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A four parameter integral identity and a few consequences

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Abstract. The identity

$$\int_0^\infty e^{-\alpha x}\frac{e^{-a\sqrt{x^2+2\beta x+b^2}}}{\sqrt{x^2+2\beta x+b^2}} = \int_0^\infty e^{-\beta x}\frac{e^{-b\sqrt{x^2+2\alpha x+a^2}}}{\sqrt{x^2+2\alpha x+a^2}}$$

is derived, applied to the Struve function $\mathbf{H_0}$ and used to deduce the reduction formula

$$\int_0^\infty \frac{F(\sqrt{x^2+2\beta x+b^2}+x)}{\sqrt{x^2+2\beta x+b^2}}dx = \int_0^\infty f(t)e^{\beta t}E_1[(\beta+b)t]dt.$$

where F is arbitrary.

1. Introduction

Integrals rational in two different quadratic surds, as shown by Legendre [**4, 5**], can be evaluated, or at least reduced to three standard forms, by elliptic substitutions. An attempt to investigate what happens in non-rational cases led, in a previous study (Glasser, unpublished), to a curious connection between a Jacobian elliptic function and a modified Bessel function. The examination of a second example has resulted in the striking integral identity

$$\int_0^\infty \frac{dx}{\sqrt{(x+\beta)(x+b)}}e^{-\sqrt{\alpha(x+\beta)}-\sqrt{a(x+b)}} = \int_0^\infty \frac{dx}{\sqrt{(x+a)(x+\alpha)}}e^{-\sqrt{b(x+a)}-\sqrt{\beta(x+\alpha)}} \tag{1.1}$$

involving four parameters restricted only to the convergence of the integrals. In this note (1.1) will be proven and several examples and consequences will be presented.

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2. Derivation

We begin with the integral

$$I(\alpha,\beta,a,b)=\int_0^\infty \frac{x\; e^{-(\alpha\sqrt{x^2+\beta^2}+a\sqrt{x^2+b^2})}}{\sqrt{(x^2+\beta^2)(x^2+b^2)}}dx \tag{2.1}$$

From [**3**] one has

$$\int_1^\infty e^{-ax}J_0(\beta\sqrt{x^2-1})dx=\sqrt{\frac{2}{\pi}}(a^2+\beta^2)^{-1/4}K_{1/2}(\sqrt{a^2+\beta^2}), \tag{2.2}$$

which after a simple change of variable and noting the exponential form of $K_{1/2}$, can be rearranged to read

$$\frac{e^{-u\sqrt{A^2+x^2}}}{\sqrt{A^2+x^2}}=\int_0^\infty \frac{tJ_0(xt)}{\sqrt{t^2+u^2}}e^{-A\sqrt{t^2+u^2}}dx. \tag{2.3}$$

Inserting this twice into (2.1) produces a triple integral of which one is

$$\int_0^\infty xJ_0(tx)J_0(t'x)dx=\delta(t-t'). \tag{2.4}$$

Therefore, one integration remains, which is $I(\beta,\alpha,b,a)$. The replacement of x by $\sqrt{x}$ then gives (1.1).

3. Discussion

The integrals in (1.1) can be rewritten in various ways. With $u = x+\beta$ the left-hand side becomes

$$\int_\beta^\infty \frac{e^{-\sqrt{\alpha u}}}{\sqrt{u}}\frac{e^{-\sqrt{a(u+b-\beta)}}}{\sqrt{u+b-\beta}}du=\int_{\sqrt{\beta}}^\infty dx e^{-x\sqrt{\alpha}}\frac{e^{-\sqrt{a(x^2+b-\beta)}}}{\sqrt{x^2+b-\beta}}dx. \tag{3.1}$$

Then by introducing $x-\sqrt{\beta}$ as the new integration variable and canceling the common exponential pre factor which occurs after a similar manipulation of the right-hand side, we obtain

$$\int_0^\infty e^{-\alpha x}\frac{e^{-a\sqrt{x^2+2\beta x+b^2}}}{\sqrt{x^2+2\beta x+b^2}}dx=\int_0^\infty e^{-\beta x}\frac{e^{-b\sqrt{x^2+2\alpha x+a^2}}}{\sqrt{x^2+2\alpha x+a^2}}dx. \tag{3.2}$$

For example, setting $b=\alpha=0$ in (3.2) gives

$$\int_0^\infty \frac{e^{-a\sqrt{x(x+2\beta)}}}{\sqrt{x(x+2\beta)}}dx=\int_0^\infty \frac{e^{-\beta x}}{\sqrt{x^2+a^2}}dx \tag{3.3}$$

where the right-hand side is a tabulated Laplace transform [**2**]. Hence, after a simple manipulation we have the representation for the Struve function

$$\int_0^\infty \frac{e^{-\alpha x\sqrt{x^2+1}}}{\sqrt{x^2+1}}dx=\int_0^\infty e^{-\frac{1}{2}\alpha\sinh 2\theta}d\theta=\frac{\pi}{4}[\mathbf{H}_0(\tfrac{1}{2}\alpha)-Y_0(\tfrac{1}{2}\alpha)]. \tag{3.4}$$

By setting $a = \alpha$ in (3.2) one finds

$$\int_0^\infty e^{-a(x+\sqrt{x^2+2\beta x+b^2})} \frac{dx}{\sqrt{x^2+2\beta x+b^2}} = e^{a\beta} E_1\left(a(b+\beta)\right), \tag{3.5}$$

where $E_1(t)$ denotes the exponential integral function [**1**]. In addition, (3.5) leads to the identity, for any function f with Laplace transform F

$$\int_0^\infty \frac{F[x+\sqrt{x^2+2\beta x+b^2}]}{\sqrt{x^2+2\beta x+b^2}} dx = \int_0^\infty f(t) e^{\beta t} E_1[(\beta+b)t] dt. \tag{3.6}$$

Thus, e.g. for $0 < \nu < 1$

$$\int_0^\infty \frac{dx}{(\sqrt{x^2+2\beta x+1}+x)^\nu \sqrt{x^2+2\beta x+1}} = \frac{\pi}{\beta^\nu \sin \pi\nu} + \frac{(\beta+1)^{-1}}{\nu-1}\, {}_2F_1(1,1;2-\nu;\frac{1}{\beta+1}), \tag{3.7}$$

and, more generally, for $0 < b < c$,

$$\int_0^{(c^2-b^2)/2(\beta+c)} \frac{dx}{\sqrt{x^2+2\beta x+b^2}} F[x+\sqrt{x^2+2\beta x+b^2}] = \int_0^{(c-b)/(\beta+b)} \frac{dt}{t+1} F[(b+\beta)t+b]. \tag{3.8}$$

To conclude, we examine the form (2.1) takes when the surds are rationalized by the elliptic substitution [**5**] $x = sc(u,k)$, where $k = (1-\beta^2/b^2)^{1/2}$. Then

$$\sqrt{1+x^2} = nc(u,k) \text{ and } \sqrt{1+k'^2x^2} = dc(u,k) \tag{3.9}$$

where $k' = \beta/b$. In this case

$$I(\alpha,\beta,a,b) = k' \int_0^{K(k)} \frac{sn(u,k)}{cn(u,k)} \exp\left[-\frac{\alpha\beta}{cn(u,k)} - ab\frac{dn(u,k)}{cn(u,k)}\right] du \tag{3.10}$$

which is therefore invariant under the replacement $k' \to l' = \alpha/a$.

References

[1] M. Abramowitz and I. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables.* Dover, New York, 1972.

[2] A. Erdélyi. *Tables of Integral Transforms*, volume I. McGraw-Hill, New York, 1st edition, 1954.

[3] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products.* Edited by D. Zwillinger and V. Moll. Academic Press, New York, 8th edition, 2015.

[4] A. M. Legendre. *Traite des Fonctions Elliptiques et des Integrales Euleriennes; three volumes.* Firmin Didot Freres, Paris, 1825, 1826, 1828-1832.

[5] H. P. McKean and V. Moll. *Elliptic Curves: Function Theory, Geometry, Arithmetic.* Cambridge University Press, New York, 1997.

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