paths of length n. Then a new graph obtained by joining $u_{\frac{n}{2}}$ and $v_{\frac{n}{2}+1}$ (or $u_{\frac{n+1}{2}}$ and $v_{\frac{n+1}{2}}$) if n is even (or odd). Then the graph is called as **H-graph** of length n .

Definition 2.3. [1] The **Corona** $G_1 \odot G_2$ of two graphs G_1 and G_2(where G_i has p_i points and q_i lines) is defined as the graph G obtained by taking one copy of G_1 and p_1 copies of G_2, and then joining by a line the i^{th} point of G_1 to every point in the i^{th} copy of G_2. It follows from the definition of the corona that $G_1 \odot G_2$ has $p_1(1 + p_2)$ points and $q_1 + p_1q_2 + p_1p_2$ lines.

Definition 2.4. The graph $H \bigodot mK_1$ or $H \odot \bar{K}_m$ is called $m-$**crown** with H-graph of length n.

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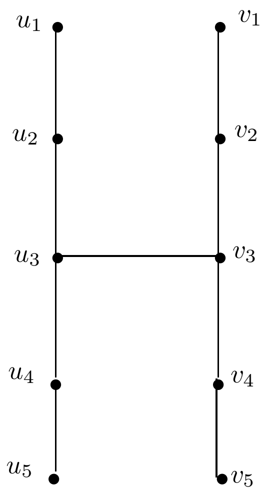
Definition 2.5. The graph $H_{(n)}^{(a)}$ is a graph obtained from the H-graph by attaching an arbitrary number of pendant vertices at any vertex on the two paths of n vertices, $1 \leqslant i \leqslant n$.

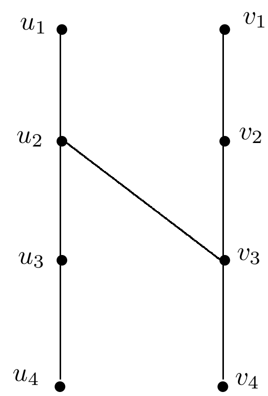
Definition 2.6. The graph $H'_{(n)}$ is a graph obtained from the H-graph by attaching i pendant vertices at each i^{th} vertex on the two paths of n vertices, $1 \leqslant i \leqslant n$.

Theorem 2.1. *Any H-graph is one modulo N graceful for every positive integer N.*

Proof: Let $P_1(u_1, u_2, \ldots, u_n)$ and $P_2(v_1, v_2, \ldots, v_n)$ be two distinct paths with $n \geqslant 3$ vertices in a H-graph. The H-graph has $2n$ vertices and $2n-1$ edges. Naming of the vertices is as shown in Figure.



n is odd



n is even

Case (i) n is odd

Let $n = 2k+1, k \geqslant 1$

Define

$\phi(u_{2i}) = N(i-1)$ for $i = 1, 2, \ldots, k$

$\phi(u_{2i-1}) = 3Nk + 1 + N(k+1-i)$ for $i = 1, 2, \ldots, k+1$

$\phi(v_{2i}) = 2Nk + 1 + N(k-i)$ for $i = 1, 2, \ldots, k$

$\phi(v_{2i-1}) = N(k+i-1)$ for $i = 1, 2, \ldots, k+1$

From the definition of ϕ it is clear that

$\{\phi(u_{2i}), i = 1, 2, \ldots, k\} \cup \{\phi(u_{2i-1}), i = 1, 2, \ldots, k+1\} \cup \{\phi(v_{2i}), i = 1, 2, \ldots, k\} \cup \{\phi(v_{2i-1}),\ i = 1, 2, \ldots, k+1\}$

$$= \{0, N, 2N, \ldots, N(k-1)\} \cup \{4Nk+1, N(4k-1)+1, \ldots, 3Nk+1\}$$
$$\cup \{N(3k-1)+1, N(3k-2)+1, \ldots, 2Nk+1)\} \cup \{Nk, N(k+1), \ldots, 2kN\}$$

Thus it is clear that the vertices have distinct labels. Therefore ϕ is $1-1$.

We compute the edge labelling in the following sequence.

$\mid \phi(u_{k+1}) - \phi(v_{k+1}) \mid = 2Nk + 1$

For $i = 1, 2, \ldots, k$

$\mid \phi(u_{2i-1}) - \phi(u_{2i}) \mid = 2N(2k-i+1) + 1$

For $i = 1, 2, \ldots, k$

$\mid \phi(u_{2i+1}) - \phi(u_{2i}) \mid = N(4k-2i+1) + 1$

For $i = 1, 2, \ldots, k$

$\mid \phi(v_{2i}) - \phi(v_{2i-1}) \mid = N(2k-2i+1) + 1$

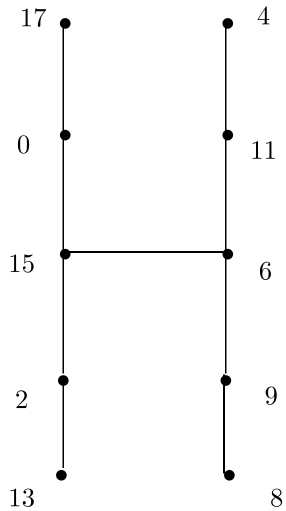
For $i = 1, 2, \ldots, k$

$\mid \phi(v_{2i}) - \phi(v_{2i+1}) \mid = 2N(k-i) + 1$

This shows that the edges have the distinct labels
$\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$.

It is clear from the above labelling that the function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$ is such that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$where $\phi^*(uv)=|\phi(u) - \phi(v)|$. Hence the H-graph is one modulo N graceful.
Clearly ϕ defines a one modulo N graceful labelling of H-graph.

EXAMPLE 2.7. Odd graceful labelling of H-graph. $(n = 5)$



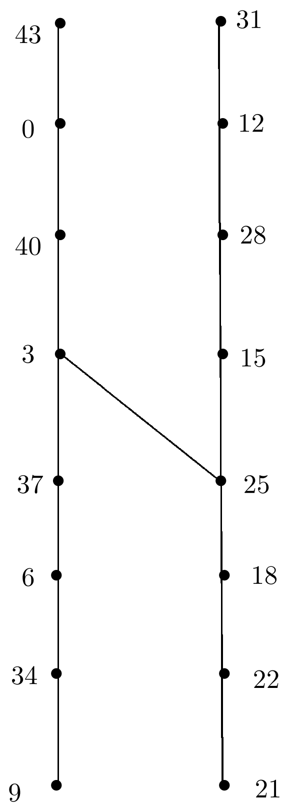
Case (ii) n is even
Let $n = 2k, k \geqslant 1$
Define $\phi(u_{2i}) = N(i-1)$ for $i = 1, 2, \ldots, k$
$\phi(u_{2i-1}) = 3Nk - N + 1 + N(k-i)$ for $i = 1, 2, \ldots, k$
$\phi(v_{2i}) = Nk + N(i-1)$ for $i = 1, 2, \ldots, k$
$\phi(v_{2i-1}) = 2Nk + 1 + N(k-i) - N$ for $i = 1, 2, \ldots, k$

From the definition of ϕ it is clear that
$\{\phi(u_{2i}), i = 1, 2, \ldots, k\} \cup \{\phi(u_{2i-1}), i = 1, 2, \ldots, k\} \cup \{\phi(v_{2i}), i = 1, 2, \ldots, k\} \cup \{\phi(v_{2i-1}), i = 1, 2, \ldots, k\}$
$= \{0, N, 2N, \ldots, N(k-1)\} \cup \{2N(2k-1)+1, N(4k-3)+1, \ldots, N(3k-1)+1\}$
$\cup \{Nk, N(k+1), ..., N(2k-1)\} \cup \{N(3k-2)+1, N(3k-3)+1, ..., N(2k-1)+1\}$
Thus it is clear that the vertices have distinct labels. Therefore ϕ is $1-1$.

We compute the edge labelling in the following sequence.
$|\ \phi(u_k) - \phi(v_{k+1})\ | = N(2k-1)+1$
For $i = 1, 2, \ldots, k$
$|\ \phi(u_{2i-1}) - \phi(u_{2i})\ | = 2N(2k-i)+1$
For $i = 1, 2, \ldots, k-1$
$|\ \phi(u_{2i+1}) - \phi(u_{2i})\ | = N(4k-2i-1)+1$
For $i = 1, 2, \ldots, k$
$|\ \phi(v_{2i-1}) - \phi(v_{2i})\ | = 2N(k-i)+1$
For $i = 1, 2, \ldots, k-1$
$|\ \phi(v_{2i+1}) - \phi(v_{2i})\ | = N(2k-2i-1)+1$
This shows that the edges have the distinct labels
$\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$.

It is clear from the above labelling that the function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$ is such that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$where $\phi^*(uv)=|\phi(u) - \phi(v)|$. Hence the H-graph is one modulo N graceful.
Clearly ϕ defines a one modulo N graceful labelling of H-graph for every positive integer N.

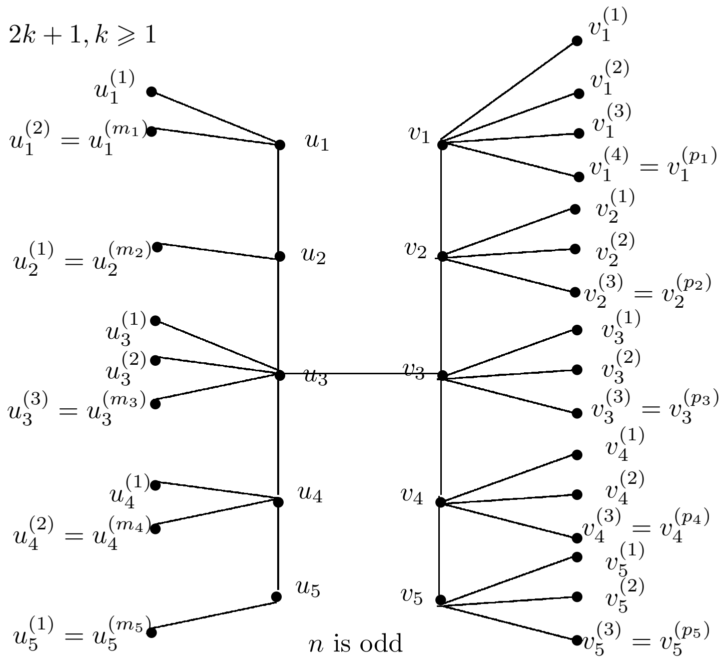
Example 2.8. One modulo 3 graceful labelling of H-graph. $(n = 8)$



Theorem 2.2. *$H^{(a)}_{(n)}$ is one modulo N graceful for every positive integer N.*

Proof. Let G=$H^{(a)}_{(n)}$ where "a" is a arbitrary number of pendant vertices at any vertex on the two paths of n vertices, $1 \leqslant i \leqslant n$. Let $P_1(u_1, u_2, \ldots, u_n)$ and $P_2(v_1, v_2, \ldots, v_n)$ be two distinct paths with $n(\geqslant 3)$ vertices in a H-graph. Let $u_i^{(j)}$,$1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m_i$ and $v_i^{(j)}$,$1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant p_i$. Let $M = m_1 + m_2 + \cdots + m_n + p_1 + p_2 + \cdots + p_n$. The H-graph has $2n + M$ vertices and $2n + M - 1$ edges.
Case (i) n is odd

Let $n = 2k+1, k \geqslant 1$



n is odd

Define

$\phi(u_1) = N\{2(n-1)+M\}+1$

$\phi(u_{2i}) = N(m_1+m_3+\cdots+m_{2i-1}+i-1)$ for $i = 1,2,\ldots,k$

$\phi(u_{2i-1}) = N\{2(n-1)+M\}+1-N(m_2+m_4+\cdots+m_{2i-2}+i-1)$ for $i = 2,3,\ldots,k+1$

$\phi(v_{2i}) = N\{2(n-1)+M\}+1-N(m_2+m_4+\cdots+m_{2k}+k)-N(p_1+p_3+\cdots+p_{2i-1}+i)$ for $i = 1,2,\ldots,k$

$\phi(v_1) = N(m_1+m_3+\cdots+m_{2k+1}+k)$

$\phi(v_{2i-1}) = \phi(v_1)+N(p_2+p_4+\cdots+p_{2i-2}+i-1)$ for $i = 2,3,\ldots,k+1$

For $j = 1,2,\ldots,m_i$

$\phi(u_1^{(j)}) = N(j-1)$

$\phi(u_{2i-1}^{(j)}) = N(m_1+m_3+\cdots+m_{2i-3}+i-1)+N(j-1)$ for $i = 2,3,\ldots,k+1$

$\phi(u_2^{(j)}) = N\{2(n-1)+M-1\}-N(j-1)+1$

$\phi(u_{2i}^{(j)}) = N\{2(n-1)+M-1\}+1-N(m_2+m_4+\cdots+m_{2i-2}+i-1)-N(j-1)$ for $i = 2,3,\ldots,k$

For $j = 1,2,\ldots,p_i$

$\phi(v_2^{(j)}) = N(m_1+m_3+\cdots+m_{2k+1}+k+1)+N(j-1)$

$\phi(v_{2i}^{(j)}) = N(m_1+m_3+\cdots+m_{2k+1}+k+1)+N(p_2+p_4+\cdots+p_{2i-2}+i-1)+N(j-1)$ for $i = 2,3,\ldots,k$

$\phi(v_1^{(j)}) = N\{2(n-1)+M-1\}+1-N(m_2+m_4+\cdots+m_{2k}+k)-N(j-1)$

$\phi(v_{2i-1}^{(j)}) = N\{2(n-1)+M-1\}+1-N(m_2+m_4+\cdots+m_{2k}+k)-N(p_1+p_3+\cdots+p_{2i-3}+i-1)-N(j-1)$ for $i = 2,3,\ldots,k+1$

From the definition of ϕ it is clear that

$\{\phi(u_1)\} \cup \{\phi(u_{2i}), i = 1,2,\ldots,k\} \cup \{\phi(u_{2i-1}), i = 2,3,\ldots,k+1\} \cup \{\phi(v_{2i}), i = 1,2,\ldots,k\} \cup \{\phi(v_1)\} \cup \{\phi(v_{2i-1}), i = 2,3,\ldots,k+1\} \cup \{\phi(u_1^{(j)}), j = 1,2,\ldots,m_i\} \cup \{\phi(u_{2i}^{(j)}), i = 2,3,\ldots,k$ and $j = 1,2,\ldots,m_i\} \cup \{\phi(u_{2i-1}^{(j)}), i = 2,3,\ldots,k+1$ and $j = 1,2,\ldots,m_i\} \cup \{\phi(u_2^{(j)}), j = 1,2,\ldots,m_i\} \cup \{\phi(v_1^{(j)}), j = 1,2,\ldots,p_i\} \cup \{\phi(v_2^{(j)}), j = 1,2,\ldots,p_i\} \cup \{\phi(v_{2i}^{(j)}), i = 2,3,\ldots,k$ and $j = 1,2,\ldots,p_i\} \cup \{\phi(v_{2i-1}^{(j)}), i = 2,3,\ldots,k+1$ and $j = 1,2,\ldots,p_i\}$
={N(2n-2+M)+1} $\cup\{Nm_1, N(m_1+m_3+1),\ldots,N(m_1+m_3+\cdots+m_{2k-1}+k-1)\} \cup \{N(2n-3+M-m_2)+1, N(2n-4+M-m_2-m_4)+1,\ldots,N(2n-2+M-m_2-m_4-\cdots-m_{2k}-k)+1\} \cup \{N(2n-3+M-m_2-m_4-\cdots-m_{2k}-$

$k - p_1) + 1, N(2n - 4 + M - m_2 - m_4 - \cdots - m_{2k} - k - p_1 - p_3) + 1, \ldots, N(2n - 2 + M - m_2 - m_4 - \cdots - m_{2k} - 2k - p_1 - p_3 - \cdots - p_{2k-1}) + 1\} \cup \{N(m_1 + m_3 + \cdots + m_{2k+1} + k)\} \cup \{N(m_1 + m_3 + \cdots + m_{2k+1} + p_2 + k + 1), N(m_1 + m_3 + \cdots + m_{2k+1} + p_2 + p_4 + k + 2), \ldots, N(m_1 + m_3 + \cdots + m_{2k+1} + p_2 + p_4 + \cdots + p_{2k} + 2k)\} \cup \{0, N, \ldots, N(m_i - 1)\} \cup \{N(2n-3+M) + 1, N(2n-4+M) + 1, \ldots, N(2n-2+M-m_i) + 1\} \cup \{N(m_1 + 1), N(m_1 + 2), \ldots, N(m_1 + m_2), N(m_1 + m_3 + 2), N(m_1 + m_3 + 3), \ldots, N(m_1 + 2m_3 + 1), \ldots, N(m_1 + m_3 + \cdots + m_{2k-1} + k), N(m_1 + m_3 + \cdots + m_{2k-1} + k) \ldots, N(m_1 + m_3 + \cdots + m_{2k-1} + m_{k+1} + k - 1)\} \cup \{N(2n-4+M-m_2)+1, N(2n-5+M-m_2)+1, \ldots, N(2N-3+M-m_2-m_k)+1, N(2n+M-m_2-m_4-5)+1, N(2n+M-m_2-m_4-6)+1, \ldots, N(2n+M-m_2-m_4-m_k-4)+1, \ldots, N(2n-2+M-m_2-m_4-\cdots-m_{2k-2}-k)+1, N(2n-3+M-m_2-m_4-\cdots-m_{2k-2}-k)+1, \ldots, N(2n-1+M-m_2-m_4-\cdots-m_{2k-2}-m_k-k)+1\} \cup \{N(m_1+m_3+\cdots+m_{2k+1}+k+1), N(m_1+m_3+\cdots+m_{2k+1}+k+2), \ldots, N(m_1+m_3+\cdots+m_{2k+1}+p_2+k)\} \cup \{N(2n-3+M-m_2-m_4-\cdots-m_{2k}-k)+1, N(2n-4+M-m_2-m_4-\cdots-m_{2k}-k)+1, \ldots, N(2n-2+M-m_2-m_4-\cdots-m_{2k}-k-p_1)+1\} \cup \{N(m_1+m_3+\cdots+m_{2k+1}+k+2+p_2), N(m_1+m_3+\cdots+m_{2k+1}+k+3+p_2), \ldots, N(m_1+m_3+\cdots+m_{2k+1}+2p_2+k), N(m_1+m_3+\cdots+m_{2k+1}+k+3+p_2+p_4), N(m_1+m_3+\cdots+m_{2k+1}+k+4+p_2+p_4), \ldots, N(m_1+m_3+\cdots+m_{2k+1}+p_2+p_4+p_3+k+2) \ldots, N(m_1+m_3+\cdots+m_{2k+1}+p_2+p_4+\cdots+p_{2k-2}+2k), N(m_1+m_3+\cdots+m_{2k+1}+p_2+p_4+\cdots+p_{2k-2}+2k+1), \ldots, N(m_1+m_3+\cdots+m_{2k+1}+p_2+p_4+\cdots+p_{2k-2}+2k+p_k-1)\} \cup \{N(2n-4+M-m_2-m_4-\cdots-m_{2k}-k-p_1)+1, N(2n-5+M-m_2-m_4-\cdots-m_{2k}-k-p_1)+1, \ldots, N(2n-3+M-m_2-m_4-\cdots-m_{2k}-k-p_1-p_2)+1, \cup \{N(2n-5+M-m_2-m_4-\cdots-m_{2k}-k-p_1-p_3)+1, N(2n-6+M-m_2-m_4-\cdots-m_{2k}-k-p_1-p_3)+1, \ldots, N(2n-4+M-m_2-m_4-\cdots-m_{2k}-k-p_1-2p_3)+1, \ldots, N(2n-3+M-m_2-m_4-\cdots-m_{2k}-2k-p_1-p_3-\cdots-p_{2k-1})+1, N(2n-4+M-m_2-m_4-\cdots-m_{2k}-2k-p_1-p_3-\cdots-p_{2k-1})+1, \ldots, N(2n-2+M-m_2-m_4-\cdots-m_{2k}-2k-p_1-p_3-\cdots-p_{2k-1}-p_{k+1})+1\}\}$

Thus it is clear that the vertices have distinct labels. Therefore ϕ is $1-1$.

We compute the edge labels as follows:

$\phi^*(u_1u_2) = \mid \phi(u_1) - \phi(u_2) \mid = N(2n-2+M-m_1)+1$

$\phi^*(v_2v_1) = \mid \phi(v_2) - \phi(v_1) \mid = N(2n-3+M-m_2-m_4-\cdots-m_{2k}-2k-p_1-m_1-m_3-\cdots-m_{2k+1})+1$

$\phi^*(u_{k+1}v_{k+1}) = \mid \phi(u_{k+1}) - \phi(v_{k+1}) \mid = N(2n+M-m_2-m_4-\cdots-m_k-2k-2-m_1-m_3-\cdots-m_{2k+1}-p_2-p_4-\cdots-p_k)+1$

For $i = 2, 3, \ldots, k$

$\phi^*(u_{2i-1}u_{2i}) = \mid \phi(u_{2i-1}) - \phi(u_{2i}) \mid = N(2n+M-m_2-m_4-\cdots-m_{2i-2}-2i-m_1-m_3-\cdots-m_{2i-1})+1$

For $i = 1, 2, \ldots, k$

$\phi^*(u_{2i+1}u_{2i}) = \mid \phi(u_{2i+1}) - \phi(u_{2i}) \mid = N(2n-1+M-m_2-m_4-\cdots-m_{2i}-2i-m_1-m_3-\cdots-m_{2i-1})+1$

For $i = 2, 3, \ldots, k$

$\phi^*(v_{2i}v_{2i-1}) = \mid \phi(v_{2i}) - \phi(v_{2i-1}) \mid = N[2n-1+(p_1+p_2+\cdots+p_{2k+1})-(p_1+p_3+\cdots+p_{2i-1})-(p_2+p_4+\cdots+p_{2i-2})-2k-2i]+1$

For $i = 1, 2, \ldots, k$

$\phi^*(v_{2i}v_{2i+1}) = \mid \phi(v_{2i}) - \phi(v_{2i+1}) \mid = N[2n-2+(p_1+p_2+\cdots+p_{2k+1})-(p_1+p_3+\cdots+p_{2i-1})-(p_2+p_4+\cdots+p_{2i})-2k-2i]+1$

For $j = 1, 2, \ldots, m_i$

$\phi^*(u_1u_1^{(j)}) = \mid \phi(u_1) - \phi(u_1^{(j)}) \mid = N(2n+M-j-1)+1$

$\phi^*(u_2^{(j)}u_2) = \mid \phi(u_2^{(j)}) - \phi(u_2) \mid = N(2n-2+M-j-m_1)+1$

For $i = 2, 3, \ldots, k+1$ and $j = 1, 2, \ldots, m_i$

$\phi^*(u_{2i-1}u_{2i-1}^{(j)}) = \mid \phi(u_{2i-1}) - \phi(u_{2i-1}^{(j)}) \mid = N[2n+1+M-(m_2+m_4+\cdots+m_{2i-2})-2i-(m_1+m_3+\cdots+m_{2i-3})-j]+1$

For $i = 2, 3, \ldots, k$ and $j = 1, 2, \ldots, m_i$

$\phi^*(u_{2i}^{(j)}u_{2i}) = \mid \phi(u_{2i}^{(j)}) - \phi(u_{2i}) \mid = N[2n+M-(m_2+m_4+\cdots+m_{2i-2})-2i-(m_1+m_3+\cdots+m_{2i-1})-j]+1$

For $j = 1, 2, \ldots, p_i$

$\phi^*(v_1^{(j)} v_1) = \mid \phi(v_1)^{(j)} - \phi(v_1) \mid = N(2n - 2 + p_1 + p_2 + \cdots + p_{2k+1} - 2k - j) + 1$

$\phi^*(v_2 v_2^{(j)}) = \mid \phi(v_2) - \phi(v_2^{(j)}) \mid = N(2n - 3 + p_2 + p_3 + \cdots + p_{2k+1} - 2k - j) + 1$

For $i = 2, 3, \ldots, k+1$ and $j = 1, 2, \ldots, p_i$

$\phi^*(v_{2i-1}^{(j)} v_{2i-1}) = \mid \phi(v_{2i-1}^{(j)}) - \phi(v_{2i-1}) \mid = N[2n + (p_1 + p_2 + \cdots + p_{2k+1}) - (p_1 + p_3 + \cdots + p_{2i-3}) - (p_2 + p_4 + \cdots + p_{2i-2}) - 2k - 2i - j) + 1$

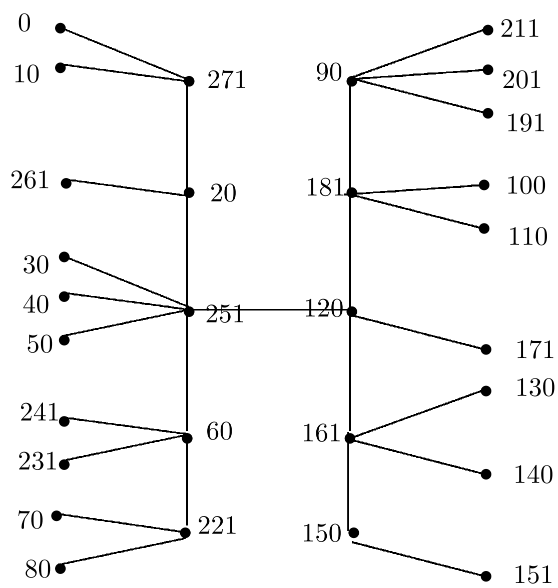
For $i = 2, 3, \ldots, k$ and $j = 1, 2, \ldots, p_i$

$\phi^*(v_{2i} v_{2i}^{(j)}) = \mid \phi(v_{2i}) - \phi(v_{2i}^{(j)}) \mid = N[2n - 1 + (p_1 + p_2 + \cdots + p_{2k+1}) - (p_1 + p_3 + \cdots + p_{2i-1}) - (p_2 + p_4 + \cdots + p_{2i-2}) - 2k - 2i - j) + 1$

This shows that the edges have the distinct labels $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$, where $q = 2n + m - 1$.

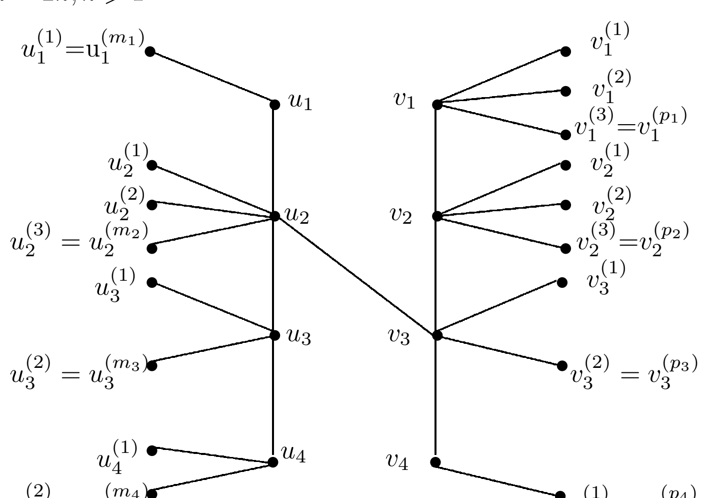
Hence $H_{(n)}^{(a)}$ is one modulo N graceful, for every positive integer N.

Example 2.9. One modulo 10 graceful labelling of $H_{(5)}^{(a)}$.



Case (ii) n is even

Let $n = 2k, k \geqslant 1$



n is even

Define

$\phi(u_1) = N\{2(n-1)+M\}+1$

$\phi(u_{2i}) = N(m_1+m_3+\cdots+m_{2i-1}+i-1)$ for $i=1,2,\ldots,k$

$\phi(u_{2i-1}) = N\{2(n-1)+M\}+1-N(m_2+m_4+\cdots+m_{2i-2}+i-1)$ for $i=2,3,\ldots,k$

$\phi(v_2) = N(m_1+m_3+\cdots+m_{2k-1}+p_{1+}k+1)$

$\phi(v_{2i}) = N(m_1+m_3+\cdots+m_{2k-1}+k-1)+N(p_1+p_3+\cdots+p_{2i-1}+i)$ for $i=2,3,\ldots,k$

$\phi(v_1) = N\{2(n-1)+M\}+1-N(m_2+m_4+\cdots+m_{2k}+k)$

$\phi(v_{2i-1}) = \phi(v_1)-N(p_2+p_4+\cdots+p_{2i-2}+i-1)$ for $i=2,3,\ldots,k$

For $j=1,2,\ldots,m_i$

$\phi(u_1^{(j)}) = N(j-1)$

$\phi(u_{2i-1}^{(j)}) = N(m_1+m_3+\cdots+m_{2i-3}+i-1)+N(j-1)$ for $i=2,3,\ldots,k$

$\phi(u_2^{(j)}) = N\{2(n-1)+M-1\}+N(j-1)+1$

$\phi(u_{2i}^{(j)}) = N\{2(n-1)+M-1\}+1-N(m_2+m_4+\cdots+m_{2i-2}+i-1)-N(j-1)$ for $i=2,3,\ldots,k$

For $j=1,2,\ldots,p_i$

$\phi(v_2^{(j)}) = N\{2(n-1)+M-1\}+1-N(m_2+m_4+\cdots+m_{2k}+k)-N(j-1)$

$\phi(v_{2i}^{(j)}) = N\{2(n-1)+M-1\}+1-N(m_2+m_4+\cdots+m_{2k}+k)-N(j-1)-N(p_2+p_4+\cdots+p_{2i-2}+i-1)$ for $i=2,3,\ldots,k$

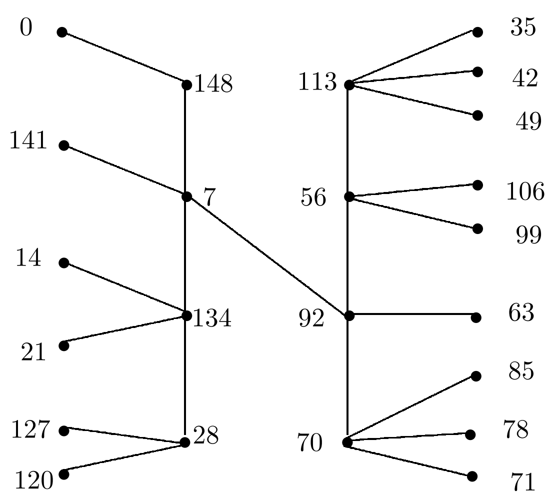
$\phi(v_1^{(j)}) = N(m_1+m_3+\cdots+m_{2k-1}+k)+N(j-1)$

$\phi(v_{2i-1}^{(j)}) = N(m_1+m_3+\cdots+m_{2k-1}+k)+N(j-1)+N(p_1+p_3+\cdots+p_{2i-3}+i-1)+N(j-1)$ for $i=2,3,\ldots,k$

The proof is similar to the proof as in the above case.

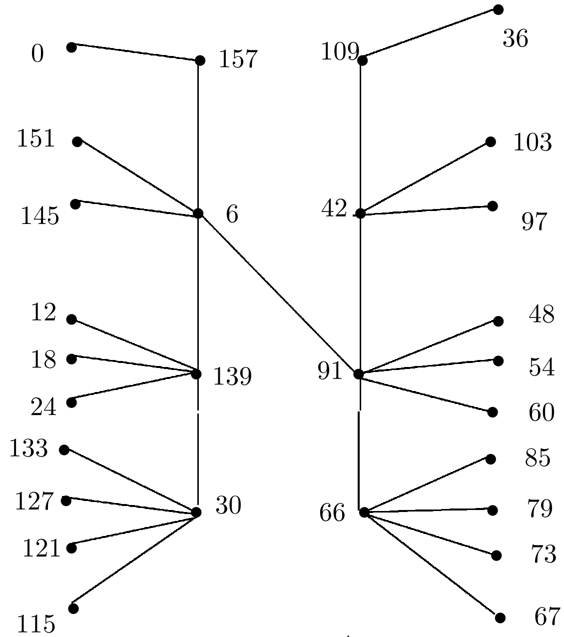
Hence $H_{(n)}^{(a)}$ is one modulo N graceful, for every positive integer N. □

EXAMPLE 2.10. One modulo 7 graceful labelling of $H_{(4)}^{(a)}$.

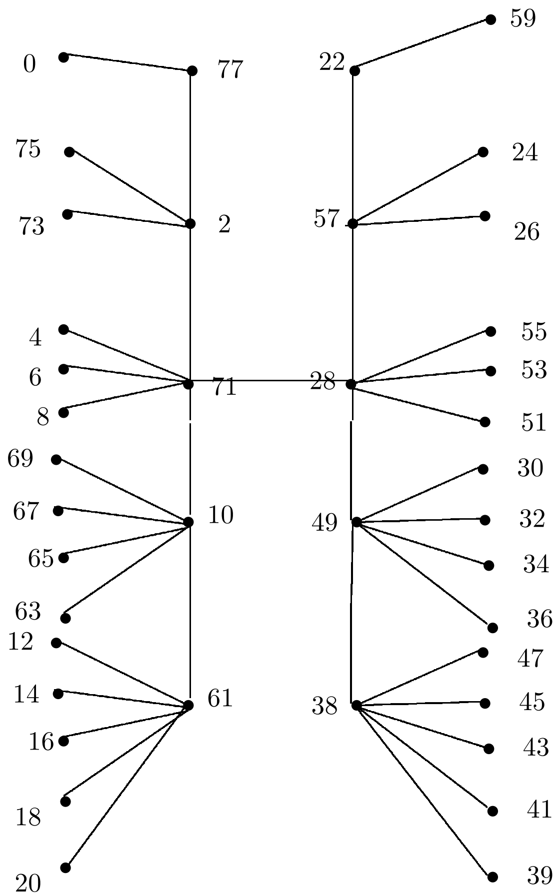


Note: If we attach i pendant vertices at each i^{th} vertex on the two paths of n vertices then we obtain the graph $H'_{(n)}$.Therefore we have a one modulo N graceful labelling $H'_{(n)}$ as a deduction from the above labelling .

EXAMPLE 2.11. One modulo 6 graceful labelling of $H'_{(4)}$.



EXAMPLE 2.12. Odd graceful labelling of $H'_{(5)}$



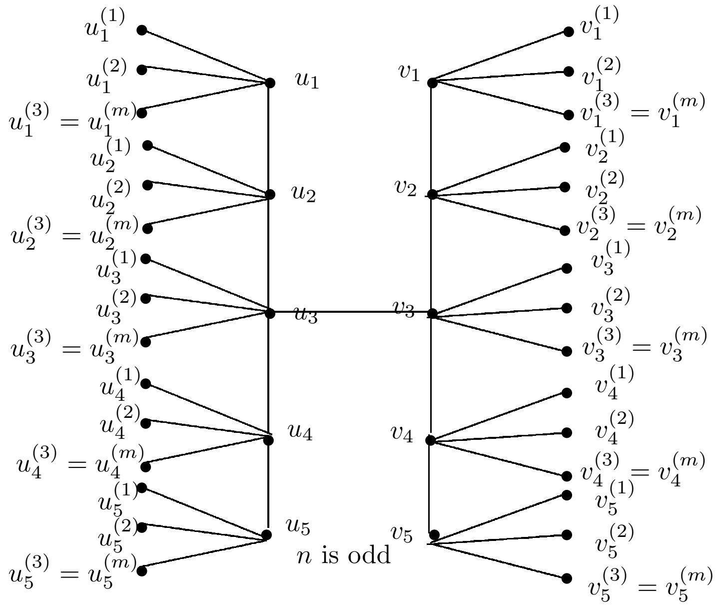
THEOREM 2.3. *$H \bigodot mK_1$ is one modulo N graceful for every positive integer N.*

PROOF. Let $P_1(u_1, u_2, \ldots, u_n)$ and $P_2(v_1, v_2, \ldots, v_n)$ be two distinct paths with $n(\geqslant 3)$ vertices in a H-graph. The H-graph has $2n(m+1)$ vertices and $2n(m+1)-1$

edges.

Case (i) n is odd

Let $n = 2k+1, k \geqslant 1$



Define

$\phi(u_{2i}) = Nm + N(m+1)(i-1)$ for $i = 1,2,\ldots,k$

$\phi(u_{2i-1}) = 2N(2k+nm) + 1 - N(m+1)(i-1)$ for $i = 1,2,\ldots,k+1$

$\phi(v_{2i}) = 2N(2k+nm) + 1 - N(m+1)(k+i)$ for $i = 1,2,\ldots,k$

$\phi(v_{2i-1}) = Nm + N(m+1)(k+i-1)$ for $i = 1,2,\ldots,k+1$

For $j = 1,2,\ldots,m$

$\phi(u_{2i}^{(j)}) = 2N(2k+nm) - (N-1) - N(m+1)(i-1) - N(j-1)$ for $i = 1,2,\ldots,k$

$\phi(u_{2i-1}^{(j)}) = N(j-1) + N(m+1)(i-1)$ for $i = 1,2,\ldots,k+1$

$\phi(v_{2i}^{(j)}) = N(m+1)(k+i) + N(j-1)$ for $i = 1,2,\ldots,k$

$\phi(v_{2i-1}^{(j)}) = 2N(2k+nm) - (N-1) - N(m+1)(k+i-1) - N(j-1)$
for $i = 1,2,\ldots,k+1$

From the definition of ϕ it is clear that

$\{\phi(u_{2i}), i = 1,2,\ldots,k\} \cup \{\phi(u_{2i-1}), i = 1,2,\ldots,k+1\} \cup \{\phi(v_{2i}), i = 1,2,\ldots,k\} \cup \{\phi(v_{2i-1}), i = 1,2,\ldots,k+1\} \cup \{\phi(u_{2i}^{(j)}), i = 1,2,\ldots,k$ and $j = 1,2,\ldots,m\} \cup \{\phi(u_{2i-1}^{(j)}), i = 1,2,\ldots,k+1$ and $j = 1,2,\ldots,m\} \cup \{\phi(v_{2i}^{(j)}),\ i = 1,2,\ldots,k$ and $j = 1,2,\ldots,m\} \cup \{\phi(v_{2i-1}^{(j)}), i = 1,2,\ldots,k+1$ and $j = 1,2,\ldots,m\}$

$$\begin{aligned}
= \{&Nm, N(2m+1),\ldots,N(mk+k-1)\} \cup \{2N(2k+nm)+1,\\
&N(4k+2nm-m-1)+1,\ldots,N(3k+2nm-km)+1\}\\
&\cup\{N(3k+2nm-km-m-1)+1, N(3k+2nm-km-2m-2)+1,\ldots,\\
&2N(k+nm-km)+1\} \cup \{N(m+mk+k)+1, N(2m+mk+k+1),\ldots,\\
&N(m+2km+2k)\} \cup \{N(4k+2nm-1)+1, N(4k+2mn-m-2)+1,\ldots,\\
&N(3k+2nm-km+m)+1, N(4k+2nm-2)+1, N(4k+2mn-m-3)+1,\ldots,\\
&N(3k+2nm-km+m-1)+1, N(4k+2mn-m)+1, N(4k+2nm-2m-1)\\
&+1,\ldots,N(3k+2nm-km+1)+1\ \} \cup \{0, N(m+1),\ldots,Nk(m+1), N, N(m+2)\\
&\ldots, N(mk+k+1), N(m-1), 2Nm,\ldots,N(mk+k+m-1)\ \}
\end{aligned}$$

$\cup\{N(mk+m+k+1), N(mk+2m+k+2), \ldots, N(2mk+2k),$
$N(mk+m+k+2), N(mk+2m+k+3)\ldots, N(2mk+2k+1), \ldots,,$
$N(mk+2m+k), N(mk+3m+k+1), \ldots, N(2mk+2k+m-1)\ \}$
$\cup\{N(3k+2nm-mk-1)+1, N(3k+2nm-mk-m-2)+1, \ldots,$
$N(2k+2nm-2km-1)+1, N(3k+2nm-mk-2)+1,$
$N(3k+2nm-mk-m-3)+1\ldots, N(2k+2nm-2km-2)+1, \ldots,$
$N(3k+2nm-mk-m)+1, N(3k+2nm-mk-2m-1)+1, \ldots,$
$N(2k+2nm-2km-m)+1\}$

Thus it is clear that the vertices have distinct labels. Therefore ϕ is $1-1$.
We compute the edge labels as follows:

$\phi^*(u_{k+1}v_{k+1}) = \mid \phi(u_{k+1}) - \phi(v_{k+1}) \mid = N(2k+2nm-2mk-m)+1$

For $i = 1, 2, \ldots, k$

$\phi^*(u_{2i-1}u_{2i}) = \mid \phi(u_{2i-1}) - \phi(u_{2i}) \mid = N(4k+2nm+m-2mi-i)+1$

For $i = 1, 2, \ldots, k$

$\phi^*(u_{2i+1}u_{2i}) = \mid \phi(u_{2i+1}) - \phi(u_{2i}) \mid = N(4k+2nm-2mi-2i+1)+1$

For $i = 1, 2, \ldots, k$

$\phi^*(v_{2i}v_{2i-1}) = \mid \phi(v_{2i}) - \phi(v_{2i-1}) \mid = N(2k+2nm-2mk-2mi-2i+1)+1$

For $i = 1, 2, \ldots, k$

$\phi^*(v_{2i}v_{2i+1}) = \mid \phi(v_{2i}) - \phi(v_{2i+1}) \mid = N(2k+2nm-2mk-2mi-2i-m)+1$

For $i = 1, 2, \ldots, k+1$ and $j = 1, 2, \ldots, m$

$\phi^*(u_{2i-1}u_{2i-1}^{(j)}) = \mid \phi(u_{2i-1}) - \phi(u_{2i-1}^{(j)}) \mid = N(4k+2nm-2mi+2m-2i-j+3)+1$

For $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, m$

$\phi^*(u_{2i}^{(j)}u_{2i}) = \mid \phi(u_{2i}^{(j)}) - \phi(u_{2i}) \mid = N(4k+2nm-2mi-2i-j+m+2)+1$
For $i = 1, 2, \ldots, k+1$ and $j = 1, 2, \ldots, m$

$\phi^*(v_{2i-1}^{(j)}v_{2i-1}) = \mid \phi(v_{2i-1}^{(j)}) - \phi(v_{2i-1}) \mid = N(2k+2nm-2mk-2mi+m+2-$
$2i-j)+1$
For $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, m$

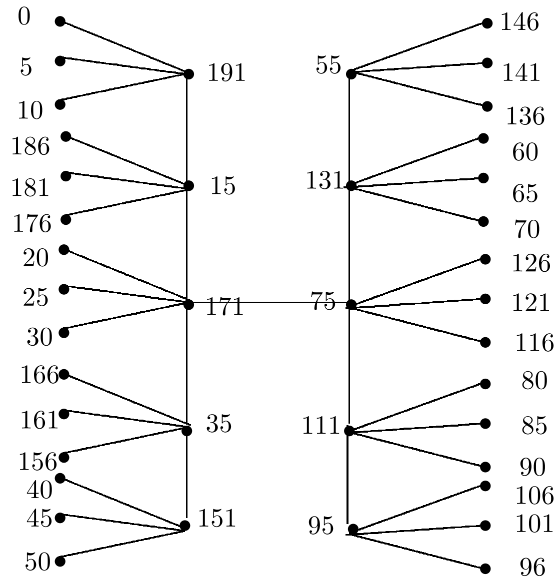
$\phi^*(v_{2i}v_{2i}^{(j)}) = \mid \phi(v_{2i}) - \phi(v_{2i}^{(j)}) \mid = N(2k+2nm-2mk-2mi-2i-j+1)+1$

This shows that the edges have the distinct labels $\{1, N+1, 2N+1, \ldots,$

$N(q-1)+1\}$, where $q = 2n(m+1)-1$.

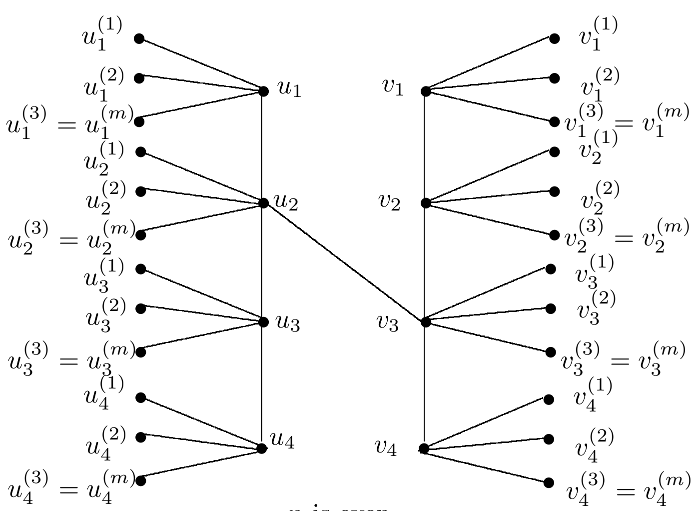
Hence $H \bigodot mK_1$ is one modulo N graceful, for every positive integer N.

EXAMPLE 2.13. One modulo 5 graceful labelling of $H \bigodot 3K_1$. $(n=5)$



Case (ii) n is even

Let $n = 2k, k \geqslant 1$



n is even

Define

$$\phi(u_{2i}) = Nm + N(m+1)(i-1) \quad \text{for } i = 1, 2, \ldots, k$$

$$\phi(u_{2i-1}) = 2N(2k + nm - 1) + 1 - N(m+1)(i-1) \text{ for } i = 1, 2, \ldots, k$$

$$\phi(v_{2i}) = Nm + N(m+1)(k+i-1) \quad \text{for } i = 1, 2, \ldots, k$$

$$\phi(v_{2i-1}) = 2N(2k + nm - 1) + 1 - N(m+1)(k+i-1) \text{ for } i = 1, 2, \ldots, k$$

For $j = 1, 2, \ldots, m$

$$\phi(u_{2i}^{(j)}) = 2N(2k+nm-1) - (N-1) - N(m+1)(i-1) - N(j-1) \quad \text{for } i = 1, 2, \ldots, k$$

$\phi(u_{2i-1}^{(j)}) = N(j-1) + N(m+1)(i-1) \quad \text{for } i = 1, 2, \ldots, k$

$\phi(v_{2i}^{(j)}) = 2N(2k+nm-1)-(N-1)-N(m+1)(k+i-1)-N(j-1) \quad \text{for } i = 1, 2, \ldots, k$

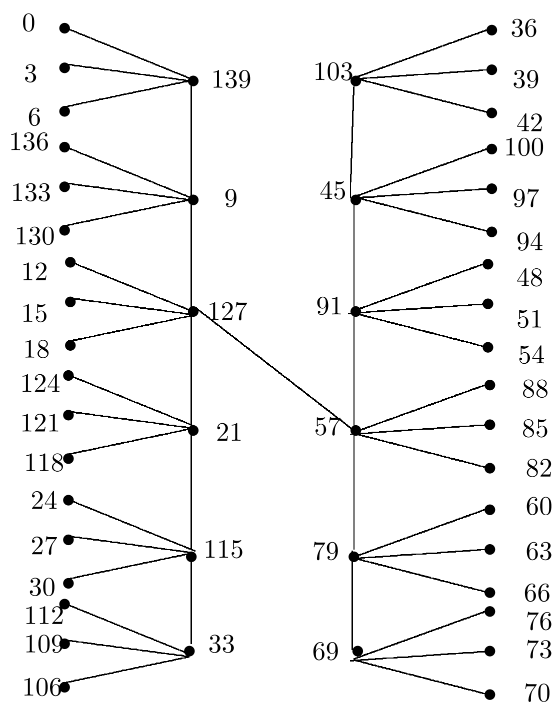
$\phi(v_{2i-1}^{(j)}) = N(m+1)(k+i-1) + N(j-1) \quad \text{for } i = 1, 2, \ldots, k$

The proof is similar to the proof as in the above case.

Hence $H \bigodot mK_1$ is one modulo N graceful, for every positive integer N.

□

Example 2.14. One modulo 3 graceful labelling of $H \bigodot 3K_1$. $(n = 6)$



3. Conclusion

For further work, we have planned to investigate common properties of various graph labelling schemes with generalized one modulo N graceful labelling and to classify them.

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