SCIENTIA
Series A: *Mathematical Sciences*, Vol. 32 (2022), 31-69
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

The integrals in Gradshteyn and Ryzhik. Part 31: Forms containing binomials

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Abstract. The table of Gradshteyn and Ryzhik contains many integrals that involve the expression $z_k = a + bx^k$. All the entries containing this form are evaluated in detail.

1. Introduction

The evaluation of integrals is found in the elementary courses where the student is exposed to a variety of `methods of integration`. The topics range from integration by parts (with the statement that *if your integrand is the product of two functions, this may work*, also includes some `standard substitutions` (with the statement that *if your integrand contains the term* $\sqrt{1-x^2}$*, try the substitution* $x = \sin\theta$ *and it will simplify*; the `method of partial fractions` used to integrate rational functions (provided the polynomial in the denominator is factored) and sometimes the student is exposed to the `Weierstrass substitution` $u = \tan\frac{x}{2}$ which transforms rational functions in $\sin x$ and $\cos x$ into a rational function of u.

Historically the evaluation of integrals have been collected in volumes such as [**2, 3**] where the results were stated in the form

$$\int_0^\infty \frac{x^{\mu-1}\ln x}{\beta+x}\,dx = \frac{\pi\beta^{\mu-1}}{\sin\mu\pi}(\ln\beta - \pi\cot\mu\pi) \tag{1.1}$$

but without an indication of how the formula was obtained. The modern version of these entries were collected in the table by I. S. Gradsheyn and I. M. Ryzhik [**7**]. In the process of his thesis at Tulane University [**5**], George Boros began to develop

2000 *Mathematics Subject Classification.* Primary 33.
Key words and phrases. Integrals, closed-form formulas, rational integrands, Gradshteyn and Ryzhik, integration by parts, recurrences.

techniques for a systematic verification of these entries. This task was motivated by some incorrect evaluations appearing in that edition of the table.

The latest edition of the table of integrals by I. S. Gradshteyn and I. M. Ryzhik [**8**] contains a large selection of integrals (as did the previous editions). One of the difficulties in using them is that, in view of the large possible options for changes of variables, the integrand of interest might not match the entry in the table. A description of this phenomena is presented in [**6**] in the context of the evaluation of

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}. \tag{1.2}$$

This integral may be evaluated in terms of some hypergeometric functions. Naturally, it is possible that the entries are incorrect, as in the entry **3.248.5** of [**7**]:

$$I_{wrong} = \int_0^\infty \frac{dx}{(1+x^2)^{\frac{3}{2}}\sqrt{\varphi(x)+\sqrt{\varphi(x)}}} = \frac{\pi}{2\sqrt{6}} \quad \text{with} \quad \varphi(x) = 1 + \frac{4x^2}{3(x^2+1)^2}.$$

The correct integrand was found in [**4**] as

$$I_{correct} := \int_0^\infty \frac{dx}{(1+x^2)^{\frac{3}{2}}\sqrt{\varphi(x)+\sqrt{\varphi(x)^{\mathbf{3}}}}} = \frac{\pi}{2\sqrt{6}}.$$

The correct value of the original problem has been obtained by J. Arias de Reyna in terms of elliptic functions,

The present work is part of a project dedicated to proving all these evaluations and to provide context for them. This project started with [**9**] and the latest work in this series is [**1**]. The present paper is the first in the series by participants in the Zoom-summer program `PolyMath Jr.` (June-2021). The students were given the list of entries and provided proofs that were checked and written by the senior author and a graduate assistant (V. Sharma).

The entries discussed here appear in Sections 2.11 to 2.15. The integrands considered here have the form $x^n z_k^m$, where $z_k = a + bx^k$ and n, m are integers. A small variation, including two linear factors in x, appears at the end of the paper. The results include explicit closed form evaluations as well as some reduction formulas.

2. Entries in Section $2.11 - 2.13$. Forms containing the binomial $a + bx^k$

Section 2.110 contains reduction formulas for $z_k = a + bx^k$ and an explicit expression for the general case.

2.1. Entry 2.110.1.

(2.1)

$$\begin{aligned}\int x^n z_k^m \, dx &= \frac{x^{n+1} z_k^m}{km+n+1} + \frac{amk}{km+n+1} \int x^n z_k^{m-1} \, dx \\ &= \frac{x^{n+1}}{m+1} \sum_{s=0}^{p} \frac{(ak)^s (m+1)m(m-1) \cdots (m-s+1) z_k^{m-s}}{[mk+n+1][(m-1)k+n+1] \cdots [(m-s)k+n+1]} \\ &+ \frac{(ak)^{p+1}(m+1)m(m-1) \cdots (m-p+1)(m-p)}{[mk+n+1][(m-1)k+n+1] \cdots [(m-p)k+n+1]} \int x^n z_k^{m-p-1} \, dx\end{aligned}$$

Proof. In the proof of the first part, we illustrate a procedure to choose the terms u and v in the integration by parts formula

$$\int u \, dv = uv - \int v \, du. \tag{2.2}$$

In later proofs this will be referred as `the usual procedure`. The idea is simple: in order to prove the first identity (2.1) impose conditions on u and v so that

$$u \, dv = x^n z_k^m \quad \text{and} \quad uv = \frac{x^{n+1} z_k^m}{(km+n+1)}. \tag{2.3}$$

This equations are now used to determine v and one hopes that the term $v \, du$ matches the last term in the first identity. The second formula follows from iteration of the first one.

Dividing both equations in (2.3) leads to $\frac{dv}{v} = (km+n+1)\frac{dx}{x}$ and this gives $v = x^{km+n+1}$. The formula for uv implies that $u = \frac{z_k^m}{(km+n+1)x^{km}}$. From here it follows that $dv = (km+n+1)x^{km+n}$ and after some simplifications one obtains $du = -\frac{amk}{km+n+1}\frac{z_k^{m-1}}{x^{mk+1}}$. Therefore $v \, du = -\frac{amk}{km+n+1} z_k^{m-1} x^n$, proving (2.1). The second formula is obtained by iterating the first one. It can be proven directly by an inductive argument. □

2.2. Entry 2.110.2.

$$\int x^n z_k^m \, dx = -\frac{x^{n+1} z_k^{m+1}}{ak(m+1)} + \frac{km+k+n+1}{ak(m+1)} \int x^n z_k^{m+1} \, dx \tag{2.4}$$

Proof. This entry is verified by integration by parts. The choice of u and dv is now motivated by the appearance of the coefficient $(n+km+k+1)$ on the right-hand side. Therefore take $u = x^{n+km+k+1}$ and $dv = x^{-km-k-1} z_k^m \, dx$. Then $du = (n+km+k+1)x^{n+km+k} dx$ is direct and some elementary manipulation produce $d\left(x^{-km-k} z_k^{m+1}\right) = -ak(m+1)x^{-km-k-1} z_k^m$. Integration by parts now gives the stated formula. □

2.3. Entry 2.110.3.

$$\int x^n z_k^m \, dx = \frac{x^{n+1} z_k^m}{n+1} - \frac{bkm}{n+1} \int x^{n+k} z_k^{m-1} \, dx \tag{2.5}$$

Proof. Integrate by parts with $u = z_k^m$ and $dv = x^n$. Then $du = mbkx^{k-1}z_k^{m-1}$ and $v = \frac{1}{n+1}x^{n+1}$. The results follows directly. □

2.4. Entry 2.110.4.

$$\int x^n z_k^m \, dx = \frac{x^{n+1-k} z_k^{m+1}}{bk(m+1)} - \frac{n+1-k}{bk(m+1)} \int x^{n-k} z_k^{m+1} \, dx \tag{2.6}$$

Proof. Integrate by parts with $u = x^{n+1-k}$ and $dv = x^{k-1}z_k^m$. Then $du = (n+1-k)x^{n-k}$ and $v = z_k^{m+1}/(bk(m+1))$. This gives the result. □

2.5. Entry 2.110.5.

$$\int x^n \, z_k^m \, dx = \frac{x^{n+1-k} z_k^{m+1}}{b(km+n+1)} - \frac{a(n+1-k)}{b(km+n+1)} \int x^{n-k} z_k^m \, dx \tag{2.7}$$

Proof. Entry 2.110.4 gives

$$\int x^n z_k^m \, dx = \frac{x^{n+1-k} z_k^{m+1}}{bk(m+1)} - \frac{n+1-k}{bk(m+1)} \int x^{n-k} z_k^{m+1} \, dx. \tag{2.8}$$

The current entry comes from writing $z_k^{m+1} = z_k^m(a + bx^k) = az_k^m + bx^k z_k^m$ in the last integral of (2.8) and distributing the terms. □

2.6. Entry 2.110.6.

$$\int x^n z_k^m \, dx = \frac{x^{n+1} z_k^{m+1}}{a(n+1)} - \frac{b(km+k+n+1)}{a(n+1)} \int x^{n+k} z_k^m \, dx \tag{2.9}$$

Proof. Replace n by $n+k$ in Entry 2.110.5. □

2.7. Entry 2.110.7.

$$\int x^n (nx^b + c)^k \, dx = \frac{n^k}{b} \sum_{i=0}^{k} \frac{(-1)^i \, k! \, \Gamma\left(\frac{n+1}{b}\right) \left(x^b + \frac{c}{n}\right)^{k-i}}{(k-i)! \, \Gamma\left(\frac{n+1}{b} + i + 1\right)} x^{n+1+ib} \tag{2.10}$$

Proof. This is a special case of Entry 2.110.8. See the details at the end of the proof for that entry. Note that this entry has been modified from the way it currently appears in [**8**]. □

2.8. Entry 2.110.8.

$$\int x^n z_k^m \, dx = \frac{b^m}{k} \sum_{i=0}^{m} \frac{(-1)^i m! \, J! \, \left(x^k + \frac{a}{b}\right)^{m-i} x^{k(J+i+1)}}{(m-i)! \, (J+i+1)!} \tag{2.11}$$

where $J = \frac{n+1}{k} - 1$.

Proof. Write $\int x^n z_k^n \, dx = b^m I(n,m)$ with

$$I(n,m) = \int x^n \left(x^k + \frac{a}{b}\right)^m \, dx. \tag{2.12}$$

Integrate by parts with $u = (x^k + a/b)^m$ and $dv = x^n \, dx$ to obtain

$$\begin{aligned} I(n,m) &= \frac{1}{n+1} x^{n+1} \left(x^k + \frac{a}{b}\right)^m - \frac{mk}{n+1} \int x^{n+k} \left(x^k + \frac{a}{b}\right)^{m-1} \, dx \\ &= \frac{1}{n+1} x^{n+1} \left(x^k + \frac{a}{b}\right)^m - \frac{mk}{n+1} I(n+k, m-1). \end{aligned} \tag{2.13}$$

Iterating the relation gives

$$\begin{aligned} I(n,k) = \frac{x^{n+1}}{n+1} \left(x^k + \frac{a}{b}\right)^m &- \frac{mk}{(n+1)(n+k+1)} x^{n+k+1} \left(x^k + \frac{a}{b}\right)^{m-1} \\ &+ \frac{m(m-1)k^2}{(n+1)(n+1+k)} I(n+2k, m-2) \end{aligned} \tag{2.14}$$

Further iteration produces the formula

$$\begin{aligned} I(n,m) = \sum_{j=0}^{\ell} (-1)^j k^j x^{n+1+jk} \left(x^k + \frac{a}{b}\right)^{m-j} & \frac{m!}{(m-j)!} \frac{1}{\prod_{r=0}^{j} (n+1+rk)} \\ + (-1)^{\ell+1} \frac{m!}{(m-\ell-1)!} k^{\ell+1} & \frac{I(n+(\ell+1)k, m-\ell-1)}{\prod_{r=0}^{\ell} (n+1+rk)}. \end{aligned} \tag{2.15}$$

The proof of this identity is an elementary induction argument based upon (2.13). Since the parameter ℓ is arbitrary, choose $\ell = m-1$ and use $I(n,0) = x^{n+1}/(n+1)$. Then notice that the extra term in (2.15) corresponds to the term $j = m$ in the sum. The final product is

$$I(n,m) = \frac{1}{k} \sum_{j=0}^{m} \frac{m!}{(m-j)!} \frac{(-1)^j x^{n+1+jk} \left(x^k + \frac{a}{b}\right)^{m-j}}{\prod_{r=0}^{j} \left(\frac{n+1}{k} + r\right)}. \tag{2.16}$$

The expression above is written as it appears in the table by writing

$$\prod_{r=0}^{j} \left(\frac{n+1}{k} + r\right) = \frac{(J+1+j)!}{J!} \quad \text{with } J = \frac{n+1}{k} - 1. \tag{2.17}$$

In order to obtain Entry 2.110.7 make the change of parameters $\{a, b, k, m\} \mapsto \{c, n, b, k\}$. □

3. Section 2.111. Forms containing the binomial $z_1 = a + bx$

3.1. Entry 2.111.1.

$$\int z_1^m \, dx = \frac{z_1^{m+1}}{b(m+1)} \tag{3.1}$$

and for $m = -1$

$$\int \frac{dx}{z_1} = \frac{1}{b} \ln z_1 \tag{3.2}$$

Proof. Let $u = a + bx$ to obtain

$$\int z_1^m \, dx = \int (a+bx)^m \, dx = \frac{1}{b} \int u^m \, dx = \frac{u^m}{b(m+1)}. \tag{3.3}$$

This gives the result for $m \neq -1$. In the case $m = -1$, the same change of variables yields $\displaystyle\int \frac{dx}{z_1} = \int \frac{dx}{a+bx} = \frac{1}{b} \int \frac{du}{u}$ and this gives the stated formula. □

3.2. Entry 2.111.2.

$$\int \frac{x^n \, dx}{z_1^m} = \frac{x^n}{z_1^{m-1}(n+1-m)b} - \frac{na}{(n+1-m)b} \int \frac{x^{n-1} \, dx}{z_1^m} \tag{3.4}$$

Proof. In order to integrate by parts, write the problem as

$$\begin{aligned}
\int x^n (a+bx)^{-m} \, dx &= \frac{1}{(1-m)b} \int x^n \frac{d}{dx} \left[(a+bx)^{1-m} \right] \, dx \\
&= \frac{1}{(1-m)b} \frac{x^n}{z_1^{m-1}} - \frac{n}{b(1-m)} \int \frac{x^{n-1} \, dx}{(a+bx)^{m-1}} \\
&= \frac{1}{(1-m)b} \frac{x^n}{z_1^{m-1}} - \frac{n}{(1-m)b} \int \frac{x^{n-1}(a+bx)}{(a+bx)^m} \, dx \\
&= \frac{x^n}{b(1-m)z_1^{m-1}} - \frac{na}{b(1-m)} \int \frac{x^{n-1} \, dx}{(a+bx)^m} \, dx - \frac{n}{(1-m)} \int \frac{x^n \, dx}{z_1^m}
\end{aligned}$$

Now bring the last term to the right-hand side to obtain the result. □

3.3. Entry 2.111.3. For $n = m - 1$ in Entry 2.111.2 we use

$$\int \frac{x^{m-1} \, dx}{z_1^m} = -\frac{x^{m-1}}{z_1^{m-1}(m-1)b} + \frac{1}{b} \int \frac{x^{m-2} \, dx}{z_1^{m-1}} \tag{3.5}$$

and for $m = 1$ we have

$$\int \frac{x^n \, dx}{z_1} = \frac{x^n}{nb} - \frac{ax^{n-1}}{(n-1)b^2} + \frac{a^2 x^{n-2}}{(n-2)b^3} - \cdots + (-1)^{n-1} \frac{a^{n-1}x}{1 \cdot b^n} + \frac{(-1)^n a^n}{b^{n+1}} \ln z_1 \tag{3.6}$$

PROOF. For the first formula, integrate by parts with $u = x^{m-1}$ and $dv = z_1^{-m}dx$. The result follows directly. For the second formula, written as

$$\int \frac{x^n\,dx}{z_1} = \sum_{k=0}^{n-1} \frac{(-1)^k a^k x^{n-k}}{(n-k)b^{k+1}} + \frac{(-1)^n a^n}{b^{n+1}} \ln z_1, \tag{3.7}$$

observe that from $x^n = x^{n-1} \times x$ and writing the x on the right in terms of $z_1 = a+bx$ leads to $\frac{x^n}{z_1} = \frac{1}{b}x^{n-1} - \frac{a}{b}\frac{x^{n-1}}{z_1}$. Denoting the integral on the left of (3.6) by I_n this produces

$$I_n = \frac{1}{bn}x^n - \frac{a}{b}I_{n-1}. \tag{3.8}$$

This is now to prove the identity (3.6) by induction. The case $n = 1$ is direct. Replacing the inductive form of I_{n-1} on the right-hand side of (3.8) one checks that the logarithmic terms match and that the proof amounts to the identity

$$\sum_{k=0}^{n-1} \frac{(-1)^k a^k x^{n-k}}{(n-k)b^{k+1}} = \frac{1}{bn}x^n - \sum_{k=0}^{n-2} \frac{(-1)^k a^{k+1} x^{n-1-k}}{(n-1-k)b^{k+2}}. \tag{3.9}$$

The term $k = 0$ on the left-hand side matches the first term on the right. A simple shift of indices verifies the rest. □

3.4. Entry 2.111.4.

$$\int \frac{x^n\,dx}{z_1^2} = \sum_{k=1}^{n-1} (-1)^{k-1} \frac{ka^{k-1}x^{n-k}}{(n-k)b^{k+1}} + (-1)^{n-1}\frac{a^n}{b^{n+1}z_1} + (-1)^{n+1}\frac{na^{n-1}}{b^{n+1}} \ln z_1 \tag{3.10}$$

PROOF. Differentiate Entry 2.111.3 with respect to the parameter a. □

3.5. Entry 2.111.5.

$$\int \frac{x\,dx}{z_1} = \frac{x}{b} - \frac{a}{b^2} \ln z_1 \tag{3.11}$$

PROOF. The change of variables $t = a + bx$ gives

$$\int \frac{x\,dx}{z_1} = \frac{1}{b^2}\left(\int dt - a\int \frac{dt}{t}\right) = \frac{1}{b^2}\left(t - a\ln t\right) \tag{3.12}$$

and this gives the evaluation, with the extra constant term a/b^2. Since these are indefinite integrals, the extra constant plays no role. □

3.6. Entry 2.111.6.

$$\int \frac{x^2\,dx}{z_1} = \frac{x^2}{2b} - \frac{ax}{b^2} + \frac{a^2}{b^3} \ln z_1 \tag{3.13}$$

PROOF. Let $t = a + bx$ to obtain $\int \frac{x^2\,dx}{z_1} = \frac{1}{b^3}\int \frac{(t-a)^2}{t}\,dt$. The result follows by expanding the square and integrating. □

4. Section 2.113

4.1. Entry 2.113.1.

$$\int \frac{dx}{z_1^2} = -\frac{1}{bz_1} \tag{4.1}$$

Proof. The change of variables $u = a + bx$ gives

$$\int \frac{dx}{z_1^2} = \int \frac{dx}{(a+bx)^2} = \frac{1}{b}\int u^{-2}\,du = -\frac{1}{bu}, \tag{4.2}$$

and this is the evaluation. □

4.2. Entry 2.113.2.

$$\int \frac{x\,dx}{z_1^2} = -\frac{x}{bz_1} + \frac{1}{b^2}\ln z_1 = \frac{a}{b^2 z_1} + \frac{1}{b^2}\ln z_1 \tag{4.3}$$

Proof. Integrating by parts with $u = x$ and $dv = 1/z_1^2 = 1/(a+bx)^2$ gives

$$\int \frac{x\,dx}{z_1^2} = -\frac{x}{b(a+bx)} + \frac{1}{b}\int \frac{dx}{a+bx}. \tag{4.4}$$

Now make the change of variables $u = a + bx$ to evaluate the last integral. □

4.3. Entry 2.113.3.

$$\int \frac{x^2\,dx}{z_1^2} = \frac{x}{b^2} - \frac{a^2}{b^3 z_1} - \frac{2a}{b^3}\ln z_1 \tag{4.5}$$

Proof. Since the answers are all expressed in terms of z_1, this entry should be written as $\int \frac{x^2\,dx}{z_1^2} = \frac{z_1}{b^3} - \frac{a^2}{b^3 z_1} - \frac{2a}{b^3}\ln z_1$. Integrating by parts with $u = x^2$ and $dv = (a+bx)^{-2}\,dx$ gives $\int \frac{x^2\,dx}{(a+bx)^2} = -\frac{x^2}{b(a+bx)} + \frac{2}{b}\int \frac{x\,dx}{a+bx}$. Now write $x = \frac{1}{b}(a+bx) - \frac{a}{b}$ to obtain $\int \frac{x\,dx}{a+bx} = \frac{1}{b}\int dx - \frac{a}{b}\int \frac{dx}{a+bx}$. The result follows from the evaluation of these two integrals. □

Remark 4.1. Every entry in this section can be evaluated as was done for Entry 2.113.1. For instance, in the last example, the change of variables $u = a + bx$ yields $\int \frac{x^2\,dx}{z_1^2} = \frac{1}{b^3}\int \frac{(u-a)^2\,du}{u^3}$ and the integrand can be expanded to produce integrals with a pure power of u.

5. Section 2.114

5.1. Entry 2.114.1.

$$\int \frac{dx}{z_1^3} = -\frac{1}{2bz_1^2} \tag{5.1}$$

Proof. The change of variables $t = a + bx$ gives the result. □

5.2. Entry 2.114.2.

$$\int \frac{x\,dx}{z_1^3} = -\left(\frac{x}{b} + \frac{a}{2b^2} \right) \frac{1}{z_1^2} \tag{5.2}$$

PROOF. The change of variables $u = a + bx$ gives

$$I = \int \frac{u-a}{b^2u^3}\,du = \frac{1}{b^2} \int \frac{du}{u^2} - \frac{a}{b^2} \int \frac{du}{u^3}. \tag{5.3}$$

Now evaluate both integrals to obtain the result. □

5.3. Entry 2.114.3.

$$\int \frac{x^2\,dx}{z_1^3} = \left(\frac{2ax}{b^2} + \frac{3a^2}{2b^3} \right) \frac{1}{z_1^2} + \frac{1}{b^3} \ln z_1 \tag{5.4}$$

PROOF. The change of variables $t = a + bx$ gives

$$\int \frac{x^2\,dx}{z_1^3} = \frac{1}{b^3} \left(\int \frac{dt}{t} - 2a \int \frac{dt}{t^2} + a^2 \int \frac{dt}{t^3} \right). \tag{5.5}$$

The result follows by computing the previous elementary integrals. □

5.4. Entry 2.114.4.

$$\int \frac{x^3\,dx}{z_1^3} = \left(\frac{x^3}{b} + \frac{2a}{b^2}x^2 - \frac{2a^2x}{b^3} - \frac{5a^3}{2b^4} \right) \frac{1}{z_1^2} - \frac{3a}{b^4} \ln z_1 \tag{5.6}$$

PROOF. The change of variables $t = a + bx$ gives

$$\int \frac{x^3\,dx}{z_1^3} = \frac{1}{b^4} \int dt - \frac{3a}{b^4} \int \frac{dt}{t} + \frac{3a^2}{b^4} \int \frac{dt}{t^2} - \frac{a^3}{b^4} \int \frac{dt}{t^3} = \frac{t}{b^4} - \frac{3a}{b^4} \ln t - \frac{3a^2}{b^4 t} + \frac{a^3}{2b^4t^2}.$$

The term involving logarithm matches with the answer, so we need to work on the non-logarithmic part. This can be written as $\frac{\left(2t^2a + 2bxt^2\right) - 6a^2t + a^3}{2b^4t^2}$. Now observe that $\left(2t^2a + 2bxt^2\right) - 2xt^2b = 2at^2$ and so the term $2at^2 + 2bxt^2$ can be replaced by $2xt^2b$, since this affects the result by a constant. The modified answer matches the stated result. □

6. Section 2.115

6.1. Entry 2.115.1.

$$\int \frac{dx}{z_1^4} = -\frac{1}{3bz_1^3} \tag{6.1}$$

PROOF. The change of variables $t = a + bx$ gives the result. □

6.2. Entry 2.115.2.

$$\int \frac{x\,dx}{z_1^4} = -\left(\frac{x}{2b} + \frac{a}{6b^2}\right)\frac{1}{z_1^3} \tag{6.2}$$

Proof. Integration by parts and entry 2.115.1 give $\int \frac{x}{z_1^4}\,dx = -\frac{x}{3bz_1^3} + \frac{1}{3b}\int \frac{dx}{z_1^3}$. The last integral is $-1/2bz_1^2$, see Entry 2.114.1. The result follows from here. □

6.3. Entry 2.115.3.

$$\int \frac{x^2\,dx}{z_1^4} = -\left(\frac{x^2}{b} + \frac{ax}{b^2} + \frac{a^2}{3b^3}\right)\frac{1}{z_1^3} \tag{6.3}$$

Proof. Integrate by parts, with $u = x^2$ and $dv = z_1^{-4}\,dx$ to obtain $\int \frac{x^2\,dx}{z_1^4} = -\frac{x^2}{3bz_1^3} + \frac{2}{3b}\int \frac{x\,dx}{z_1^3}$. Replace the last integral (evaluated in Entry 2.114.2) to obtain the result. □

6.4. Entry 2.115.4.

$$\int \frac{x^3\,dx}{z_1^4} = \left(\frac{3ax^2}{b^2} + \frac{9a^2x}{2b^3} + \frac{11a^3}{6b^4}\right)\frac{1}{z_1^3} + \frac{1}{b^4}\ln z_1 \tag{6.4}$$

Proof. The change of variables $t = a + bx$ gives

$$\int \frac{x^2\,dx}{z_1^4} = \frac{1}{b^4}\int \frac{(t-a)^3}{t^4}\,dt = \frac{1}{b^4}\left(\ln t + \frac{3a}{t} - \frac{3a^2}{2t^2} + \frac{a^3}{3t^3}\right).$$

This can be transformed to the stated result. □

7. Section 2.116

7.1. Entry 2.116.1.

$$\int \frac{dx}{z_1^5} = -\frac{1}{4bz_1^4} \tag{7.1}$$

Proof. The change of variables $t = a + bx$ gives the result. □

7.2. Entry 2.116.2.

$$\int \frac{x\,dx}{z_1^5} = -\left(\frac{x}{3b} + \frac{a}{12b^2}\right)\frac{1}{z_1^4} \tag{7.2}$$

Proof. Let $t = a + bx$ to obtain

$$\int \frac{x\,dx}{z_1^5} = \frac{1}{b^2}\int \left(\frac{1}{t^4} - \frac{a}{t^5}\right)dt = \frac{1}{b^2}\left(\frac{a}{4t^4} - \frac{1}{3t^3}\right)$$

and this reduces to the stated form. □

7.3. Entry 2.116.3.

$$\int \frac{x^2\,dx}{z_1^5} = -\left(\frac{x^2}{2b} + \frac{ax}{3b^2} + \frac{a^2}{12b^3} \right) \frac{1}{z_1^4} \tag{7.3}$$

PROOF. Let $t = a + bx$ to obtain

$$\int \frac{x^2\,dx}{z_1^5} = \frac{1}{b^3} \int \left(\frac{1}{t^3} - \frac{2a}{t^4} + \frac{a^2}{t^5} \right) dt. \tag{7.4}$$

Evaluating the three integrals and using $x = (t-a)/b$ gives the result. □

7.4. Entry 2.116.4.

$$\int \frac{x^3\,dx}{z_1^5} = -\left(\frac{x^3}{b} + \frac{3ax^2}{2b^2} + \frac{a^2x}{b^3} + \frac{a^3}{4b^4} \right) \frac{1}{z_1^4} \tag{7.5}$$

PROOF. Let $t = a + bx$ to obtain a sum of four integrals which are pure powers in t. Evaluating these integrals gives the result. □

8. Section 2.117

8.1. Entry 2.117.1.

$$\int \frac{dx}{x^n z_1^m} = -\frac{1}{(n-1)ax^{n-1}z_1^{m-1}} + \frac{b(2-n-m)}{a(n-1)} \int \frac{dx}{x^{n-1}z_1^m} \tag{8.1}$$

PROOF. Integrate by parts and then, the usual procedure, yields $\frac{dv}{v} = -\frac{(n-1)a}{xz_1}\,dx$. Then (8.1) gives $v = \left(\frac{z_1}{x}\right)^{n-1}$ and then (8.1) follows from $u = -\frac{1}{a(n-1)z_1^{m+n-2}}$ and $du = \frac{b(m+n-2)}{a(n-1)} z_1^{-m-n+1}$. □

8.2. Entry 2.117.2.

$$\int \frac{dx}{z_1^m} = -\frac{1}{(m-1)bz_1^{m-1}} \tag{8.2}$$

PROOF. Let $u = a+bx$ to obtain $\int \frac{dx}{z_1^m} = \frac{1}{b}\int u^{-m}du$. This gives the result. □

8.3. Entry 2.117.3.

$$\int \frac{dx}{xz_1^m} = \frac{1}{z_1^{m-1}a(m-1)} + \frac{1}{a}\int \frac{dx}{xz_1^{m-1}} \tag{8.3}$$

PROOF. Divide the identity $\frac{1}{xz_1} - \frac{1}{ax} = -\frac{b}{a}\frac{1}{z_1}$ by z_1^{m-1} and use $\frac{d}{dx}\frac{1}{z_1^{m-1}} = \frac{b(1-m)}{z_1^m}$ to obtain $\left(\frac{1}{xz_1} - \frac{1}{ax}\right)\frac{1}{z_1^{m-1}} = \frac{1}{a(m-1)}\frac{d}{dx}z_1^{1-m}$. Integrate this relation to conclude. □

8.4. Entry 2.117.4.

$$\int \frac{dx}{x^n z_1} = \sum_{k=1}^{n-1} \frac{(-1)^k b^{k-1}}{(n-k)a^k x^{n-k}} + \frac{(-1)^n b^{n-1}}{a^n} \ln \frac{z_1}{x} \tag{8.4}$$

PROOF. The method of partial fractions begins with the expansion

$$\frac{1}{x^n(a+bx)} = \sum_{j=1}^{n} \frac{A_j}{x^j} + \frac{B}{a+bx}. \tag{8.5}$$

Multiply by $x^n(a+bx)$ and collect terms to obtain

$$1 = aA_n + \sum_{j=1}^{n-1} \left[aA_j + bA_{j+1} \right] x^{n-j} + (bA_1 + B)x^n. \tag{8.6}$$

This produces the system of equations

$$aA_n = 1, \quad A_j = -\frac{b}{a} A_{j+1} \text{ for } 1 \leqslant n-1, \quad B = -bA_1. \tag{8.7}$$

An inductive proof shows that $A_j = \dfrac{(-1)^{n-j} b^{n-j}}{a^{n-j+1}}$ for $1 \leqslant j \leqslant n$. Therefore the partial fraction decomposition is

$$\frac{1}{x^n(a+bx)} = \sum_{j=2}^{n} \frac{(-1)^{n-j} b^{n-j}}{a^{n-j+1}} \frac{1}{x^j} + \frac{(-1)^{n-1} b^{n-1}}{a^n} \left(\frac{1}{x} - \frac{b}{a+bx} \right). \tag{8.8}$$

Now integrate to produce the result. □

9. Section 2.118

9.1. Entry 2.118.1.

$$\int \frac{dx}{xz_1} = -\frac{1}{a} \ln \frac{z_1}{x} \tag{9.1}$$

PROOF. Integrate the identity $\dfrac{1}{x(a+bx)} = \dfrac{1}{ax} - \dfrac{b}{a(a+bx)}$ to obtain the result. □

9.2. Entry 2.118.2.

$$\int \frac{dx}{x^2 z_1} = -\frac{1}{ax} + \frac{b}{a^2} \ln \frac{z_1}{x} \tag{9.2}$$

PROOF. The method of partial fractions gives

$$\frac{1}{x^2(a+bx)} = \frac{1}{a}\frac{1}{x^2} - \frac{b}{a^2}\frac{1}{x} + \frac{b^2}{a^2}\frac{1}{a+bx} \tag{9.3}$$

and the result follows by integration. □

9.3. Entry 2.118.3.

$$\int \frac{dx}{x^3 z_1} = -\frac{1}{2ax^2} + \frac{b}{a^2 x} - \frac{b^2}{a^3} \ln \frac{z_1}{x} \tag{9.4}$$

Proof. The identity

$$\frac{1}{x^3(a+bx)} = \frac{1}{a}\frac{1}{x^3} - \frac{b}{a^2}\frac{1}{x^2} + \frac{b^2}{a^3}\frac{1}{x} - \frac{b^3}{a^3}\frac{1}{a+bx} \tag{9.5}$$

gives the result follows by integration. □

10. Section 2.119

10.1. Entry 2.119.1.

$$\int \frac{dx}{x z_1^2} = \frac{1}{a z_1} - \frac{1}{a^2} \ln \frac{z_1}{x} \tag{10.1}$$

Proof. The change of variables $u = a + bx$ gives $\displaystyle\int \frac{dx}{x z_1^2} = \int \frac{du}{(u-a)u^2}$. The evaluation now comes by integrating the partial fraction decomposition

$$\frac{1}{(u-a)u^2} = -\frac{1}{au^2} - \frac{1}{a^2 u} + \frac{1}{a^2(u-a)}. \tag{10.2}$$

□

10.2. Entry 2.119.2.

$$\int \frac{dx}{x^2 z_1^2} = -\left(\frac{1}{ax} + \frac{2b}{a^2} \right) \frac{1}{z_1} + \frac{2b}{a^3} \ln \frac{z_1}{x} \tag{10.3}$$

Proof. The evaluation follows from the partial fraction expansion

$$\frac{1}{x^2(a+bx)^2} = \frac{1}{a^2x^2} - \frac{2b}{a^3 x} + \frac{b^2}{a^2(a+bx)^2} + \frac{2b^2}{a^3(a+bx)}. \tag{10.4}$$

□

10.3. Entry 2.119.3.

$$\int \frac{dx}{x^3 z_1^2} = \left(-\frac{1}{2ax^2} + \frac{3b}{2a^2x} + \frac{3b^2}{a^3} \right) \frac{1}{z_1} - \frac{3b^2}{a^4} \ln \frac{z_1}{x} \tag{10.5}$$

Proof. The evaluation follows from the partial fraction expansion

$$\frac{1}{x^3(a+bx)^2} = \frac{1}{a^2x^3} - \frac{2b}{a^3x^2} + \frac{3b^2}{a^4 x} - \frac{b^3}{a^3(a+bx)^2} - \frac{3b^3}{a^4(a+bx)}. \tag{10.6}$$

□

11. Section 2.121

11.1. Entry 2.121.1.

$$\int \frac{dx}{xz_1^3} = \left(\frac{3}{2a} + \frac{bx}{a^2} \right) \frac{1}{z_1^2} - \frac{1}{a^3} \ln \frac{z_1}{x} \tag{11.1}$$

Proof. The change of variables $t = a+bx$ gives $\displaystyle\int \frac{dx}{xz_1^3} = \int \frac{dt}{t^3(t-a)}$. The result now follows from integrating the partial fraction expansion

$$\frac{1}{t^3(t-a)} = -\frac{1}{at^3} - \frac{1}{a^2t^2} - \frac{1}{a^3t} + \frac{1}{a^3(t-a)}. \tag{11.2}$$

□

11.2. Entry 2.121.2.

$$\int \frac{dx}{x^2z_1^3} = -\left(\frac{1}{ax} + \frac{9b}{2a^2} + \frac{3b^2x}{a^3} \right) \frac{1}{z_1^2} + \frac{3b}{a^4} \ln \frac{z_1}{x} \tag{11.3}$$

Proof. This evaluation follows directly by integrating the partial fraction decomposition

$$\frac{1}{x^2(a+bx)^3} = \frac{1}{a^3x^2} - \frac{3b}{a^4x} + \frac{b^2}{a^2(a+bx)^3} + \frac{2b^2}{a^3(a+bx)^2} + \frac{3b^2}{a^4(a+bx)}.$$

□

11.3. Entry 2.121.3.

$$\int \frac{dx}{x^3z_1^3} = \left(-\frac{1}{2ax^2} + \frac{2b}{a^2x} + \frac{9b^2}{a^3} + \frac{6b^3x}{a^4} \right) \frac{1}{z_1^2} - \frac{6b^2}{a^5} \ln \frac{z_1}{x} \tag{11.4}$$

Proof. This evaluation follows directly by integrating the partial fraction decomposition

$$\frac{1}{x^3(a+bx)^3} = \frac{1}{a^3x^3} - \frac{3b}{a^4x^2} + \frac{6b^2}{a^5x} - \frac{b^3}{a^3(a+bx)^3} - \frac{3b^3}{a^4(a+bx)^2} - \frac{6b^3}{a^5(a+bx)}.$$

□

12. Section 2.122

12.1. Entry 2.122.1.

$$\int \frac{dx}{xz_1^4} = \left(\frac{11}{6a} + \frac{5bx}{2a^2} + \frac{b^2x^2}{a^3} \right) \frac{1}{z_1^3} - \frac{1}{a^4} \ln \frac{z_1}{x} \tag{12.1}$$

Proof. Integrate by parts with $u = 1/x$ and $dv = dx/z_1^4$ to obtain

$$\int \frac{dx}{xz_1^4} = -\frac{1}{3bxz_1^3} - \frac{1}{3b} \int \frac{dx}{x^2z_1^3}. \tag{12.2}$$

Now use entry 2.121.2 to write this as

$$\int \frac{dx}{xz_1^4} = -\frac{1}{a^4} \ln\left(\frac{z_1}{x}\right) - \frac{1}{3bxz_1^3} + \frac{1}{3b}\left(\frac{1}{ax} + \frac{9b}{2a^2} + \frac{3b^2x}{a^3} \right) \frac{1}{z_1^2}. \tag{12.3}$$

Now take the terms without the logarithm, leave the z_1^3 in the denominator and replace z_1 by $a+bx$ to obtain the stared formula. □

12.2. Entry 2.122.2.

$$\int \frac{dx}{x^2 z_1^4} = -\left(\frac{1}{ax} + \frac{22b}{3a^2} + \frac{10b^2x}{a^3} + \frac{4b^3x^2}{a^4} \right) \frac{1}{z_1^3} + \frac{4b}{a^5} \ln \frac{z_1}{x} \tag{12.4}$$

PROOF. Let $u = 1/x$ and $dv = dx/(xz_4)$ using Entry 2.122.1 to evaluate the primitive of $1/(xz_1^4)$ to produce

$$\int \frac{dx}{x^2 z^4} = \frac{1}{x}\left(\frac{11}{6a} + \frac{5bx}{2a^2} + \frac{b^2x^2}{a^3} \right) \frac{1}{z_1^3} - \frac{1}{a^4x} \ln\left(\frac{z_1}{x} \right) + \int \left(\frac{11}{6a} + \frac{5bx}{2a^2} + \frac{b^2x^2}{a^4} \right) \frac{dx}{x^2 z_1^3} - \frac{1}{a^4} \int \frac{1}{x^2} \ln\left(\frac{z_1}{x} \right) dx. \tag{12.5}$$

The first integral is evaluated using entries 2.117.2. 2.121.1, 2.121.2 and the last one is evaluate by the substitution $t = a/x + b$. □

12.3. Entry 2.122.3.

$$\int \frac{dx}{x^3 z_1^4} = \left(-\frac{1}{2ax^2} + \frac{5b}{2a^2x} + \frac{55b^2}{3a^3} + \frac{25b^3x}{a^4} + \frac{10b^4x^2}{a^5} \right) \frac{1}{z_1^3} - \frac{10b^2}{a^6} \ln \frac{z_1}{x} \tag{12.6}$$

PROOF. The reduction formula in Entry 2.117.1 gives

$$\int \frac{dx}{x^3 z_1^4} = -\frac{1}{2ax^2 z_1^3} - \frac{5b}{2a} \int \frac{dx}{x^2 z_1^4}. \tag{12.7}$$

Replace the value for this last integral appearing in Entry 2.122.2 to obtain the result. □

13. Section 2.123

13.1. Entry 2.123.1.

$$\int \frac{dx}{x z_1^5} = \left(\frac{25}{12a} + \frac{13bx}{3a^2} + \frac{7b^2x^2}{2a^3} + \frac{b^3x^3}{a^4} \right) \frac{1}{z_1^4} - \frac{1}{a^5} \ln \frac{z_1}{x} \tag{13.1}$$

PROOF. Write $\dfrac{1}{z_1^5} = \dfrac{d}{dx}\left(-\dfrac{1}{4bz_1^4} \right)$, integrate by parts and use Entry 2.122.2 to obtain the result. □

13.2. Entry 2.123.2.

$$\int \frac{dx}{x^2 z_1^5} = -\left(\frac{1}{ax} + \frac{125b}{12a^2} + \frac{65b^2x}{3a^3} + \frac{35b^3x^2}{2a^4} + \frac{5b^4x^3}{a^5} \right) \frac{1}{z_1^4} + \frac{5b}{a^6} \ln \frac{z_1}{x} \tag{13.2}$$

PROOF. Entry 2.117.1 gives $\displaystyle\int \frac{dx}{x^2 z_1^5} = -\frac{1}{axz_1^4} - \frac{5b}{a} \int \frac{dx}{xz_1^5}$. The result now follows from Entry 2.123.1. □

13.3. Entry 2.123.3.

$$\int \frac{dx}{x^3 z_1^5} = \left(-\frac{1}{2ax^2} + \frac{3b}{a^2 x} + \frac{125b^2}{4a^3} + \frac{65b^3 x}{a^4} + \frac{105b^4 x^2}{2a^5} + \frac{15b^5 x^3}{a^6}\right)\frac{1}{z_1^4} - \frac{15b^2}{a^7}\ln\frac{z_1}{x} \tag{13.3}$$

PROOF. Entry 2.117.1 gives $\displaystyle\int \frac{dx}{x^3 z_1^5} = -\frac{1}{2ax^2 z_1^4} - \frac{3b}{a}\int \frac{dx}{x^2 z_1^5}$. The result now follows from Entry 2.123.2. ☐

14. Section 2.124. Forms containing the form $z_2 = a + bx^2$

14.1. Entry 2.124.1.

$$\int \frac{dx}{z_2} = \begin{cases} \frac{1}{\sqrt{ab}}\arctan\left(x\sqrt{\frac{b}{a}}\right) & \text{if } ab > 0 \\ \frac{1}{2i\sqrt{ab}}\ln\frac{a+xi\sqrt{ab}}{a-xi\sqrt{ab}} & \text{if } ab < 0 \end{cases} \tag{14.1}$$

PROOF. Assume first $ab > 0$. Then $\displaystyle\int \frac{dx}{a+bx^2} = a\int \frac{dx}{a^2+abx^2}$. The change of variables $t = \sqrt{ab}x/a$ gives $\displaystyle\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}}\int \frac{dt}{1+t^2}$. This is the result.

Now assume $ab < 0$ and write $ab = -c^2$ with $c > 0$. Then the change of variables $t = cx/a$ gives $\displaystyle\int \frac{dx}{a+bx^2} = \frac{1}{c}\int \frac{dt}{1-t^2}$ and the partial fraction decomposition $\displaystyle\frac{1}{1-t^2} = \frac{1}{2(1+t)} - \frac{1}{2(1-t)}$ gives $\displaystyle\int \frac{dx}{a+bx^2} = \frac{1}{2\sqrt{-ab}}\ln\left(\frac{a+\sqrt{-ab}\,x}{a-\sqrt{-ab}\,x}\right)$. This is a better way to write the answer. ☐

14.2. Entry 2.124.2.

$$\int \frac{x\,dx}{z_2^m} = -\frac{1}{2b(m-1)z_2^{m-1}} \tag{14.2}$$

PROOF. This follows directly from the substitution $t = a + bx^2$. ☐

15. Section 2.125. Forms containing the form $z_3 = a + bx^3$

`Notation`: $\alpha = \sqrt[3]{\frac{a}{b}}$

15.1. Entry 2.125.1.

$$\int \frac{x^n\,dx}{z_3^m} = \frac{x^{n-2}}{z_3^{m-1}(n+1-3m)b} - \frac{(n-2)a}{b(n+1-3m)}\int \frac{x^{n-3}\,dx}{z_3^m} \tag{15.1}$$

PROOF. In order to choose the terms to integrate by parts, set

$$u\,dv = \frac{x^n}{z_3^m}\,dx \quad \text{and} \quad uv = \frac{x^{n-2}}{\alpha z_3^{m-1}} \tag{15.2}$$

with $\alpha = (n+1-3m)b$. Divide these two equations, integrate and use the expression for uv to obtain $v = z_3^{\alpha/3b}$ and $u = \dfrac{x^{n-2}}{\alpha z_3^{(n-2)/3}}$. The result now follows from differentiation and the expression for v above:

$$v\,du = \frac{a(n-2)}{\alpha}\frac{x^{n-3}}{z^m}\,dx = \frac{(n-2)a}{(n+1-3m)b}\frac{x^{n-3}}{z^m}\,dx. \tag{15.3}$$

□

15.2. Entry 2.125.2.

$$\int \frac{x^n\,dx}{z_3^m} = \frac{x^{n+1}}{3a(m-1)z_3^{m-1}} - \frac{n+4-3m}{3a(m-1)}\int \frac{x^n\,dx}{z_3^{m-1}} \tag{15.4}$$

Proof. The same procedure as the one use in the proof of Entry 2.125.1 show that the choices $u\,dv = \dfrac{x^n}{z_3^m}$ and $uv = \dfrac{x^{n+1}}{3a(m-1)z^{m-1}}$ and dividing these two relations gives $v = \dfrac{b^{m-1}x^{3(m-1)}}{z_3^{m-1}}$, from which one obtains the value $u = \dfrac{b^{1-m}}{3a(m-1)}x^{4+n-3m}$ and finally $v\,du = \dfrac{(4+n-3m)}{3a(m-1)}\dfrac{x^n\,dx}{z_3^{m-1}}$ as claimed on the right-hand side of (15.4). The proof is complete. □

16. Section 2.126

16.1. Entry 2.126.1.

$$\begin{aligned} \int \frac{dx}{z_3} &= \frac{\alpha}{3a}\left\{ \frac{1}{2}\ln\frac{(x+\alpha)^2}{x^2-\alpha x+\alpha^2} + \sqrt{3}\arctan\frac{x\sqrt{3}}{2\alpha - x}\right\} \\ &= \frac{\alpha}{3a}\left\{ \frac{1}{2}\ln\frac{(x+\alpha)^2}{x^2-\alpha x+\alpha^2} + \sqrt{3}\arctan\frac{2x-\alpha}{\alpha\sqrt{3}}\right\} \end{aligned} \tag{16.1}$$

Proof. Let $x = \alpha t$ to obtain

$$\int \frac{dx}{z_3} = \int \frac{dx}{a+bx^3} = \frac{\alpha}{a}\int \frac{dt}{t^3+1}. \tag{16.2}$$

The factorization $t^3+1 = (t+1)(t^2-t+1)$ gives the partial fraction decomposition

$$\frac{1}{t^3+1} = \frac{1}{3(1+t)} - \frac{1}{6}\frac{2t-1}{t^2-t+1} + \frac{1}{2(t^2-t+1)}. \tag{16.3}$$

Completing the square in the second integral gives the first evaluation. To verify the second identity simply differentiate both arctangent terms to see that they differ by a constant. This completes the proof. □

16.2. Entry 2.126.2.

$$\int \frac{x\,dx}{z_3} = -\frac{1}{3b\alpha}\left\{\frac{1}{2}\ln\frac{(x+\alpha)^2}{x^2-\alpha x+\alpha^2} - \sqrt{3}\arctan\frac{2x-\alpha}{\alpha\sqrt{3}}\right\} \tag{16.4}$$

Proof. The same change of variables in the proof of Entry 2.126.1 and the partial fraction decomposition $\frac{1}{t(1+t^3)} = \frac{1}{t} - \frac{1}{3(1+t)} - \frac{2t-1}{3(t^2-t+1)}$ give the result. □

16.3. Entry 2.126.3.

$$\int \frac{x^2\,dx}{z_3} = \frac{1}{3b}\ln(1+x^3\alpha^{-3}) = \frac{1}{3b}\ln z_3 \tag{16.5}$$

Proof. Let $t = a + bx^3$ to obtain $\int \frac{x^2\,dx}{z_3} = \frac{1}{3b}\int\frac{dt}{t} = \frac{1}{3b}\ln t$. This gives the second form of the answer. To obtain the first one, write

$$a + bx^3 = a\left(1+\frac{b}{a}x^3\right) = a\left(1+\alpha^{-3}x^3\right). \tag{16.6}$$

□

16.4. Entry 2.126.4.

$$\int \frac{x^3\,dx}{z_3} = \frac{x}{b} - \frac{a}{b}\int\frac{dx}{z_3} \tag{16.7}$$

Proof. This is Entry 2.125.1 (see formula (15.1)) with $n = 3$ and $m = 1$. □

16.5. Entry 2.126.5.

$$\int \frac{x^4\,dx}{z_3} = \frac{x^2}{2b} - \frac{a}{b}\int\frac{x\,dx}{z_3} \tag{16.8}$$

Proof. This is Entry 2.125.1 (see formula (15.1)) with $n = 4$ and $m = 1$. □

17. Section 2.127

17.1. Entry 2.127.1.

$$\int \frac{dx}{z_3^2} = \frac{x}{3az_3} + \frac{2}{3a}\int\frac{dx}{z_3} \tag{17.1}$$

Proof. This is the case $n = 0,\ m = 2$ in Entry 2.125.2; see (15.4). □

17.2. Entry 2.127.2.

$$\int \frac{x\,dx}{z_3^2} = \frac{x^2}{3az_3} + \frac{1}{3a}\int\frac{x\,dx}{z_3} \tag{17.2}$$

Proof. This is the case $n = 1,\ m = 2$ in Entry 2.125.2; see (15.4). □

17.3. Entry 2.127.3.

$$\int \frac{x^2\,dx}{z_3^2} = -\frac{1}{3bz_3} \tag{17.3}$$

PROOF. The change of variables $t = a + bx^3$ gives $\displaystyle\int \frac{x^2\,dx}{z_3^2} = \frac{1}{3b}\int \frac{dt}{t^2}$. Evaluate the last integral to obtain the result. □

17.4. Entry 2.127.4.

$$\int \frac{x^3\,dx}{z_3^2} = -\frac{x}{3bz_3} + \frac{1}{3b}\int \frac{dx}{z_3} \tag{17.4}$$

PROOF. Put $n = 3$ and $m = 2$ in Entry 2.125.1 to obtain $\displaystyle\int \frac{x^3\,dx}{z_3^2} = -\frac{x}{2bz_3} + \frac{a}{2b}\int \frac{dx}{z_3^2}$. Now use Entry 2.127.1 in the form $\displaystyle\int \frac{dx}{z_3^2} = \frac{x}{3az_3} + \frac{2}{3a}\int \frac{dx}{z_3}$ to obtain the result. □

18. Section 2.128

18.1. Entry 2.128.1.

$$\int \frac{dx}{x^n z_3^m} = -\frac{1}{(n-1)ax^{n-1}z_3^{m-1}} - \frac{b(3m+n-4)}{a(n-1)}\int \frac{dx}{x^{n-3}z_3^m} \tag{18.1}$$

PROOF. The usual procedure with $u\,dv = \dfrac{1}{x^n z_3^n}$ and $uv = -\dfrac{1}{(n-1)ax^{n-1}z_3^{m-1}}$ and using Entry 2.110.7 produce $v = \dfrac{z_3^{(n-1)/3}}{x^{n-1}}$ and $u = -\dfrac{1}{(n-1)a}z_3^{-m+1-(n-1)/3}$. From here, the value $v\,du$ gives the right-hand side of the formula. □

18.2. Entry 2.128.2.

$$\int \frac{dx}{x^n z_3^m} = \frac{1}{3a(m-1)x^{n-1}z_3^{m-1}} + \frac{n+3m-4}{3a(m-1)}\int \frac{dx}{x^n z_3^{m-1}} \tag{18.2}$$

PROOF. This expression is obtained from Entry 2.125.2 by replacing n by $-n$. □

19. Section 2.129

19.1. Entry 2.129.1.

$$\int \frac{dx}{xz_3} = \frac{1}{3a}\ln\frac{x^3}{z_3} \tag{19.1}$$

PROOF. The change of variables $t = a + bx^2$ and a partial fractions decomposition give

$$\int \frac{dx}{xz_3} = \frac{1}{3}\int \frac{dt}{t(t-a)} = \frac{1}{3a}\left(\int \frac{dt}{t-a} - \int \frac{dt}{t}\right) = \frac{1}{3a}\left(\ln(t-a) - \ln t\right) = \frac{1}{3}\ln\left(\frac{t-a}{t}\right)$$

and this is the result. (Recall that an arbitrary constant can be added to the integral). □

19.2. Entry 2.129.2.

$$\int \frac{dx}{x^2 z_3} = -\frac{1}{ax} - \frac{b}{a}\int \frac{x\,dx}{z_3} \tag{19.2}$$

PROOF. The usual procedure gives $u\,dv = \frac{1}{x^2 z_3}$ and $uv = -\frac{1}{ax}$. This gives $v = z_3^{1/3}/x$ and then $u = -\frac{1}{a}z_3^{-1/3}$. It follows that $v\,du = \frac{bx}{az_3}$ and integration by parts gives the formula. □

19.3. Entry 2.129.3.

$$\int \frac{dx}{x^3 z_3} = -\frac{1}{2ax^2} - \frac{b}{a}\int \frac{dx}{z_3} \tag{19.3}$$

PROOF. Use the identity $1 = \frac{a + bx^3 - bx^3}{a}$ to write $\int \frac{dx}{x^3 z_3} = \frac{1}{a}\int \frac{dx}{x^3} - \frac{b}{a}\int \frac{dx}{z_3}$. Evaluate the first integral on the right to obtain the result. □

20. Section 2.131

20.1. Entry 2.131.1.

$$\int \frac{dx}{xz_3^2} = \frac{1}{3az_3} + \frac{1}{3a^2}\ln\frac{x^3}{z_3} \tag{20.1}$$

PROOF. The reduction formula in Entry 2.128.2 gives

$$\int \frac{dx}{xz_3^2} = \frac{1}{3az_3} + \frac{1}{a}\int \frac{dx}{xz_3}. \tag{20.2}$$

The integral on the right in the formula above is evaluated in Entry 2.129.1. □

20.2. Entry 2.131.2.

$$\int \frac{dx}{x^2 z_3^2} = -\left(\frac{1}{ax} + \frac{4bx^2}{3a^2}\right)\frac{1}{z_3} - \frac{4b}{3a^2}\int \frac{x\,dx}{z_3} \tag{20.3}$$

PROOF. The reduction formula in Entry 2.128.2 gives

$$\int \frac{dx}{x^2 z_3^2} = \frac{1}{3axz_3} + \frac{4}{3a}\int \frac{dx}{x^2 z_3}. \tag{20.4}$$

Using Entry 2.129.2 this can be written as $\int \frac{dx}{x^2 z_3^2} = \frac{1}{3axz_3} - \frac{4}{3a^2 x} - \frac{4b}{3a^2}\int \frac{x\,dx}{z_3}$, which gives the result. □

20.3. Entry 2.131.3.

$$\int \frac{dx}{x^3 z_3^2} = -\left(\frac{1}{2ax^2} + \frac{5bx}{6a^2} \right) \frac{1}{z_3} - \frac{5b}{3a^2} \int \frac{dx}{z_3} \tag{20.5}$$

PROOF. Use the reduction formula in Entry 2.128.1 to obtain

$$\int \frac{dx}{x^3 z_3^2} = -\frac{1}{2ax^2 z_3} - \frac{5b}{2a} \int \frac{dx}{z_3^2}. \tag{20.6}$$

The last integral is given in Entry 2.127.1 as $\int \frac{dx}{z_3^2} = \frac{x}{3az_3} + \frac{2}{3a} \int \frac{dx}{z_3}$. Now replace in (20.6) to obtain the result. □

21. Section 2.132. Forms containing the form $z_4 = a + bx^4$

Notation. $\alpha = \sqrt[4]{\frac{a}{b}}$, $\alpha' = \sqrt[4]{-\frac{a}{b}}$

21.1. Entry 2.132.1.

(21.1)

$$\begin{aligned} \int \frac{dx}{z_4} &= \frac{\alpha}{4a\sqrt{2}} \left\{ \ln \frac{x^2 + \alpha x\sqrt{2} + \alpha^2}{x^2 - \alpha x\sqrt{2} + \alpha^2} + 2\arctan\left(\frac{\alpha x\sqrt{2}}{\alpha^2 - x^2} \right) \right\} \quad \text{for } ab > 0 \\ &= \frac{\alpha'}{4a} \left\{ \ln \frac{x + \alpha'}{x - \alpha'} + 2\arctan\left(\frac{x}{\alpha'} \right) \right\} \quad \text{for } ab < 0. \end{aligned}$$

PROOF. For $ab > 0$, the change of variables $x = \alpha t$ gives

$$\int \frac{dx}{z_4} = \frac{1}{b\alpha^2} \int \frac{dt}{t^4 + 1}. \tag{21.2}$$

The factorization $t^4 + 1 = (t^2 + \sqrt{2}t + 1)(t^2 - \sqrt{2}t + 1)$ gives the result using the method of partial fractions. For $ab < 0$, let $x = \alpha' t$ to obtain $\int \frac{dx}{z_4} = -\frac{\alpha'}{a} \int \frac{dt}{t^4 - 1}$. The result now follows as from the factorization $t^4 - 1 = (t-1)(t+1)(t^2+1)$. □

21.2. Entry 2.132.2.

$$\begin{aligned} \int \frac{x\,dx}{z_4} &= \tfrac{1}{2\sqrt{ab}} \arctan\left(x^2 \sqrt{\tfrac{b}{a}} \right) \quad \text{for } ab > 0 \\ &= \tfrac{1}{4\sqrt{-ab}} \ln\left(\tfrac{a + x^2\sqrt{-ab}}{a - x^2\sqrt{-ab}} \right) \quad \text{for } ab < 0. \end{aligned} \tag{21.3}$$

PROOF. The change of variables $t = x^2$ reduces this integral to Entry 2.124.1. □

21.3. Entry 2.132.3.

(21.4)

$$\begin{aligned} \int \frac{x^2\,dx}{z_4} &= \frac{1}{4b\alpha\sqrt{2}} \left\{ \ln \frac{x^2 - \alpha x\sqrt{2} + \alpha^2}{x^2 + \alpha x\sqrt{2} + \alpha^2} + 2\arctan\left(\frac{\alpha x\sqrt{2}}{\alpha^2 - x^2} \right) \right\} \quad \text{for } ab > 0 \\ &= -\frac{1}{4b\alpha'} \left\{ \ln \frac{x + \alpha'}{x - \alpha'} - 2\arctan\left(\frac{x}{\alpha'} \right) \right\} \quad \text{for } ab < 0. \end{aligned}$$

Proof. Scale as in the previous two entries and use the partial fraction decompositions

$$\begin{aligned} \frac{t^2}{t^4+1} &= \frac{t}{2\sqrt{2}(t^2-\sqrt{2}t+1)} - \frac{t}{2\sqrt{2}(t^2+\sqrt{2}t+1)} \\ \frac{t^2}{t^4-1} &= \frac{1}{4(t-1)} - \frac{1}{4(t+1)} + \frac{1}{2(t^2+1)} \end{aligned} \tag{21.5}$$

to verify the evaluation. □

21.4. Entry 2.132.4.

$$\int \frac{x^3\,dx}{z_4} = \frac{1}{4b}\ln z_4 \tag{21.6}$$

Proof. This follows directly from the substitution $z_4 = a + bx^4$. □

22. Section 2.133

22.1. Entry 2.133.1.

$$\int \frac{x^n\,dx}{z_4^m} = \frac{x^{n+1}}{4a(m-1)z_4^{m-1}} + \frac{4m-n-5}{4a(m-1)}\int \frac{x^n\,dx}{z_4^{m-1}} \tag{22.1}$$

Proof. The usual procedure to integrate by parts with $u\,dv = \frac{x^n}{z_4^m}$ and $uv = \frac{x^{n+1}}{4a(m-1)z_4^{m-1}}$ gives $v = \left(\frac{x^4}{z_4}\right)^{m-1}$. From here it follows that $u = \frac{1}{4a(m-1)x^{n-4m+5}}$ and $du = \frac{n-4m+5}{4a(m-1)}x^{n-4m+4}\,dx$. This gives the result. □

22.2. Entry 2.133.2.

$$\int \frac{x^n\,dx}{z_4^m} = \frac{x^{n-3}}{z_4^{m-1}(n+1-4m)b} - \frac{(n-3)a}{b(n+1-4m)}\int \frac{x^{n-4}\,dx}{z_4^m} \tag{22.2}$$

Proof. Integrate by parts choosing $u\,dv = \frac{x^n}{z_4^m}$ and $u\,v = \frac{x^{n-3}}{z_4^{m-1}(n+1-4m)b}$ (by the usual procedute). This produces $v = z_4^{(n+1-4m)/4}$ and then $u = \frac{x^{n-3}}{z_4^{(n-3)/4}(n+1-4m)b}$. From here compute $v\,du$ to obtain the result. □

23. Section 2.134

23.1. Entry 2.134.1.

$$\int \frac{dx}{z_4^2} = \frac{x}{4az_4} + \frac{3}{4a}\int \frac{dx}{z_4} \tag{23.1}$$

Proof. Put $n = 0$ and $m = 2$ in Entry 2.133.1. □

23.2. Entry 2.134.2.

$$\int \frac{x\,dx}{z_4^2} = \frac{x^2}{4az_4} + \frac{1}{2a}\int \frac{x\,dx}{z_4} \tag{23.2}$$

Proof. This is the case $n=1$ and $m=2$ of Entry 2.133.1. □

23.3. Entry 2.134.3.

$$\int \frac{x^2\,dx}{z_4^2} = \frac{x^3}{4az_4} + \frac{1}{4a}\int \frac{x^2\,dx}{z_4} \tag{23.3}$$

Proof. This is the case $n=m=2$ in Entry 2.133.1. □

23.4. Entry 2.134.4.

$$\int \frac{x^3\,dx}{z_4^2} = \frac{x^4}{4az_4} = -\frac{1}{4bz_4} \tag{23.4}$$

Proof. The first expression follows from the change of variables $t=a+bx^4$. The second one from $\dfrac{x^4}{4at} = \dfrac{1}{4ab} - \dfrac{1}{4bt}$. □

24. Section 2.135

24.1. Entry 2.135.

$$\int \frac{dx}{x^n z_4^m} = -\frac{1}{(n-1)ax^{n-1}z_4^{m-1}} - \frac{b(4m+n-5)}{(n-1)a}\int \frac{dx}{x^{n-4}z_4^m} \tag{24.1}$$

and for $n=1$

$$\int \frac{dx}{xz_4^m} = \frac{1}{a}\int \frac{dx}{xz_4^{m-1}} - \frac{b}{a}\int \frac{x^3\,dx}{z_4^m} \tag{24.2}$$

Proof. The usual procedure with $u\,dv = \dfrac{1}{x^n z_4^m}$ and $uv = -\dfrac{1}{(n-1)ax^{n-1}z_4^{m-1}}$ yields $v = \dfrac{z_4^{(n-1)/4}}{x^{n-1}}$ and $u = -\dfrac{1}{a(n-1)}z_4^{-m+1-(n-1)/4}$. From here one computes $v\,du$ to complete the evaluation. The case $n=1$ follows directly from the identity $\dfrac{1}{z_4^{m-1}} = \dfrac{a+bx^4}{z_4^m} = \dfrac{a}{z_4^m} + \dfrac{bx^4}{z_4^m}$. Now divide by x and integrate. □

25. Section 2.136

25.1. Entry 2.136.1.

$$\int \frac{dx}{xz_4} = \frac{\ln x}{a} - \frac{\ln z_4}{4a} = \frac{1}{4a}\ln\frac{x^4}{z_4} \tag{25.1}$$

PROOF. Start with $I = \int \frac{dx}{x(a+bx^4)} = \frac{1}{4b}\int \frac{4bx^3\,dx}{x^4(a+bx^4)}$ and make the change of variables $t = a+bx^4$ and use partial fraction decompositions to obtain

$$I = \frac{1}{4}\int \frac{dt}{t(t-a)} = \frac{1}{4a}\left(\int \frac{dt}{t-a} - \int \frac{dt}{t}\right) = \frac{\ln(t-a)-\ln t}{4a} = \frac{\ln(bx^4)-\ln z_4}{4a}.$$

The b factor in the first term can be eliminated since it only contributes a constant. □

25.2. Entry 2.136.2.

$$\int \frac{dx}{x^2 z_4} = -\frac{1}{ax} - \frac{b}{a}\int \frac{x^2\,dx}{z_4} \tag{25.2}$$

PROOF. This is the special case $n=2$, $m=1$ in Entry 2.135. □

26. Section 2.14. Forms containing the binomial $1 \pm x^n$

`Notation`. $\alpha = \sqrt[4]{\frac{a}{b}}$, $\alpha' = \sqrt[4]{-\frac{a}{b}}$

26.1. Entry 2.141.1.

$$\int \frac{dx}{1+x} = \ln(1+x) \tag{26.1}$$

PROOF. This follows directly from the `logarithm` as the primitive of $1/x$. □

26.2. Entry 2.141.2.

$$\int \frac{dx}{1+x^2} = \arctan x = -\text{arctan}\left(\frac{1}{x}\right) \tag{26.2}$$

PROOF. This is an elementary integral. The change of variables $x = \tan t$ proves it. The second form comes from the fact that $\arctan(x) + \arctan\left(\frac{1}{x}\right)$ is constant. □

26.3. Entry 2.141.3.

$$\int \frac{dx}{1+x^3} = \frac{1}{3}\ln\frac{1+x}{\sqrt{1-x+x^2}} + \frac{1}{\sqrt{3}}\text{arctan}\,\frac{x\sqrt{3}}{2-x} \tag{26.3}$$

PROOF. The partial fraction decomposition is written as

$$\frac{1}{1+x^3} = \frac{1}{3}\frac{1}{x+1} + \frac{1}{3}\frac{2-x}{x^2-x+1} = \frac{1}{3}\frac{1}{x+1} - \frac{1}{6}\frac{2x-1}{x^2-x+1} + \frac{1}{2}\frac{1}{x^2-x+1}.$$

The first two integrals are direct

$$\int \frac{dx}{x+1} = \ln(x+1) \quad \text{and} \quad \int \frac{2x-1}{x^2-x+1} = \ln(x^2-x+1), \tag{26.4}$$

while the last one follows by writing $x^2-x+1 = (x-\frac{1}{2})^2+\frac{3}{4}$ and making the substitution $x = \frac{1}{2}+\frac{\sqrt{3}}{2}\tan\theta$ to obtain $\frac{1}{2}\int \frac{dx}{x^2-x+1} = \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)$. The form gives here follows from the identity $\arctan\left(\frac{x\sqrt{3}}{2-x}\right) = \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + \frac{\pi}{6}$, that can be verified using the addition theorem for the tangent function. ☐

26.4. Entry 2.141.4.

$$\int \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}} \ln \frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}} \arctan \frac{x\sqrt{2}}{1-x^2} \tag{26.5}$$

Proof. Start with the partial fraction decomposition

$$\frac{1}{1+x^4} = \frac{1}{2\sqrt{2}} \frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{1}{2\sqrt{2}} \frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}. \tag{26.6}$$

Details are given for the first integral, the second one is similar. Start with

$$\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} = \frac{1}{2} \frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} + \frac{1}{2} \frac{\sqrt{2}}{x^2+\sqrt{2}x+1} \tag{26.7}$$

and integrating produces

$$\begin{aligned} \int \frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} dx &= \frac{1}{2} \int \frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} dx + \frac{1}{2} \int \frac{\sqrt{2}}{x^2+\sqrt{2}x+1} dx \\ &= \frac{1}{2} \ln(x^2+\sqrt{2}x+1) + \frac{\sqrt{2}}{2} \int \frac{dx}{x^2+\sqrt{2}x+1}. \end{aligned}$$

To evaluate the second integral write $x^2+\sqrt{2}x+1 = \left(x+\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2$ and make $x = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\tan\theta$ to obtain $\int \frac{dx}{x^2+\sqrt{2}x+1} = \sqrt{2}\arctan(\sqrt{2}x+1)$. It follows that

$$\int \frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} dx = \frac{1}{2} \ln(x^2+\sqrt{2}x+1) + \arctan(\sqrt{2}x+1). \tag{26.8}$$

Similarly

$$\int \frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1} dx = \frac{1}{2} \ln(x^2-\sqrt{2}x+1) + \arctan(\sqrt{2}x-1). \tag{26.9}$$

The simplification of the arctangent terms follows from their addition theorem. ☐

26.5. Entry 2.142.

$$\begin{aligned} \int \frac{dx}{1+x^n} &= -\frac{2}{n} \sum_{k=0}^{\frac{n}{2}-1} P_k \cos\left(\frac{2k+1}{n}\pi\right) + \frac{2}{n} \sum_{k=0}^{\frac{n}{2}-1} Q_k \sin\left(\frac{2k+1}{n}\pi\right) && \text{for } n \text{ even} \\ &= \frac{\ln(1+x)}{n} - \frac{2}{n} \sum_{k=0}^{\frac{n-3}{2}} P_k \cos\left(\frac{2k+1}{n}\pi\right) + \frac{2}{n} \sum_{k=0}^{\frac{n-3}{2}} Q_k \sin\left(\frac{2k+1}{n}\pi\right) && \text{for } n \text{ odd} \end{aligned}$$

where

$$P_k = \frac{1}{2}\ln\left(x^2 - 2x\cos\left(\frac{2k+1}{2}\pi\right) + 1\right)$$

$$Q_k = \arctan\left(\frac{x\sin\left(\frac{2k+1}{n}\pi\right)}{1 - x\cos\left(\frac{2k+1}{n}\pi\right)}\right) = \arctan\left(\frac{x - \cos\left(\frac{2k+1}{n}\pi\right)}{\sin\left(\frac{2k+1}{n}\pi\right)}\right)$$

PROOF. The roots of $x^n + 1 = 0$ are given by

$$x_{n,k} = \exp\left(\frac{(2k+1)\pi i}{n}\right) = \cos\left(\frac{2k+1}{n}\pi\right) + i\sin\left(\frac{2k+1}{n}\pi\right), \quad \text{for } 0 \leqslant k \leqslant n-1. \tag{26.10}$$

The method of partial fractions yields

$$\frac{1}{x^n+1} = \sum_{k=0}^{n-1} \frac{A_{n,k}}{x - x_{n,k}} \tag{26.11}$$

with $A_{n,k} = \lim_{x \to x_{n,k}} \frac{x - x_{n,k}}{x^n + 1} = -\frac{x_{n,k}}{n}$. This is obtained by using $x_{n,k}^n = -1$. Then

$$\frac{1}{x^n+1} = -\frac{1}{n}\sum_{k=0}^{n-1} \frac{x_{n,k}}{x - x_{n,k}} \tag{26.12}$$

Assume now that n is even. Then, for $n/2 \leqslant k \leqslant n-1$, one has $0 \leqslant n-1-k \leqslant n/2-1$ and

$$x_{n,n-1-k} = \exp\left[(2(n-1-k)+1)\pi i/n\right] = \exp\left[-(2k+1)\pi i/n\right] = \overline{\exp\left[(2k+1)\pi i/n\right]} = \overline{x_{n,k}}$$

and therefore

$$\begin{aligned} \frac{1}{x^n+1} &= -\frac{1}{n}\sum_{k=0}^{\frac{n}{2}-1}\left[\frac{x_{n,k}}{x - x_{n,k}} + \frac{\overline{x_{n,k}}}{x - \overline{x_{n,k}}}\right] \\ &= \frac{1}{n}\sum_{k=0}^{\frac{n}{2}-1} \frac{-2R_{n,k}x + 2}{x^2 - 2R_{n,k}x + 1}, \\ &= \frac{1}{n}\sum_{k=0}^{\frac{n}{2}-1}(-R_{n,k})\frac{2x - 2R_{n,k}}{x^2 - 2R_{n,k}x + 1} + \sum_{k=0}^{\frac{n}{2}-1} \frac{2I_{n,k}^2}{x^2 - 2R_{n,k}x + 1} \end{aligned} \tag{26.13}$$

using $|x_{n,k}| = 1$ and with the notation $R_{n,k} = \operatorname{Re} x_{n,k},\ I_{n,k} = \operatorname{Im} x_{n,k}$. The form in which the rational function $1/(x^n+1)$ has been written is to facilitate the integrals:

$$\int \frac{2x - 2R_{n,k}}{x^2 - 2R_{n,k}x + 1} = \ln(x^2 - 2xR_{n,k} + 1) \quad \text{and} \quad \int \frac{I_{n,k}\,dx}{x^2 - 2R_{n,k}x + 1} = \operatorname{Arctan}\left(\frac{x - R_{n,k}}{I_{n,k}}\right).$$

This completes the evaluation when n is even. In the case n odd, the index $k = (n-1)/2$ yields $x_{n,k} = -1$, producing the term $\ln(1+x)$. The rest of the proof is the same as for n even (the symmetry of the roots is now at $k = \frac{1}{2}(n-3)$). □

27. Section 2.143

27.1. Entry 2.143.1.

$$\int \frac{dx}{1-x} = -\ln(1-x) \tag{27.1}$$

PROOF. This follows from the change of variables $t = 1-x$ and the definition of `logarithm` as the primitive of $1/x$. □

27.2. Entry 2.143.2.

$$\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\frac{1+x}{1-x} = \text{arctanh } x \quad -1 < x < 1 \tag{27.2}$$

PROOF. This comes by integrating the partial fraction decomposition

$$\frac{1}{1-x^2} = \frac{1}{2(x+1)} - \frac{1}{2(x-1)}. \tag{27.3}$$

□

27.3. Entry 2.143.3.

$$\int \frac{dx}{x^2-1} = \frac{1}{2}\ln\frac{x-1}{x+1} = -\text{arccoth } x \quad x > -1,\ x < -1 \tag{27.4}$$

PROOF. This comes by integrating the partial fraction decomposition $\frac{1}{x^2-1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$. □

27.4. Entry 2.143.4.

$$\int \frac{dx}{1-x^3} = \frac{1}{3}\ln\frac{\sqrt{1+x+x^2}}{1-x} + \frac{1}{\sqrt{3}}\arctan\frac{x\sqrt{3}}{2+x} \tag{27.5}$$

PROOF. Write the partial fraction decomposition

$$\frac{1}{1-x^3} = \frac{1}{3(1-x)} + \frac{x+2}{3(x^2+x+1)} \tag{27.6}$$

in the form $\frac{1}{1-x^3} = -\frac{1}{3}\left(\frac{-1}{1-x}\right) + \frac{1}{6}\left(\frac{2x+1}{x^2+x+1}\right) + \frac{1}{2}\frac{1}{x^2+x+1}$. The first two integrals are logarithmic and the last one is evaluated by completing the square in the denominator. The argument of the arctangent may be replaced by $(2x+1)/\sqrt{3}$. □

27.5. Entry 2.143.5.

$$\int \frac{dx}{1-x^4} = \frac{1}{4}\ln\frac{1+x}{1-x} + \frac{1}{2}\arctan x = \frac{1}{2}(\text{arctanh } x + \arctan x) \tag{27.7}$$

PROOF. The evaluation follows directly from the partial fraction decomposition

$$\frac{1}{1-x^4} = \frac{1}{4(1-x)} + \frac{1}{4(1+x)} + \frac{1}{2(1+x^2)}. \tag{27.8}$$

□

28. Section 2.144

28.1. Entry 2.144.1.

$$\int \frac{dx}{1-x^n} = \frac{1}{n}\ln\frac{1+x}{1-x} - \frac{2}{n}\sum_{k=1}^{\frac{n}{2}-1} P_k \cos\frac{2k\pi}{n} + \frac{2}{n}\sum_{k=1}^{\frac{n}{2}-1} Q_k \sin\frac{2k\pi}{n} \quad \text{for } n \text{ even} \tag{28.1}$$

where

$$P_k = \frac{1}{2}\ln\left(x^2 - 2x\cos\frac{2k\pi}{n} + 1\right), \quad Q_k = \arctan\left(\frac{x-\cos\frac{2k\pi}{n}}{\sin\frac{2k\pi}{n}}\right) \tag{28.2}$$

PROOF. The method of partial fractions require the roots of $x^n = 1$. These are given by

$$x_{n,k} = \exp\left(\frac{2\pi i k}{n}\right) = \cos\left(\frac{2\pi k}{n}\right) + i\,\sin\left(\frac{2\pi k}{n}\right), \quad 0 \leqslant k \leqslant n-1. \tag{28.3}$$

Observe that $x_{n,0} = 1$ and $x_{n,n/2} = -1$. Now the partial fraction decomposition is written as

$$\begin{aligned} \frac{1}{1-x^n} &= \frac{A_{n,0}}{x-1} + \sum_{k=1}^{\frac{n}{2}-1} \frac{A_{n,k}}{x-x_{n,k}} + \frac{A_{n,n/2}}{x+1} + \sum_{k=\frac{n}{2}+1}^{n-1} \frac{A_{n,k}}{x-x_{n,k}} \\ &= \frac{A_{n,0}}{x-1} + \sum_{k=1}^{\frac{n}{2}-1} \frac{A_{n,k}}{x-x_{n,k}} + \frac{A_{n,n/2}}{x+1} + \sum_{k=1}^{\frac{n}{2}-1} \frac{A_{n,\frac{n}{2}+k}}{x-x_{n,\frac{n}{2}+k}}. \end{aligned} \tag{28.4}$$

A direct calculation of the coefficients $A_{n,k}$, the formulas for the poles $x_{n,k}$ and the fact that $|x_{n,k}| = 1$ give

$$\frac{1}{1-x^n} = \frac{1}{n(1-x)} + \frac{1}{n(1+x)} - \frac{1}{n}\sum_{k=1}^{\frac{n}{2}-1} \frac{(x_{n,k} + \overline{x_{n,k}})\,x - 2}{(x-x_{n,k})(x-\overline{x_{n,k}})}. \tag{28.5}$$

For the purpose of integration, it is convenient to write this as

$$\begin{aligned} \frac{1}{1-x^n} &= \frac{1}{n(1-x)} + \frac{1}{n(1+x)} \\ &\quad - \frac{1}{n}\sum_{k=1}^{\frac{n}{2}-1} \operatorname{Re}(x_{nk}) \frac{2x - 2\operatorname{Re}(x_{n,k})}{x^2 - 2x\operatorname{Re}(x_{n,k}) + 1} \\ &\quad + \frac{2}{n}\sum_{k=1}^{\frac{n}{2}-1} \frac{\operatorname{Im}^2(x_{n,k})}{x^2 - 2x\operatorname{Re}(x_{n,k}) + 1}. \end{aligned} \tag{28.6}$$

Integrating gives the result. ∎

28.2. Entry 2.144.2.

$$\int \frac{dx}{1-x^n} = -\frac{1}{n}\ln(1-x) - \frac{2}{n}\sum_{k=1}^{\frac{n-1}{2}} P_k \cos\frac{2k\pi}{n} + \frac{2}{n}\sum_{k=1}^{\frac{n-1}{2}} Q_k \sin\frac{2k\pi}{n} \quad \text{for } n \text{ odd} \tag{28.7}$$

where

$$P_k = \frac{1}{2}\ln\left(x^2 - 2x\cos\frac{2k\pi}{n} + 1\right), \quad Q_k = \arctan\left(\frac{x-\cos\frac{2k\pi}{n}}{\sin\frac{2k\pi}{n}}\right) \tag{28.8}$$

PROOF. The result follows from the partial fraction expansion

$$\frac{1}{1-x^n} = \sum_{k=0}^{n-1} \frac{A_{n,k}}{x-x_{n,k}}, \tag{28.9}$$

where $\{x_{n,k} : 0 \leqslant k \leqslant n-1\}$ is the set of roots of $x^n = 1$. The standard computation gives $A_{n,k} = -\frac{1}{n}x_{n,k}$ and the stated formula follow by pairing the pair of roots $x_{n,k}$ and $x_{n,n-k}$ for $1 \leqslant k \leqslant n-1$. (See the solution to Entry 2.142 for full details of a similar example). The single term comes when $k=0$ corresponding to the root $x=1$. This produces the term $\ln(1-x)$ in the formula. □

29. Section 2.145

29.1. Entry 2.145.1.

$$\int \frac{x\,dx}{1+x} = x - \ln(1+x) \tag{29.1}$$

PROOF. Let $u = 1+x$, the integrand becomes $1 - 1/u$ and now integrate. □

29.2. Entry 2.145.2.

$$\int \frac{x\,dx}{1+x^2} = \frac{1}{2}\ln(1+x^2) \tag{29.2}$$

PROOF. This follows directly from the change of variables $u = 1+x^2$. □

29.3. Entry 2.145.3.

$$\int \frac{x\,dx}{1+x^3} = -\frac{1}{6}\ln\frac{(1+x)^2}{1-x+x^2} + \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} \tag{29.3}$$

PROOF. The partial fraction decomposition

$$\frac{x}{1+x^3} = -\frac{1}{3}\frac{1}{x+1} + \frac{1}{3}\frac{x+1}{x^2-x+1} \tag{29.4}$$

is written in a more convenient form as

$$\frac{x}{1+x^3} = -\frac{\frac{1}{3}}{x+1} + \frac{1}{6}\frac{2x-1}{x^2-x+1} + \frac{\frac{1}{2}}{x^2-x+1}. \tag{29.5}$$

Therefore

$$\int \frac{x\,dx}{x^3+1} = -\frac{1}{3}\ln(x+1) + \frac{1}{6}\ln(x^2-x+1) + \frac{1}{2}\int\frac{dx}{x^2-x+1}.$$

To compute the last integral write $x^2 - x + 1 = (x - \frac{1}{2})^2 + \frac{3}{4} = (x - \frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2$ and the change of variables $x = \frac{1}{2} + \frac{\sqrt{3}}{2}\tan\theta$ gives $\displaystyle\int \frac{dx}{x^2 - x + 1} = \frac{2}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)$. Now put all the pieces together to get the result. □

29.4. Entry 2.145.4.

$$\int \frac{x\,dx}{1+x^4} = \frac{1}{2}\arctan x^2 \tag{29.6}$$

Proof. This evaluation follows directly from the change of variables $u = x^2$. □

29.5. Entry 2.145.5.

$$\int \frac{x\,dx}{1-x} = -\ln(1-x) - x \tag{29.7}$$

Proof. This evaluation follows directly from the change of variables $u = 1 - x$. □

29.6. Entry 2.145.6.

$$\int \frac{x\,dx}{1-x^2} = -\frac{1}{2}\ln(1-x^2) \tag{29.8}$$

Proof. Make the change of variables $u = 1 - x^2$. □

29.7. Entry 2.145.7.

$$\int \frac{x\,dx}{1-x^3} = -\frac{1}{6}\ln\frac{(1-x)^2}{1+x+x^2} - \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} \tag{29.9}$$

Proof. A similar argument to the one used in the proof of formula (29.3) produces this evaluation. □

29.8. Entry 2.145.8.

$$\int \frac{x\,dx}{1-x^4} = \frac{1}{4}\ln\frac{1+x^2}{1-x^2} \tag{29.10}$$

Proof. The change of variables $u = x^2$ gives $\displaystyle\int \frac{x\,dx}{1-x^4} = \frac{1}{2}\int\frac{du}{1-u^2}$. Now integrate the decomposition $\displaystyle\frac{1}{1-u^2} = \frac{\frac{1}{2}}{1+u} + \frac{\frac{1}{2}}{1-u}$ to complete the evaluation. □

30. Section 2.146

30.1. Entry 2.146.1.

$$\begin{aligned}\int \frac{x^{m-1}\,dx}{1+x^{2n}} &= -\frac{1}{2n}\sum_{k=1}^{n}\cos\frac{m\pi(2k-1)}{2n}\ln\left(1 - 2x\cos\frac{2k-1}{2n}\pi + x^2\right) \\ &\quad + \frac{1}{n}\sum_{k=1}^{n}\sin\frac{m\pi(2k-1)}{2n}\arctan\left(\frac{x - \cos\frac{2k-1}{2n}\pi}{\sin\frac{2k-1}{2n}\pi}\right) \quad m < 2n.\end{aligned} \tag{30.1}$$

PROOF. The proof is based on the ideas given in detail for Entry 2.144.1. Let $x = x_{2n,k} = \exp\left((2k-1)\pi i/n\right),\ 1 \leqslant k \leqslant 2n$ be the roots of $x^{2n} = -1$. Consider the partial fraction decomposition

$$\frac{x^{m-1}}{1+x^{2n}} = \sum_{k=1}^{2n} \frac{A_{2n,k}}{x - x_{2n,k}}. \tag{30.2}$$

The procedure described in the proof of Entry 2.144.1 yields

$$A_{2n,k} = -\frac{x_{2n,k}^{m}}{2n}. \tag{30.3}$$

Now use the symmetry of the roots $x_{2n,k}$ and the corresponding symmetry of the coefficients $A_{2n,k}$ to obtain

This yields

$$\int \frac{x^{m-1}\,dx}{1-x^{2n}} = -\frac{1}{n}\sum_{k=1}^{n-1}\left[\int \frac{\cos\frac{m\pi(2k-1)}{2n}\left(x-\cos\frac{\pi(2k-1)}{2n}\right)\,dx}{x^2-2x\cos\frac{\pi(2k-1)}{2n}+1}\right] + \frac{1}{n}\sum_{k=1}^{n-1}\left[\int \frac{\sin\frac{m\pi(2k-1)}{2n}\sin\frac{\pi(2k-1)}{2n}\,dx}{x^2-2x\cos\frac{\pi(2k-1)}{2n}+1}\right] \tag{30.4}$$

Each of the integrals may be computed by the methods described before. This gives the evaluation. □

30.2. Entry 2.146.2.

$$\int \frac{x^{m-1}\,dx}{1+x^{2n+1}} = (-1)^{m+1}\frac{\ln(1+x)}{2n+1} - \frac{1}{2n+1}\sum_{k=1}^{n}\cos\frac{m\pi(2k-1)}{2n+1}\ln\left(1-\cos\frac{2k-1}{2n+1}\pi + x^2\right) + \frac{2}{2n+1}\sum_{k=1}^{n}\sin\frac{m\pi(2k-1)}{2n+1}\arctan\left(\frac{x-\cos\frac{2k-1}{2n+1}\pi}{\sin\frac{2k-1}{2n+1}\pi}\right) \quad m \leqslant 2n. \tag{30.5}$$

PROOF. The proof of this evaluation is similar to the one in Entry 2.146.1. The roots of $x^{2m+1} = -1$ are

$$x_{m,k} = \exp\left(\frac{\pi i(2k-1)}{2m+1}\right), \quad 1 \leqslant k \leqslant 2m+1. \tag{30.6}$$

The special case $k = m+1$ gives $x = 1$ and it produces the term $\ln(1-x)$. The remaining $2m$ roots are pairs as before to express the result in real terms. □

30.3. Entry 2.146.3.

$$\int \frac{x^{m-1}\,dx}{1-x^{2n}} = \frac{1}{2n}\left\{(-1)^{m+1}\ln(1+x) - \ln(1-x)\right\} - \frac{1}{2n}\sum_{k=1}^{n-1}\cos\frac{km\pi}{n}\ln\left(1-2x\cos\frac{k\pi}{n}+x^2\right) + \frac{1}{n}\sum_{k=1}^{n-1}\sin\frac{km\pi}{n}\arctan\left(\frac{x-\cos\frac{k\pi}{n}}{\sin\frac{k\pi}{n}}\right), \quad m < 2n. \tag{30.7}$$

Proof. The proof is obtained by the same method described above. It follows directly from the partial fraction decomposition (with $x_{n,k}$ the roots of $x^{2n}=1$):

$$\frac{x^{m-1}}{1-x^{2n}} = \frac{A_{n,m,0}}{x-1} + \sum_{k=1}^{n-1}\frac{A_{n,m,k}}{x-x_{n,k}} + \frac{A_{n,m,m}}{x+1} + \sum_{k=1}^{n-1}\frac{B_{n,m,k}}{x-\overline{x_{n,k}}}. \tag{30.8}$$

The computation of the coefficients $A_{n,m,k}$ and $B_{n,m,k}$ is exactly as before. The details are left to the reader. □

30.4. Entry 2.146.4.

$$\int \frac{x^{m-1}\,dx}{1-x^{2n+1}} = -\frac{1}{2n+1}\ln(1-x) + (-1)^{m+1}\frac{1}{2n+1}\sum_{k=1}^{n}\cos\frac{m\pi(2k-1)}{2n+1}\ln\left(1+2x\cos\frac{2k-1}{2n+1}\pi+x^2\right) (-1)^{m+1}\frac{2}{2n+1}\sum_{k=1}^{n}\sin\frac{m\pi(2k-1)}{2n+1}\arctan\left(\frac{x+\cos\frac{2k-1}{2n+1}\pi}{\sin\frac{2k-1}{2n+1}\pi}\right) \quad m \leqslant 2n \tag{30.9}$$

Proof. This is solved exactly as in the previous cases, using the roots $x_{m,k}$

$$x_{m,k} = \exp\left(\frac{2\pi i(2k+1)}{2m+1}\right), \quad \text{for } 1 \leqslant k \leqslant 2m+1. \tag{30.10}$$

The index $k=m$ gives the root $x=1$ and produces the logarithmic term $\ln(1-x)$. The usual pairing of conjugate roots gives the stated form of the integral. □

31. Section 2.147

31.1. Entry 2.147.1.

$$\int \frac{x^m\,dx}{1-x^{2n}} = \frac{1}{2}\int\frac{x^m\,dx}{1-x^n} + \frac{1}{2}\int\frac{x^m\,dx}{1+x^n} \tag{31.1}$$

Proof. The partial fraction decomposition $\dfrac{1}{1-s^2} = \dfrac{1}{2(1-s)} + \dfrac{1}{2(1+s)}$ with $s=x^n$, multiplied by x^m, gives the evaluation. □

31.2. Entry 2.147.2.

$$\int \frac{x^m\,dx}{(1+x^2)^n} = -\frac{1}{2n-m-1}\cdot\frac{x^{m-1}}{(1+x^2)^{n-1}} + \frac{m-1}{2n-m-1}\int\frac{x^{m-2}\,dx}{(1+x^2)^n} \tag{31.2}$$

Proof. Integrate by parts by writing $\dfrac{x^m\,dx}{(1+x^2)^n} = \displaystyle\int x^{m-1}\frac{d}{dx}\left(\frac{-1}{2(n-1)(1+x^2)^{n-1}}\right)$. This gives $\displaystyle\int \frac{x^m\,dx}{(1+x^2)^n} = -\frac{1}{2(n-1)}\frac{x^{m-1}}{1+x^2)^{n-1}} + \frac{m-1}{2(n-1)}\frac{x^{m-2}\,dx}{(1+x^2)^{n-1}}$. In the second integral, multiply top and bottom by $1+x^2$ and move the term coming from the factor x^2 to the left. This gives the result. □

31.3. Entry 2.147.3.

$$\int \frac{x^m}{1+x^2}\,dx = \frac{x^{m-1}}{m-1} - \int\frac{x^{m-2}}{1+x^2}\,dx \tag{31.3}$$

Proof. Write $\dfrac{x^m}{1+x^2} = \dfrac{x^{m-2}\cdot(x^2=(1+x^2)-1)}{1+x^2} = x^{m-2} - \dfrac{x^{m-2}}{1+x^2}$. The result follows by integration. □

31.4. Entry 2.147.4.

$$\begin{aligned}\int \frac{x^m\,dx}{(1-x^2)^n} &= \frac{1}{2n-m-1}\cdot\frac{x^{m-1}}{(1-x^2)^{n-1}} - \frac{m-1}{2n-m-1}\int\frac{x^{m-2}\,dx}{(1-x^2)^n}\\ &= \frac{1}{2n-2}\cdot\frac{x^{m-1}}{(1-x^2)^{n-1}} - \frac{m-1}{2n-2}\int\frac{x^{m-2}\,dx}{(1-x^2)^{n-1}}\end{aligned} \tag{31.4}$$

Proof. The second form comes directly by integration by parts writing

$$\frac{x^m}{(1-x^2)^n} = x^{m-1}\frac{d}{dx}\left(\frac{1}{2(n-1)}\frac{1}{(1-x^2)^{n-1}}\right). \tag{31.5}$$

The first form appears by multiplying the integrand by $1-x^2$ top and bottom in the integrand of the second form. □

31.5. Entry 2.147.5.

$$\int \frac{x^m\,dx}{1-x^2} = -\frac{x^{m-1}}{m-1} + \int\frac{x^{m-2}\,dx}{1-x^2} \tag{31.6}$$

Proof. Write the integrand as $\dfrac{x^m}{1-x^2} = -x^{m-2} + \dfrac{x^{m-2}}{1-x^2}$ and integrate. □

32. Section 2.148

32.1. Entry 2.148.1.

$$\int \frac{dx}{x^m(1+x^2)^n} = -\frac{1}{m-1}\frac{1}{x^{m-1}(1+x^2)^{n-1}} - \frac{2n+m-3}{m-1}\int\frac{dx}{x^{m-2}(1+x^2)^n} \tag{32.1}$$

For $m=1$

$$\int \frac{dx}{x(1+x^2)^n} = \frac{1}{2n-2}\frac{1}{(1+x^2)^{n-1}} + \int\frac{dx}{x(1+x^2)^{n-1}} \tag{32.2}$$

For $m = 1$ and $n = 1$

$$\int \frac{dx}{x(1+x^2)} = \ln \frac{x}{\sqrt{1+x^2}} \tag{32.3}$$

PROOF. Assume $m > 1$. Integrate by parts by choosing

$$u = -\frac{1}{m-1} x^{1-m} \quad \text{and} \quad dv = \frac{-2(n-1)x\,dx}{(1+x^2)^n}. \tag{32.4}$$

Then $du = x^{-m}\,dx$ and $v = (1+x^2)^{-n-1}$ and one obtains

$$\int \frac{dx}{x^m(1+x^2)^{n-1}} = -\frac{1}{(m-1)x^{m-1}(1+x^2)^{n-1}} - \frac{2(n-1)}{m-1} \int \frac{1}{x^{m-2}\,(1+x^2)^n}. \tag{32.5}$$

This is very close to the stated formula, In order to obtain the exact statement, subtract the integral $\int x^{2-m}(1+x^2)^{-n}\,dx$ to both sides of (32.5). Then one obtains the desired statement.

The integral relation with $m = 1$ follows from the identity

$$\frac{1}{(1+x^2)^{n-1}} = \frac{1+x^2}{(1+x^2)^n} = \frac{1}{(1+x^2)^n} + \frac{x^2}{(1+x^2)^n}. \tag{32.6}$$

Divide by x and integrate to obtain the result. The integral with $m = n = 1$ is computed using the partial fraction decomposition $\frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}$. □

32.2. Entry 2.148.2.

$$\int \frac{dx}{x^m(1+x^2)} = -\frac{1}{(m-1)x^{m-1}} - \int \frac{dx}{x^{m-2}(1+x^2)} \tag{32.7}$$

PROOF. Integrate the identity

$$\frac{1}{x^m(1+x^2)} = \frac{(1+x^2)-x^2}{x^m(1+x^2)} = \frac{1}{x^m} - \frac{1}{x^{m-2}(1+x^2)}.$$

□

32.3. Entry 2.148.3.

$$\int \frac{dx}{(1+x^2)^n} = \frac{1}{2n-2} \cdot \frac{x}{(1+x^2)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{dx}{(1+x^2)^{n-1}} \tag{32.8}$$

PROOF. Integrate by parts (by the usual procedure) with $u\,dv = 1/(1+x^2)^n$ and $uv = \frac{x}{2(n-1)(1+x^2)^{n-1}}$ gives $v = \frac{x^{2(n-1)}}{(1+x^2)^{n-1}}$ and $u = \frac{1}{2(n-1)x^{2n-3}}$. □

32.4. Entry 2.148.4.

$$\int \frac{dx}{(1+x^2)^n} = \frac{x}{2n-1}\sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\cdots(2n-2k+1)}{2^k(n-1)(n-2)\cdots(n-k)(1+x^2)^{n-k}} + \frac{(2n-3)!!}{2^{n-1}(n-1)!}\arctan x \tag{32.9}$$

Proof. Write the statement as

$$\int \frac{dx}{(1+x^2)^n} = \frac{x}{2n-1}\sum_{k=1}^{n-1} \frac{(2n)!(n-k)!(n-k-1)!}{(2n-2k)!2^{2k}n!(n-1)!(1+x^2)^{n-k}} + \frac{(2n-3)!!}{2^{n-1}(n-1)!}\arctan x.$$

An elementary inductive proof of this formula follows from the recurrence in Entry 2.148.3. The extra term in the sum on the right-hand side is exactly the term in the recurrence that is not under the integral sign. □

33. Section 2.149

33.1. Entry 2.149.1.

$$\int \frac{dx}{x^m(1-x^2)^n} = -\frac{1}{(m-1)x^{m-1}(1-x^2)^{n-1}} + \frac{2n+m-3}{m-1}\int \frac{dx}{x^{m-2}(1-x^2)^n} \tag{33.1}$$

for $m=1$

$$\int \frac{dx}{x(1-x^2)^n} = \frac{1}{2(n-1)(1-x^2)^{n-1}} + \int \frac{dx}{x(1-x^2)^{n-1}} \tag{33.2}$$

for $m=1$ and $n=1$

$$\int \frac{dx}{x(1-x^2)} = \ln \frac{x}{\sqrt{1-x^2}} \tag{33.3}$$

Proof. For $m>1$, use the standard procedure to integrate by parts with

$$u\,dv = \frac{1}{x^m(1-x^2)^n} \quad \text{and} \quad uv = -\frac{1}{(m-1)x^{m-1}(1-x^2)^{n-1}}. \tag{33.4}$$

This gives $v = \dfrac{(1-x^2)^{\frac{1}{2}(m-1)}}{x^{m-1}}$ and from here $v\,du = -\dfrac{2n+m-3}{(m-1)x^{m-2}(1-x^2)^n}$. With these choices, integration by parts gives the result. The other two cases also follow by integration by parts using the standard procedure. □

33.2. Entry 2.149.2.

$$\int \frac{dx}{(1-x^2)^n} = \frac{1}{2n-2}\cdot\frac{x}{(1-x^2)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{dx}{(1-x^2)^{n-1}} \tag{33.5}$$

Proof. Integrate by parts using the standard procedure with $u\,dv = \dfrac{1}{(1-x^2)^n}$ and $uv = \dfrac{x}{2(n-1)(1-x^2)^{n-1}}$. The result comes directly. □

33.3. Entry 2.149.3.

$$\int \frac{dx}{(1-x^2)^n} = \frac{x}{2n-1}\sum_{k=1}^{n-1}\frac{(2n-1)(2n-3)\cdots(2n-2k+1)}{2^k(n-1)(n-2)\cdots(n-k)(1-x^2)^{n-k}} + \frac{(2n-3)!!}{2^n\cdot(n-1)!}\ln\frac{1+x}{1-x} \tag{33.6}$$

Proof. An elementary inductive proof of this formula follows from the recurrence in Entry 2.149.2. The extra term in the sum on the right-hand side is exactly the term in the recurrence that is not under the integral sign. □

34. Section 2.15. Forms containing pairs of binomials $a+bx$ and $\alpha+\beta x$

Notation. $z = a+bx,\ t = \alpha+\beta x,\ \Delta = a\beta - \alpha b$

34.1. Entry 2.151.

$$\int z^n t^m\,dx = \frac{z^{n+1}t^m}{(m+n+1)b} - \frac{m\Delta}{(m+n+1)b}\int z^n t^{m-1}\,dx \tag{34.1}$$

Proof. The usual procedure with $u\,dv = z^n t^m$ and $u\,v = \dfrac{z^{n+1}t^m}{(m+n+1)b}$ produces $v = z^{m+n+1}$ and $du = \dfrac{m\Delta t^{m-1}}{(m+n+1)z^{m+1}}$. Now integrate by parts. □

34.2. Entry 2.152.1.

$$\int \frac{z}{t}\,dx = \frac{bx}{\beta} + \frac{\Delta}{\beta^2}\ln t \tag{34.2}$$

Proof. Write $z = \dfrac{\Delta}{\beta} + \dfrac{bt}{\beta}$ and $dx = \dfrac{dz}{b}$. Integrating yields the result. □

34.3. Entry 2.152.2.

$$\int \frac{t}{z}\,dx = \frac{\beta x}{b} - \frac{\Delta}{b^2}\ln z \tag{34.3}$$

Proof. Let $z = a+bx$. Then $\displaystyle\int \frac{t}{z}\,dx = \int \frac{\alpha+\beta\left(\frac{z-a}{b}\right)}{z}\frac{dx}{b} = \frac{1}{b^2}\int\frac{\beta z-\Delta}{z}\,dz$ yields the evaluation. □

34.4. Entry 2.153.

$$\begin{aligned}\int \frac{t^m\,dx}{z^n} &= \frac{1}{(m-n-1)b}\frac{t^m}{z^{n-1}} - \frac{m\Delta}{(m-n+1)b}\int\frac{t^{m-1}\,dx}{z^n} \\ &= \frac{1}{(n-1)\Delta}\frac{t^{m+1}}{z^{n-1}} - \frac{(m-n+2)\beta}{(n-1)\Delta}\int\frac{t^m\,dx}{z^{n-1}} \\ &= -\frac{1}{(n-1)b}\frac{t^m}{z^{n-1}} + \frac{m\beta}{(n-1)b}\int\frac{t^{m-1}}{z^{n-1}}\,dx\end{aligned} \tag{34.4}$$

PROOF. The first identity comes from the usual procedure by taking $u\,dv = t^m z^{-n}\,dx$ and $uv = \dfrac{t^m}{(m-n-1)bz^{n-1}}$. This gives $v = z^{m-n-1}$ and produces $u = \dfrac{t^m}{(m-n-1)bz^{m-2}}$. Then $v\,du = \dfrac{m\Delta}{(m-n-1)b}t^{m-1}z^{-m} + \dfrac{2}{(m-n-1)}t^m z^{-m}$. Integrate to produce the first relation. To obtain the second identity, observe that

$$\frac{d}{dx}\left(t^{m+1}z^{-n+1}\right) = t^m z^{-n}\left((m+1)\beta z - (n-1)bt\right) \tag{34.5}$$

and expressing t in terms of z as $t = (\beta z - \Delta)/b$ gives

$$\frac{d}{dx}\left(t^{m+1}z^{-n+1}\right) = (m-n+2)\beta t^m z^{-n+1} + (n-1)\Delta t^m z^{-n}. \tag{34.6}$$

Integrate to get the second identity. For the third identity, integrate by parts with $u = t^m$ and $dv = z^{-n}dx$. This gives the result. □

34.5. Entry 2.154.

$$\int \frac{dx}{zt} = \frac{1}{\Delta}\ln\frac{t}{z} \tag{34.7}$$

PROOF. The result follows directly by integrating the partial fraction decomposition $\dfrac{1}{(a+bx)(\alpha+\beta x)} = \dfrac{\beta}{\Delta}\dfrac{1}{\alpha+\beta x} - \dfrac{b}{\Delta}\dfrac{1}{a+bx}$. □

34.6. Entry 2.155.

$$\begin{aligned}\int \frac{dx}{z^n t^m} &= -\frac{1}{(m-1)\Delta}\frac{1}{t^{m-1}z^{n-1}} - \frac{(m+n-2)b}{(m-1)\Delta}\int\frac{dx}{t^{m-1}z^n} \\ &= \frac{1}{(n-1)\Delta}\frac{1}{t^{m-1}z^{n-1}} + \frac{(m+n-2)\beta}{(n-1)\Delta}\int\frac{dx}{t^m z^{n-1}}\end{aligned} \tag{34.8}$$

PROOF. For the first part use the standard procedure to integrate by parts and choose $u\,dx = t^{-m}z^{-n}\,dx$ and $uv = -\dfrac{1}{(m-1)\Delta t^{m-1}z^{n-1}}$. This gives $v = (z/t)^{m-1}$ and then $u = -\dfrac{1}{(m-1)\Delta z^{n+m-2}}$. Integration by parts gives the first formula. The second formula follows from the same procedure by choosing $udv = \dfrac{dx}{z^n t^m}$ and $uv = \dfrac{1}{(n-1)\Delta t^{m-1}z^{n-1}}$. □

34.7. Entry 2.156.

$$\int \frac{x\,dx}{zt} = \frac{1}{\Delta}\left(\frac{a}{b}\ln z - \frac{\alpha}{\beta}\ln t\right) \tag{34.9}$$

PROOF. The result follows directly by integrating the partial fraction decomposition $\dfrac{x}{(a+bx)(\alpha+\beta x)} = \dfrac{a}{\Delta}\dfrac{1}{a+bx} - \dfrac{\alpha}{\Delta}\dfrac{1}{\alpha+\beta x}$. □

Acknowledgments. This work was done as part of the Polymath Jr program, which is partially supported by NSF award DMS-2113535.

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Received 13 09 2021, revised 30 01 2022