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On some improper integrals containing products between the tails of Maclaurin series for the sine and cosine functions

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Abstract. The values of twenty-six improper integrals containing products between the tails of Maclaurin series for the sine and cosine functions are evaluated using elementary methods. The values found extend work where the values for six improper integrals containing the squares of the tails of the Maclaurin series for the sine and cosine functions were recently reported.

1. Introduction

Improper integrals containing the tails of Maclaurin series for the sine and cosine functions in various guises have [**12**, Problem 1914, p. 329], [**11**] and continue to attract attention [**6**, Section B, Problem 3], [**7**, Exercises 2.68 and 2.69, pp. 46–47], [**8, 5**]. Recently the author gave evaluations for a number of improper integrals containing the square of the tail of the Maclaurin series for the sine and cosine functions [**9**, Thms 1, 2, and 3]. Using the method of Fourier transforms the following improper integrals were evaluated:

$$I_n = \int_0^\infty \frac{1}{x^{4n+2}} \left(\cos x - \sum_{k=0}^n \frac{(-1)^k}{(2k)!} x^{2k} \right)^2 = \frac{\pi}{2(4n+1)[(2n)!]^2}, \tag{1.1}$$

$$J_n = \int_0^\infty \frac{1}{x^{4n+4}} \left(\cos x - \sum_{k=0}^n \frac{(-1)^k}{(2k)!} x^{2k} \right)^2 = \frac{\pi}{2(4n+3)[(2n+1)!]^2}, \tag{1.2}$$

$$\Lambda_n = \int_0^\infty \frac{1}{x^{4n+4}} \left(\sin x - \sum_{k=0}^n \frac{(-1)^k}{(2k+1)!} x^{2k+1} \right)^2 = \frac{\pi}{2(4n+3)[(2n+1)!]^2}, \tag{1.3}$$

$$\Pi_n = \int_0^\infty \frac{1}{x^{4n+6}} \left(\sin x - \sum_{k=0}^n \frac{(-1)^k}{(2k+1)!} x^{2k+1} \right)^2 = \frac{\pi}{2(4n+5)[(2n+2)!]^2}. \tag{1.4}$$

Here n is a non-negative integer. Two further improper integrals related to the above family of improper integrals that were left as open problems in [**9**] have recently

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been evaluated [**2**, Thms 6 and 7]. They were found using antidifferentiation and an inductive argument, and relied on making use of the cosine integral function. They are:

$$K_n = \int_0^\infty \frac{1}{x^{4n+3}} \left(\cos x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k} \right)^2 dx = \frac{2}{(4n+2)!} \left(2^{4n} \log(2) - 2^{4n} H_{4n+2} + \sum_{k=n+1}^{2n+1} \binom{4n+2}{2k} H_{2k} \right), \tag{1.5}$$

$$\Psi_n = \int_0^\infty \frac{1}{x^{4n+5}} \left(\sin x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \right)^2 dx = \frac{2}{(4n+4)!} \left(2^{4n+2} \log(2) - 2^{4n+2} H_{4n+4} + \sum_{k=n+1}^{2n+1} \binom{4n+4}{2k+1} H_{2k+1} \right). \tag{1.6}$$

Here n is a non-negative integer while H_n denotes the nth harmonic number defined by $\sum_{k=1}^n \frac{1}{k}$ along with $H_0 \equiv 0$.

Since the publication of [**9**], (1.1) to (1.4) have been evaluated using contour integration methods [**2**] and using an elementary method involving integration by parts and induction [**4**]. The improper integrals Π_0 and Ψ_0 appeared as problems in [**10, 3**] while I_n and J_n have appeared in thinly disguised form as problems in [**1**].

In this short note those results found in [**9, 2**] for improper integrals containing the squares of the tails of the Maclaurin series for the sine and cosine functions will be extended. Using elementary means, results for a number of improper integrals containing products between the tails of Maclaurin series for the sine and cosine functions are given. Though an infinite number of such improper integrals are possible, we give the first twenty-six such integrals down to a level depth equal to two. Here the level depth corresponds to the difference in the number of terms found between the product of the tails appearing in the numerator of the integrand for the improper integrals.

2. Some preliminaries

For $a > 0$ and each non-negative integer n, let

$$C_n(a, x) = \cos(ax) - \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} (ax)^{2k},$$

and

$$S_n(a, x) = \sin(ax) - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} (ax)^{2k+1}.$$

For convenience, when $a = 1$ we shall write

$$C_n(x) \equiv C_n(1, x) = \cos x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k},$$

and

$$S_n(x) \equiv S_n(1,x) = \sin x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

Treating x as fixed, derivatives with respect to a are $C_n'(a,x) = -xS_{n-1}(a,x)$, valid for $n \geqslant 1$, and $S_n'(a,x) = xC_n(a,x)$, valid for $n \geqslant 0$.

For $a > 0$ we introduce the six functions

$$I_n(a) = \int_0^\infty \frac{C_n^2(a,x)}{x^{4n+2}}\, dx = I_n a^{4n+1}, \tag{2.1}$$

$$K_n(a) = \int_0^\infty \frac{C_n^2(a,x)}{x^{4n+3}}\, dx = K_n a^{4n+2}, \tag{2.2}$$

$$J_n(a) = \int_0^\infty \frac{C_n^2(a,x)}{x^{4n+4}}\, dx = J_n a^{4n+3}, \tag{2.3}$$

$$\Lambda_n(a) = \int_0^\infty \frac{S_n^2(a,x)}{x^{4n+4}}\, dx = \Lambda_n a^{4n+3}, \tag{2.4}$$

$$\Psi_n(a) = \int_0^\infty \frac{S_n^2(a,x)}{x^{4n+5}}\, dx = \Psi_n a^{4n+4}, \tag{2.5}$$

$$\Pi_n(a) = \int_0^\infty \frac{S_n^2(a,x)}{x^{4n+6}}\, dx = \Pi_n a^{4n+5}. \tag{2.6}$$

Each of the results appearing on the right can be readily established by enforcing a substitution of $x \mapsto \frac{x}{a}$ followed by observing the results found in (1.1)–(1.6). Note that when $a = 1$ we have $I_n(1) \equiv I_n$, $J_n(1) \equiv J_n$, and so on.

3. Main results

We divide the main results into two types depending on their final form which we list separately. For the first, those improper integrals involving products between the tails that lead to a single term involving the product between a rational part and the number π are given. Down to a level depth of two there are seventeen of these improper integrals in total. They are:

$$\int_0^\infty \frac{C_n(x)S_n(x)}{x^{4n+5}}\,dx = \frac{\pi}{4[(2n+2)!]^2}, \tag{3.1}$$

$$\int_0^\infty \frac{S_n(x)S_{n-1}(x)}{x^{4n+4}}\,dx = \frac{-\pi}{4(n+1)(2n+1)(4n+1)[(2n+1)!]^2}, \tag{3.2}$$

$$\int_0^\infty \frac{C_n(x)S_n(x)}{x^{4n+3}}\,dx = \frac{\pi}{4[(2n+1)!]^2}, \tag{3.3}$$

$$\int_0^\infty \frac{C_{n-1}(x)S_n(x)}{x^{4n+3}}\,dx = \frac{-n\pi}{4(n+1)[(2n+1)!]^2}, \tag{3.4}$$

$$\int_0^\infty \frac{C_n(x)S_{n-1}(x)}{x^{4n+3}}\,dx = \frac{-\pi}{4[(2n+1)!]^2}, \tag{3.5}$$

$$\int_0^\infty \frac{S_n(x)S_{n-2}(x)}{x^{4n+2}}\,dx = \frac{n(2n-1)\pi}{2(n+1)(2n+1)(4n+1)[(2n)!]^2}, \tag{3.6}$$

$$\int_0^\infty \frac{S_n(x)S_{n-1}(x)}{x^{4n+2}}\,dx = \frac{-n\pi}{(2n+1)(4n+1)[(2n)!]^2}, \tag{3.7}$$

$$\int_0^\infty \frac{C_n(x)C_{n-1}(x)}{x^{4n+2}}\,dx = \frac{-n\pi}{2(2n+1)(4n+1)[(2n)!]^2}, \tag{3.8}$$

$$\int_0^\infty \frac{C_n(x)S_{n-1}(x)}{x^{4n+1}}\,dx = \frac{-\pi}{4[(2n)!]^2}, \tag{3.9}$$

$$\int_0^\infty \frac{C_{n-2}(x)S_n(x)}{x^{4n+1}}\,dx = \frac{(2n-1)(n-1)\pi}{4(n+1)(2n+1)[(2n)!]^2}, \tag{3.10}$$

$$\int_0^\infty \frac{C_{n-1}(x)S_n(x)}{x^{4n+1}}\,dx = \frac{-(2n-1)\pi}{4(2n+1)[(2n)!]^2}, \tag{3.11}$$

$$\int_0^\infty \frac{C_n(x)S_{n-2}(x)}{x^{4n+1}}\,dx = \frac{(2n-1)\pi}{4(2n+1)[(2n)!]^2}, \tag{3.12}$$

$$\int_0^\infty \frac{C_n(x)C_{n-1}(x)}{x^{4n}}\,dx = \frac{-(2n-1)\pi}{4n(4n-1)[(2n)!]^2}, \tag{3.13}$$

$$\int_0^\infty \frac{C_n(x)C_{n-2}(x)}{x^{4n}}\,dx = \frac{(n-1)(2n-1)\pi}{2n(2n+1)(4n-1)[(2n-1)!]^2}, \tag{3.14}$$

$$\int_0^\infty \frac{S_n(x)S_{n-2}(x)}{x^{4n}}\,dx = \frac{(n-1)(2n-1)\pi}{2n(2n+1)(4n-1)[(2n-1)!]^2}, \tag{3.15}$$

$$\int_0^\infty \frac{C_n(x)S_{n-2}(x)}{x^{4n-1}}\,dx = \frac{(n-1)\pi}{4n[(2n-1)!]^2}, \tag{3.16}$$

$$\int_0^\infty \frac{C_{n-2}(x)S_n(x)}{x^{4n-1}}\,dx = \frac{(n-1)(2n-3)\pi}{4n(2n+1)[(2n-1)!]^2}. \tag{3.17}$$

The second set of results are those improper integrals involving products between the tails that lead to two terms on evaluation. The first term involves the product between a rational part and log(2), the second term is rational only. Down to a level depth of two there are nine of these improper integrals in total. They are:

$$\int_0^\infty \frac{C_n(x)S_n(x)}{x^{4n+4}}\,dx = \frac{1}{(4n+3)!}\Big\{2^{4n+2}\log(2) - 2^{4n+2}H_{4n+4} + \sum_{k=n+1}^{2n+1}\binom{4n+4}{2k+1}H_{2k+1}\Big\}, \tag{3.18}$$

$$\int_0^\infty \frac{S_n(x)S_{n-1}(x)}{x^{4n+3}}\,dx = \frac{-1}{(4n+2)!}\Big\{2^{4n+1}\log(2) - 2^{4n+1}\left(2H_{4n+4} - H_{4n+2}\right) + \sum_{k=n+1}^{2n+1}\left[\binom{4n+4}{2k+1}H_{2k+1} - 2\binom{4n+2}{2k}H_{2k}\right]\Big\}, \tag{3.19}$$

$$\int_0^\infty \frac{C_n(x)S_{n-1}(x)}{x^{4n+2}}\,dx = \frac{-1}{(4n+1)!}\Big\{2^{4n}\log(2) - 2^{4n}H_{4n+2} + \sum_{k=n+1}^{2n+1}\binom{4n+2}{2k}H_{2k}\Big\}, \tag{3.20}$$

$$\int_0^\infty \frac{C_{n-1}(x)S_n(x)}{x^{4n+2}}\,dx = \frac{-1}{(4n+1)!}\Big\{2^{4n}\log(2) - 2^{4n}\left(4H_{4n+4} - 3H_{4n+2}\right) + \sum_{k=n+1}^{2n+1}\left[\binom{4n+4}{2k+1}H_{2k+1} - 3\binom{4n+2}{2k}H_{2k}\right]\Big\}, \tag{3.21}$$

$$\int_0^\infty \frac{C_n(x)C_{n-1}(x)}{x^{4n+1}}\,dx = \frac{-1}{(4n)!}\Big\{2^{4n-1}\log(2) - 2^{4n-1}\left(2H_{4n+2} - H_{4n}\right) + \sum_{k=n+1}^{2n+1}\binom{4n+2}{2k}H_{2k} - \sum_{k=n}^{2n-1}2\binom{4n}{2k+1}H_{2k+1}\Big\}, \tag{3.22}$$

$$\int_0^\infty \frac{S_n(x)S_{n-2}(x)}{x^{4n+1}}\,dx = \frac{1}{(4n)!}\Big\{2^{4n-1}\log(2) - 2^{4n-1}\left(H_{4n} - 8H_{4n+2} + 8H_{4n+4}\right) + \sum_{k=n+1}^{2n+1}\left[\binom{4n+4}{2k+1}H_{2k+1} - 4\binom{4n+2}{2k}H_{2k}\right] + \sum_{k=n+1}^{2n+1}2\binom{4n}{2k+1}H_{2k+1}\Big\}, \tag{3.23}$$

$$\int_0^\infty \frac{C_n(x)S_{n-2}(x)}{x^{4n}}\,dx = \frac{1}{(4n-1)!}\Big\{2^{4n-2}\log(2) + 2^{4n-2}\left(3H_{4n} - 4H_{4n+2}\right) + \sum_{k=n+1}^{2n+1}\binom{4n+2}{2k}H_{2k} - \sum_{k=n}^{2n-1}3\binom{4n}{2k+1}H_{2k+1}\Big\}, \tag{3.24}$$

$$\int_0^\infty \frac{C_{n-2}(x)S_n(x)}{x^{4n}}\,dx = \frac{1}{(4n-1)!}\Big\{2^{4n-2}\log(2) - 2^{4n-2}\left(5H_{4n} - 20H_{4n+2} + 16H_{4n+4}\right) + \sum_{k=n}^{2n-1}5\binom{4n}{2k+1}H_{2k+1} + \sum_{k=n+1}^{2n+1}\left[\binom{4n+4}{2k+1}H_{2k+1} - 5\binom{4n+2}{2k}H_{2k}\right]\Big\}, \tag{3.25}$$

$$\int_0^\infty \frac{C_n(x)C_{n-2}(x)}{x^{4n-1}}\,dx = \frac{1}{(4n-2)!}\Big\{2^{4n-3}\log(2) - 2^{4n-3}\left(H_{4n-2} - 8H_{4n} + 8H_{4n+2}\right) + \sum_{k=n+1}^{2n+1}\binom{4n+2}{2k}H_{2k} + \sum_{k=n}^{2n-1}\left[2\binom{4n-2}{2k}H_{2k} - 4\binom{4n}{2k+1}H_{2k+1}\right]\Big\}. \tag{3.26}$$

In all cases n is a non-negative integer that starts at a value corresponding to the level depth found between the differences in the number of terms in the product of the tails.

MAPLE can evaluate all twenty-six improper integrals for small values of n up to three or four, depending on the level depth of the integral. Beyond four the computations slow considerably, taking either a long time or not being able to complete. Contrasting this with MAXIMA, it is able to evaluate the seventeen improper integrals appearing in (3.1)–(3.17) to very high orders of n very quickly while the remaining nine improper integrals that contain the $\log(2)$ term when evaluated cannot even be found for the lowest order of n.

As the proofs of (3.1)–(3.26) are straightforward and rely only on elementary methods, for a basic idea of the process involved we outline the method of proof leading to (3.1). From this the remainder of the results can then be found in a similar manner, building upon previously found results where necessary as one goes along to deeper level depths. Extensions beyond level depths of two can readily be found if one so desires.

PROOF. Differentiating (2.6) with respect to a before applying $S_n'(a,x) = xC_n(a,x)$ we find

$$\int_0^\infty \frac{C_n(a,x)S_n(a,x)}{x^{4n+5}}\,dx = \frac{1}{2}\Psi_n'(a) = \frac{1}{2}(4n+5)\Psi_n a^{4n+4}.$$

Setting $a = 1$ and substituting for the value of Ψ_n found in (1.6), the result then follows. □

REMARK 3.1. As the difference in the number of terms found in the product of the tails of (3.1) is zero, the level depth for the improper integral is zero and means $n = 0, 1, 2, \ldots$. On the other hand, as the difference in the number of terms found in the product of the tails of (3.17) is two, the level depth for the improper integral is two and means $n = 2, 3, 4 \ldots$.

REMARK 3.2. The results (3.1) and (3.5) have been given by Gordon in [**2**]. They were found as a by-product of the contour integration method that was used there to evaluate (1.1)–(1.4) and was the source of inspiration for this note.

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