Symmetric tensor powers of graphs

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ABSTRACT. We introduce the symmetric tensor power of graphs and explore the fundamental properties of this operation. A wide range of intriguing phenomena occur when one considers symmetric tensor powers of familiar graphs. We conclude with a host of open questions that we hope will spur future research.

1. Introduction

A great variety of graph products, and hence graph powers, exist in the literature [12]. In this paper we introduce the symmetric tensor power of graphs, a novel graph power that displays a variety of intriguing phenomena. Although the resulting graphs often bear surprising features, the approach is well-motivated algebraically. For example, the naturalness of the symmetric tensor power is illustrated by its compatibility with graph spectra.

It is surprising that symmetric tensor powers of graphs do not seem to have been explored before since symmetric tensor powers have a long and storied history in mathematics and allied fields. For example, they appear in multilinear algebra [10], representation theory [7], abstract algebra [3, 5], quantum algebra [15], and, more recently, in function-related operator theory [8]. In quantum mechanics, the symmetric Fock space, which models the states of many-body bosonic systems, is spanned by the symmetric tensors of all ranks [17]; noncommutative q-analogues have also been considered [6]. The Hamiltonians of such systems are symmetric tensor products of certain Hermitian operators [2, 4, 9, 13, 14]. The symmetric Fock space even finds use in theoretical computer science [1].

The structure of this paper is as follows. Section 2 covers some linear-algebraic preliminaries before symmetric tensor powers of graphs are defined in Section 3 (with some computational material deferred until Appendix A). The connection between this

operation and graph spectra is considered in Section 4. We make several combinatorial observations in Section 5 and study in Section 6 several curious phenomena that arise for familiar graphs. Section 7 concerns the Wiener index of certain graphs. We conclude with a host of open questions about symmetric tensor powers of graphs, which indicates that the subject is fertile ground for future exploration, in Section 8.

2. Linear-algebraic preliminaries

Since this paper is meant to be self-contained, we begin with some linear-algebraic prerequisites. Readers who are familiar with symmetric tensor powers as a general abstract construction may proceed to Section 3. In what follows, we denote by M_n the set of real $n \times n$ matrices, $\mathbb{N} := \{1, 2, \ldots\}$ the set of natural numbers, $[n] := \{1, 2, \ldots, n\}$, and $|X|$ the cardinality of a set X.

Let $\mathcal{V}$ denote a real inner-product space with orthonormal basis $v_1, v_2, \ldots, v_n$. For $k \in \mathbb{N}$, the kth *tensor power* of $\mathcal{V}$ is the n^k-dimensional inner-product space $\mathcal{V}^{\otimes k}$ spanned by the *simple tensors* $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$, in which $(i_1, i_2, \ldots, i_k) \in [n]^k$, and endowed with the inner product that linearly extends

$$(2.1) \quad \langle v_{i_1} \otimes v_{i_2} \otimes \cdots v_{i_k}, \, v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_k} \rangle := \langle v_{i_1}, v_{j_1} \rangle \langle v_{i_2}, v_{j_2} \rangle \cdots \langle v_{i_k}, v_{j_k} \rangle.$$

In particular, the $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$ comprise an orthonormal basis of $\mathcal{V}^{\otimes k}$.

The symmetric group $\mathfrak{S}_k$ acts on $[n]^k$ by permutation. Let $\mathrm{Orb}(\mathbf{i})$ denote the orbit of $\mathbf{i} = (i_1, i_2, \ldots, i_k)$ under this action; that is, $\mathbf{j} \in \mathrm{Orb}(\mathbf{i})$ if and only if $\mathbf{j}$ is a permutation of $\mathbf{i}$. We let $\mathfrak{S}_k$ permute simple tensors in the analogous manner.

Let $\mathcal{V}^{\odot k}$ denote the subspace of $\mathcal{V}^{\otimes k}$ spanned by the *symmetric tensors*

$$(2.2) \quad v_{i_1} \odot v_{i_2} \odot \cdots \odot v_{i_k} := \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}} \otimes \cdots \otimes v_{i_{\sigma(k)}}.$$

Symmetric tensors are invariant under the action of $\mathfrak{S}_k$, so there is a representative k-tuple for (2.2) such that $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$. We have $\dim \mathcal{V}^{\odot k} = \binom{n+k-1}{k}$.

EXAMPLE 2.1. Let $n = k = 2$. Then $\mathcal{V}$ has orthonormal basis v_1, v_2 and $\mathcal{V}^{\otimes 2}$ is 4-dimensional with orthonormal basis $v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2$. The $\mathfrak{S}_2$-orbits in $\mathcal{V}^{\otimes 2}$ are $\mathrm{Orb}(v_1 \otimes v_1) = \{v_1 \otimes v_1\}$, $\mathrm{Orb}(v_1 \otimes v_2) = \{v_1 \otimes v_2, v_2 \otimes v_1\}$, and $\mathrm{Orb}(v_2 \otimes v_2) = \{v_2 \otimes v_2\}$. Thus, $v_1 \otimes v_1, \frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1), v_2 \odot v_2$ comprise a basis for $\mathcal{V}^{\odot 2}$, which is $\binom{2+2-1}{2} = \binom{3}{2} = 3$ dimensional.

For $\mathbf{i} = (i_1, i_2, \ldots, i_k) \in [n]^k$, let $\mathbf{m}(\mathbf{i}) = (m_1, m_2, \ldots, m_n)$, in which each m_ℓ is the number of occurrences of ℓ in $\mathbf{i}$. For example, if $\mathbf{i} = (1, 3, 2, 4, 3, 1) \in [5]^6$, then $\mathbf{m}(\mathbf{i}) = (2, 1, 2, 1, 0)$. We may write $\mathbf{m}$ if the dependence on $\mathbf{i}$ is clear. There are

$$\binom{k}{\mathbf{m}} := \frac{k!}{m_1! m_2! \cdots m_n!}$$

elements of $[n]^k$ that give rise to the same symmetric tensor $v_{i_1} \odot v_{i_2} \odot \cdots \odot v_{i_k}$. The quantity above equals the cardinality of $\mathrm{Orb}(\mathbf{i})$.

LEMMA 2.1. *Fix $n, k \in \mathbb{N}$ and let $N = \binom{n+k-1}{k}$. The N vectors*

$$(2.3) \qquad u_{\mathbf{i}} := \binom{k}{\mathbf{m}(\mathbf{i})}^{1/2} v_{i_1} \odot v_{i_2} \cdots \odot v_{i_k},$$

where $\mathbf{i} = (i_1, i_2, \ldots, i_k) \in [n]^k$ is nondecreasing, form an orthonormal basis for $\mathcal{V}^{\odot k}$.

PROOF. If $\mathbf{i}, \mathbf{j} \in [n]^k$ are nondecreasing and distinct, they are in different $\mathfrak{S}_k$ orbits in $[n]^k$, so (2.1) ensures that $u_{\mathbf{i}}$ and $u_{\mathbf{j}}$ are orthogonal. Since $\dim \mathcal{V}^{\odot k} = N$, we need only show that each $u_{\mathbf{i}}$ is a unit vector. To this end, observe that for each of the N nondecreasing $\mathbf{i} = (i_1, i_2, \ldots, i_k) \in [n]^k$,

$$
\begin{aligned}
\left\langle \bigodot_{\ell=1}^{k} \mathbf{i}_\ell, \bigodot_{\ell=1}^{k} \mathbf{i}_\ell \right\rangle &= \left\langle \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \bigotimes_{\ell=1}^{k} \mathbf{i}_{\sigma(\ell)}, \frac{1}{k!} \sum_{\tau \in \mathfrak{S}_k} \bigotimes_{\ell=1}^{k} \mathbf{i}_{\tau(\ell)} \right\rangle \\
&= \frac{1}{(k!)^2} \sum_{\sigma \in \mathfrak{S}_k} \left(\sum_{\tau \in \mathfrak{S}_k} \prod_{\ell=1}^{k} \left\langle \mathbf{i}_{\sigma(\ell)}, \mathbf{i}_{\tau(\ell)} \right\rangle \right) \\
&= \frac{1}{(k!)^2} \sum_{\sigma \in \mathfrak{S}_k} \left| \left\{ \tau \in \mathfrak{S}_k : \mathbf{i}_{\tau(\ell)} = \mathbf{i}_{\sigma(\ell)} \text{ for all } \ell \in [n] \right\} \right| \\
&= \frac{1}{(k!)^2} \sum_{\sigma \in \mathfrak{S}_k} m_1! m_2! \cdots m_n! = \frac{m_1! m_2! \cdots m_n!}{k!} = \binom{k}{\mathbf{m}(\mathbf{i})}^{-1}. \quad \square
\end{aligned}
$$

Let A be a linear operator on $\mathcal{V}$. For $i, j \in [n]$, the (i, j) matrix entry of A with respect to the orthonormal basis $v_1, v_2, \ldots, v_n$ is $[A]_{i,j} = \langle Av_j, v_i \rangle$. Define

$$A^{\odot k}(v_{i_1} \odot v_{i_2} \odot \cdots \odot v_{i_k}) := \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (Av_{i_{\sigma(1)}}) \otimes (Av_{i_{\sigma(2)}}) \otimes \cdots \otimes (Av_{i_{\sigma(k)}})$$

and extend this by linearity to $\mathcal{V}^{\odot k}$. The right side above is permutation invariant, so $A^{\odot k}$ is a linear transformation from $\mathcal{V}^{\odot k}$ to itself.

LEMMA 2.2. *The matrix entries of $A^{\odot k}$ with respect to the orthonormal basis (2.3) are*

$$[A^{\odot k}]_{\mathbf{i}, \mathbf{j}} = \frac{1}{\sqrt{\binom{k}{\mathbf{m}(\mathbf{i})} \binom{k}{\mathbf{m}(\mathbf{j})}}} \sum_{\substack{\mathbf{p} \in \mathrm{Orb}(\mathbf{i}) \\ \mathbf{q} \in \mathrm{Orb}(\mathbf{j})}} \prod_{\ell=1}^{k} [A]_{p_\ell, q_\ell},$$

in which $\mathbf{i}, \mathbf{j}$ run over the $N = \binom{n+k-1}{k}$ nondecreasing elements of $[n]^k$ and $\mathbf{p} = (p_1, p_2, \ldots, p_k)$, $\mathbf{q} = (q_1, q_2, \ldots, q_k)$ belong to the $\mathfrak{S}_k$ orbits of $\mathbf{i}$ and $\mathbf{j}$ in $[n]^k$, respectively. In particular, $A^{\odot k}$ is real symmetric if A is real symmetric.

PROOF. A computation using stabilizers gives

$$
\begin{aligned}
[A^{\odot k}]_{\mathbf{i},\mathbf{j}} &= \left\langle A^{\odot k} u_{\mathbf{j}}, u_{\mathbf{i}} \right\rangle \\
&= \sqrt{\binom{k}{\mathbf{m}(\mathbf{i})}\binom{k}{\mathbf{m}(\mathbf{j})}} \left\langle A^{\odot k}(v_{j_1} \odot v_{j_2} \cdot \cdots \odot v_{j_k}), v_{i_1} \odot v_{i_2} \cdot \cdots \odot v_{i_k} \right\rangle \\
&= \sqrt{\binom{k}{\mathbf{m}(\mathbf{i})}\binom{k}{\mathbf{m}(\mathbf{j})}} \sum_{\sigma,\tau \in \mathfrak{S}_k} \left\langle \bigotimes_{\ell=1}^{k} A v_{j_{\sigma(\ell)}}, \bigotimes_{\ell=1}^{k} v_{i_{\tau(\ell)}} \right\rangle \\
&= \sqrt{\binom{k}{\mathbf{m}(\mathbf{i})}\binom{k}{\mathbf{m}(\mathbf{j})}} \sum_{\sigma,\tau \in \mathfrak{S}_k} \prod_{\ell=1}^{k} \langle A v_{j_{\sigma(\ell)}}, v_{i_{\tau(\ell)}} \rangle \\
&= \sqrt{\binom{k}{\mathbf{m}(\mathbf{i})}\binom{k}{\mathbf{m}(\mathbf{j})}} \sum_{\sigma,\tau \in \mathfrak{S}_k} \prod_{\ell=1}^{k} [A]_{j_{\sigma(\ell)},i_{\tau(\ell)}} \\
&= \sqrt{\binom{k}{\mathbf{m}(\mathbf{i})}\binom{k}{\mathbf{m}(\mathbf{j})}} \sum_{\substack{\mathbf{p}\in\mathrm{Orb}(\mathbf{i})\\\mathbf{q}\in\mathrm{Orb}(\mathbf{j})}} \binom{k}{\mathbf{m}(\mathbf{i})}^{-1}\binom{k}{\mathbf{m}(\mathbf{j})}^{-1} \prod_{\ell=1}^{k} [A]_{p_\ell,q_\ell} \\
&= \frac{1}{\sqrt{\binom{k}{\mathbf{m}(\mathbf{i})}\binom{k}{\mathbf{m}(\mathbf{j})}}} \sum_{\substack{\mathbf{p}\in\mathrm{Orb}(\mathbf{i})\\\mathbf{q}\in\mathrm{Orb}(\mathbf{j})}} \prod_{\ell=1}^{k} [A]_{p_\ell,q_\ell}. \qquad \square
\end{aligned}
$$

We often identify $\mathcal{V}$ with $\mathbb{R}^n$ and $v_1, v_2, \ldots, v_n$ with the standard basis. Thus, we identify the linear operator A on $\mathcal{V}$ with its matrix representation $[a_{ij}] \in \mathrm{M}_n$ with respect to $v_1, v_2, \ldots, v_n$. Then $A^{\odot k}$ acts on the N-dimensional space $\mathcal{V}^{\odot k}$, in which $N = \binom{n+k-1}{k}$. We identify $A^{\odot k} \in \mathrm{M}_N$ with its matrix representation with respect to the orthonormal basis defined in Lemma 2.1.

EXAMPLE 2.2. Let $n = k = 2$ and $A = [a_{ij}] \in \mathrm{M}_2$. Then $A^{\odot 2} \in \mathrm{M}_3$ since $N = \binom{2+2-1}{2} = \binom{3}{2} = 3$. Proposition 2.2 and Table 1 produce

$$
(2.4) \qquad A^{\odot 2} = \begin{bmatrix} a_{11}^2 & a_{12}^2 & \sqrt{2}a_{11}a_{12} \\ a_{21}^2 & a_{22}^2 & \sqrt{2}a_{21}a_{22} \\ \sqrt{2}a_{11}a_{12} & \sqrt{2}a_{21}a_{22} & a_{11}a_{22} + a_{12}a_{21} \end{bmatrix}.
$$

For example,

$$
[A^{\odot 2}]_{1,3} = \frac{1}{\sqrt{\frac{2!}{2!0!} \cdot \frac{2!}{1!1!}}} \sum_{\substack{\mathbf{p}\in\{(1,1)\}\\\mathbf{q}\in\{(1,2),(2,1)\}}} [A]_{p_1,q_1}[A]_{p_2,q_2} = \frac{1}{\sqrt{2}}(a_{11}a_{12} + a_{12}a_{11}) = \sqrt{2}(a_{11}a_{12}).
$$

index	$\mathbf{i}$	$\mathbf{m(i)}$	$\binom{k}{\mathbf{m(i)}}$	Orb($\mathbf{i}$)
1	$(1,1)$	$(2,0)$	$\frac{2!}{2!0!}=1$	$\{(1,1)\}$
2	$(2,2)$	$(0,2)$	$\frac{2!}{0!2!}=1$	$\{(2,2)\}$
3	$(1,2)$	$(1,1)$	$\frac{2!}{1!1!}=2$	$\{(1,2),(2,1)\}$

TABLE 1. For $n=k=2$, there are $N=3$ nondecreasing elements of $[n]^k$.

index	$\mathbf{i}$	$\mathbf{m(i)}$	$\binom{k}{\mathbf{m(i)}}$	Orb($\mathbf{i}$)
1	$(1,1,1)$	$(3,0)$	$\frac{3!}{3!0!}=1$	$\{(1,1,1)\}$
2	$(2,2,2)$	$(0,3)$	$\frac{3!}{0!3!}=1$	$\{(2,2,2)\}$
3	$(1,1,2)$	$(2,1)$	$\frac{3!}{2!1!}=3$	$\{(1,1,2),(1,2,1),(2,1,1)\}$
4	$(1,2,2)$	$(1,2)$	$\frac{3!}{1!2!}=3$	$\{(1,2,2),(2,1,2),(2,2,1)\}$

TABLE 2. For $n=2$ and $k=3$, there are $N=4$ nondecreasing elements of $[n]^k$.

EXAMPLE 2.3. Let $n=2$, $k=3$, and $A=[a_{ij}]\in \mathrm{M}_3$. Then $A^{\odot 3}\in \mathrm{M}_4$ since $N=\binom{3+2-1}{3}=\binom{4}{3}=4$. Proposition 2.2 and Table 2 produce

$$(2.5) \quad A^{\odot 3}=\begin{bmatrix} a_{11}^3 & a_{12}^3 & \sqrt{3}a_{11}^2 a_{12} & \sqrt{3}a_{11}a_{12}^2 \\ a_{21}^3 & a_{22}^3 & \sqrt{3}a_{21}^2 a_{22} & \sqrt{3}a_{21}a_{22}^2 \\ \sqrt{3}a_{11}^2 a_{21} & \sqrt{3}a_{12}^2 a_{22} & 2a_{11}a_{12}a_{21}+a_{11}^2 a_{22} & a_{12}^2 a_{21}+2a_{11}a_{12}a_{22} \\ \sqrt{3}a_{11}a_{21}^2 & \sqrt{3}a_{12}a_{22}^2 & a_{12}a_{21}^2+2a_{11}a_{21}a_{22} & 2a_{12}a_{21}a_{22}+a_{11}a_{22}^2 \end{bmatrix}.$$

For example,

$$[A^{\odot 3}]_{3,4}=\frac{1}{\sqrt{\frac{3!}{2!1!}\cdot\frac{3!}{1!2!}}}\sum_{\substack{\mathbf{p}\in\{(1,1,2),(1,2,1),(2,1,1)\}\\ \mathbf{q}\in\{(1,2,2),(2,1,2),(2,2,1)\}}}[A]_{p_1,q_1}[A]_{p_2,q_2}[A]_{p_3,q_3}$$

$$=\frac{1}{3}(a_{11}a_{12}a_{22}+a_{12}a_{11}a_{22}+a_{12}a_{12}a_{21}+a_{11}a_{22}a_{12}+a_{12}a_{21}a_{12}$$

$$+a_{12}a_{22}a_{11}+a_{21}a_{12}a_{12}+a_{22}a_{11}a_{12}+a_{22}a_{12}a_{11})$$

$$=a_{12}^2 a_{21}+2a_{11}a_{12}a_{22}.$$

index	$\mathbf{i}$	$\mathbf{m(i)}$	$\binom{k}{\mathbf{m(i)}}$	$\mathrm{Orb}(\mathbf{i})$
1	$(1,1)$	$(2,0,0)$	$\frac{2!}{2!0!0!}=1$	$\{(1,1)\}$
2	$(2,2)$	$(0,2,0)$	$\frac{2!}{0!2!0!}=1$	$\{(2,2)\}$
3	$(3,3)$	$(0,0,2)$	$\frac{2!}{0!0!2!}=1$	$\{(3,3)\}$
4	$(1,2)$	$(1,1,0)$	$\frac{2!}{1!1!0!}=2$	$\{(1,2),(2,1)\}$
5	$(1,3)$	$(1,0,1)$	$\frac{2!}{1!0!1!}=2$	$\{(1,3),(3,1)\}$
6	$(2,3)$	$(0,1,1)$	$\frac{2!}{0!1!1!}=2$	$\{(2,3),(3,2)\}$

TABLE 3. For $n=3$ and $k=2$, there are $N=6$ nondecreasing elements of $[n]^k$.

EXAMPLE 2.4. Let $n=3$, $k=2$, and $A=[a_{ij}] \in \mathrm{M}_3$. Then $A^{\odot 3} \in \mathrm{M}_6$ since $N=\binom{3+2-1}{2}=\binom{4}{2}=6$. Proposition 2.2 and Table 3 give

$$(2.6)$$

$$A^{\odot 3} = \begin{bmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 & \sqrt{2}a_{11}a_{12} & \sqrt{2}a_{11}a_{13} & \sqrt{2}a_{12}a_{13} \\ a_{21}^2 & a_{22}^2 & a_{23}^2 & \sqrt{2}a_{21}a_{22} & \sqrt{2}a_{21}a_{23} & \sqrt{2}a_{22}a_{23} \\ a_{31}^2 & a_{32}^2 & a_{33}^2 & \sqrt{2}a_{31}a_{32} & \sqrt{2}a_{31}a_{33} & \sqrt{2}a_{32}a_{33} \\ \sqrt{2}a_{11}a_{21} & \sqrt{2}a_{12}a_{22} & \sqrt{2}a_{13}a_{23} & a_{12}a_{21}+a_{11}a_{22} & a_{13}a_{21}+a_{11}a_{23} & a_{13}a_{22}+a_{12}a_{23} \\ \sqrt{2}a_{11}a_{31} & \sqrt{2}a_{12}a_{32} & \sqrt{2}a_{13}a_{33} & a_{12}a_{31}+a_{11}a_{32} & a_{13}a_{31}+a_{11}a_{33} & a_{13}a_{32}+a_{12}a_{33} \\ \sqrt{2}a_{21}a_{31} & \sqrt{2}a_{22}a_{32} & \sqrt{2}a_{23}a_{33} & a_{22}a_{31}+a_{21}a_{32} & a_{23}a_{31}+a_{21}a_{33} & a_{23}a_{32}+a_{22}a_{33} \end{bmatrix}.$$

3. Symmetric tensor powers of graphs

We are now ready to define the symmetric tensor powers of a graph.

DEFINITION 3.1. Let $\mathcal{G}=(V,E,\omega)$ denote a weighted graph with vertex set $V=\{v_1,v_2,\ldots,v_n\}$, edge set E, and edge-weight function ω; nonexistent edges have weight 0. The vertices of the *kth symmetric tensor power* $\mathcal{G}^{\odot k}$ are the $N=\binom{n+k-1}{k}$ nondecreasing elements $\mathbf{i}=(i_1,i_2,\ldots,i_k)$ of $[n]^k$, each of which we may identify with the corresponding symmetric tensor $v_{i_1} \odot v_{i_2} \odot \cdots \odot v_{i_k}$ or its normalization $u_{\mathbf{i}}$ in (2.3). The edge weights are

$$(3.1) \qquad \omega^{\odot k}(\mathbf{i},\mathbf{j}) = \frac{1}{\sqrt{\binom{k}{\mathbf{m(i)}}\binom{k}{\mathbf{m(j)}}}} \sum_{\substack{\mathbf{p}\in\mathrm{Orb}(\mathbf{i}) \\ \mathbf{q}\in\mathrm{Orb}(\mathbf{j})}} \prod_{\ell=1}^{k} \omega(v_{p_\ell},v_{q_\ell}).$$

The map $\mathcal{G} \mapsto \mathcal{G}^{\odot k}$ is well defined up to graph isomorphism; see Appendix A. If A is the adjacency matrix of $\mathcal{G}$, then $A^{\odot k}$ is the adjacency matrix of $\mathcal{G}^{\odot k}$.

EXAMPLE 3.1. Let $\mathcal{G}=\mathcal{K}_3$ be the complete graph on three vertices v_1,v_2,v_3. Its adjacency matrix $A=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ acts naturally on the inner-product space $\mathcal{V}$ with

(A) The complete graph $\mathcal{K}_3$

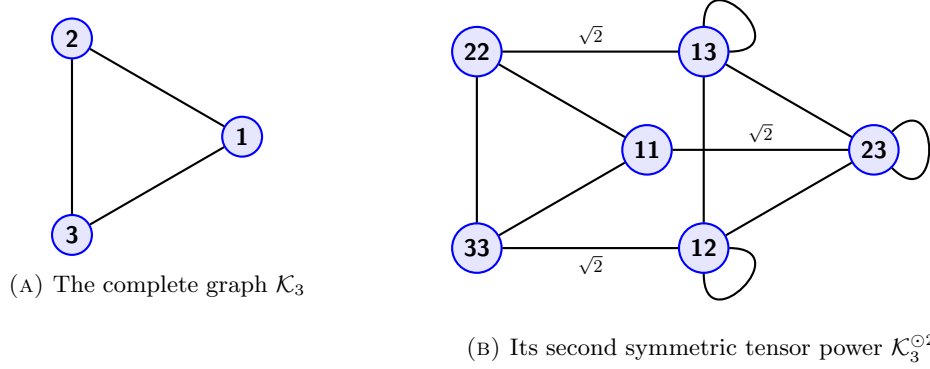(B) Its second symmetric tensor power $\mathcal{K}_3^{\odot 2}$

FIGURE 1. The complete graph $\mathcal{K}_3$ and its second symmetric tensor power. Symmetric tensor powers of unweighted loopless graphs can be weighted and may have loops.

orthonormal basis v_1, v_2, v_3. Then $\mathcal{V}^{\otimes 2}$ has orthonormal basis

$$v_1 \otimes v_1,\ v_1 \otimes v_2,\ v_1 \otimes v_3,\ v_2 \otimes v_1,\ v_2 \otimes v_2,\ v_2 \otimes v_3,\ v_3 \otimes v_1,\ v_3 \otimes v_2,\ v_3 \otimes v_3.$$

The symmetric group $\mathfrak{S}_2$ produces $N = \binom{3+2-1}{2} = 6$ orbits in $[3]^2$, which yields

$$\{v_1 \otimes v_1\}, \{v_2 \otimes v_2\}, \{v_3 \otimes v_3\}, \{v_1 \otimes v_2, v_2 \otimes v_1\}, \{v_1 \otimes v_3, v_3 \otimes v_1\}, \{v_2 \otimes v_3, v_3 \otimes v_2\}.$$

Each orbit contains a nondecreasing representative; these are, respectively,

$$v_1 \otimes v_1, \quad v_2 \otimes v_2, \quad v_3 \otimes v_3, \quad v_1 \otimes v_2, \quad v_1 \otimes v_3, \quad v_2 \otimes v_3.$$

The orthonormal basis provided by Lemma 2.1 is

$$u_{(1,1)} = v_1 \odot v_1, \qquad u_{(2,2)} = v_2 \odot v_2, \qquad u_{(3,3)} = v_3 \odot v_3,$$
$$u_{(2,3)} = \sqrt{2}(v_2 \odot v_3), \qquad u_{(1,3)} = \sqrt{2}(v_1 \odot v_3), \qquad u_{(1,2)} = \sqrt{2}(v_1 \odot v_2),$$

and Lemma 2.2 provides the adjacency matrix of $\mathcal{K}_3^{\odot 2}$:

$$A^{\odot 2} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & \sqrt{2} \\ 1 & 0 & 1 & 0 & \sqrt{2} & 0 \\ 1 & 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 1 & 1 & 1 \\ 0 & \sqrt{2} & 0 & 1 & 1 & 1 \\ \sqrt{2} & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Note that $\mathcal{K}_3$ (Figure 1a) is a subgraph of $\mathcal{K}_3^{\odot 2}$ (Figure 1b).

EXAMPLE 3.2. Let $\mathcal{P}_3$ denote the path graph on three vertices (Figure 2a). Its second symmetric tensor power $\mathcal{P}_3^{\odot 2}$ (Figure 2c) is obtained from the tensor (Kronecker) power $\mathcal{P}^{\otimes 2}$ (Figure 2b) by identifying ordered pairs in the same $\mathfrak{S}_2$ orbit, selecting nondecreasing representatives, and adding weights according to (3.1).

Examples 3.1 and 3.2 suggest that an (unweighted) graph is a subgraph of its symmetric tensor powers. The next theorem establishes this fact.

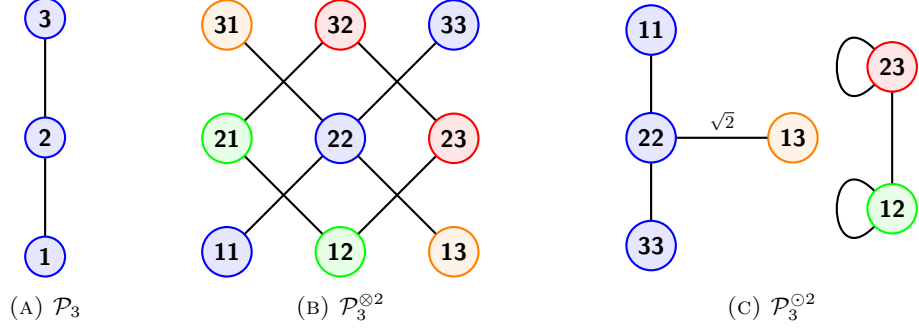(A) $\mathcal{P}_3$ (B) $\mathcal{P}_3^{\otimes 2}$ (C) $\mathcal{P}_3^{\odot 2}$

FIGURE 2. The symmetric tensor power $\mathcal{P}_3^{\odot 2}$ of the path graph $\mathcal{P}_3$ is obtained by identifying vertices in the tensor (Kronecker) power $\mathcal{P}_3^{\otimes 2}$ and adding the appropriate weights. Edge weights equal to 1 are suppressed.
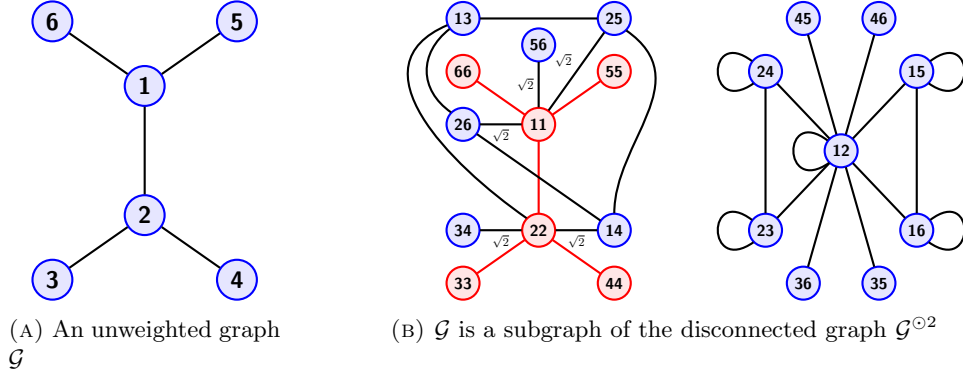

(A) An unweighted graph $\mathcal{G}$ (B) $\mathcal{G}$ is a subgraph of the disconnected graph $\mathcal{G}^{\odot 2}$

FIGURE 3. Illustration of Theorem 3.1.

THEOREM 3.1. If $\mathcal{G}$ is an unweighted graph and $k \in \mathbb{N}$, then $\mathcal{G}$ is a subgraph of $\mathcal{G}^{\odot k}$.

PROOF. Let $\mathcal{G}$ be an unweighted graph on n vertices, with adjacency matrix A. Let $\mathbf{i} = (i, i, \ldots, i) \in [n]^k$ and $\mathbf{j} = (j, j, \ldots, j) \in [n]^k$. Then $\mathbf{m}(\mathbf{i})$ and $\mathbf{m}(\mathbf{j})$ are k times the standard basis vectors in $\mathbb{R}^n$ that have their 1s in the ith and jth positions, respectively. Since $\mathrm{Orb}(\mathbf{i}) = \{\mathbf{i}\}$ and $\mathrm{Orb}(\mathbf{j}) = \{\mathbf{j}\}$, Proposition 2.2 ensures that

$$[A^{\odot k}]_{\mathbf{i},\mathbf{j}} = \frac{1}{\sqrt{\binom{k}{\mathbf{m}(\mathbf{i})}\binom{k}{\mathbf{m}(\mathbf{j})}}} \sum_{\substack{\mathbf{p} \in \mathrm{Orb}(\mathbf{i}) \\ \mathbf{q} \in \mathrm{Orb}(\mathbf{j})}} \prod_{\ell=1}^{k} [A]_{p_\ell, q_\ell} = \prod_{\ell=1}^{k} [A]_{i,j} = [A]_{i,j}$$

because the matrix entries of A belong to $\{0, 1\}$. $\qquad \square$
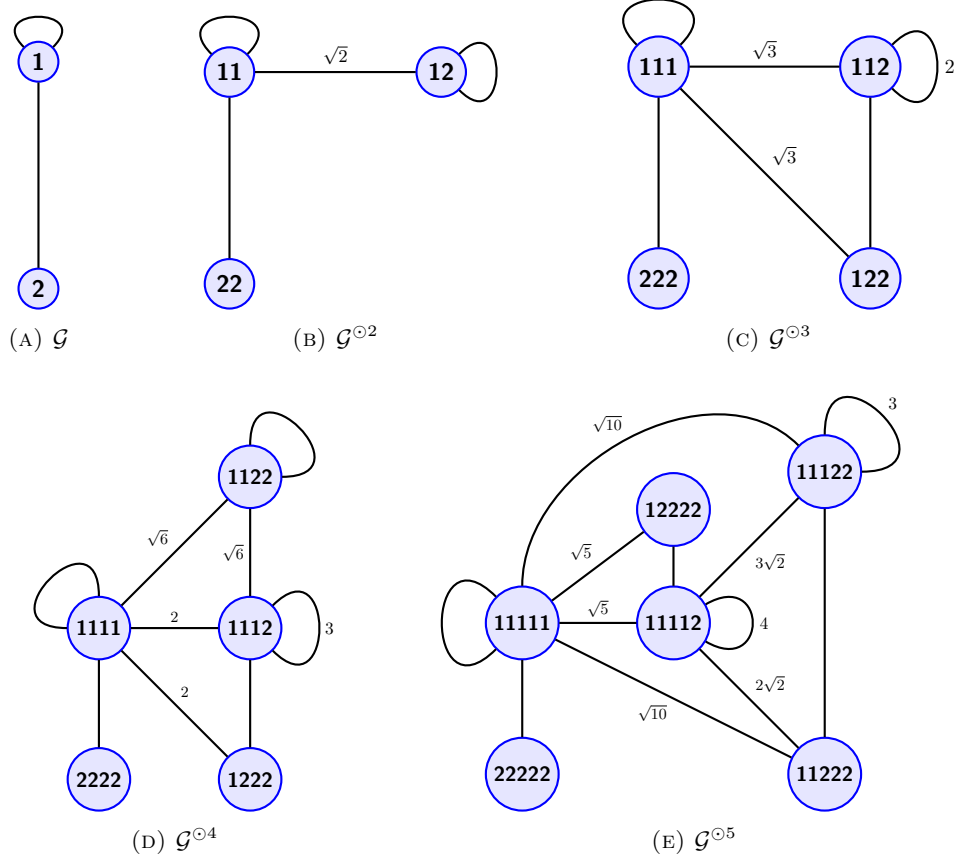
The previous theorem is illustrated in Figures 3 and 4.

FIGURE 4. Symmetric tensor powers of the "scepter" graph $\mathcal{G}$; edge weights with value 1 are suppressed. Theorem 3.1 ensures that $\mathcal{G}$ appears as a subgraph in $\mathcal{G}^{\odot k}$ for $k \in \mathbb{N}$.

PROPOSITION 3.1. (a) Let $\mathcal{G}$ be a graph that contains a loop, then as unweighted graphs $\mathcal{G}^{\odot k_1}$ is a subgraph of $\mathcal{G}^{\odot k_2}$ for all $k_1 \leqslant k_2$. (b) Let $\mathcal{G}$ be a nonempty graph, then as unweighted graphs $\mathcal{G}^{\odot k_1}$ is a subgraph of $\mathcal{G}^{\odot k_1 + 2\ell}$ for all $\ell \geqslant 0$.

PROOF. (a) We may assume the loop is at vertex v_1. Notice that if v_i is adjacent to v_j in $\mathcal{G}^{\odot k_1}$, then $\underbrace{v_1 \odot v_1 \odot \cdots \odot v_1}_{k_2 - k_1} \odot v_i$ and $\underbrace{v_1 \odot v_1 \odot \cdots \odot v_1}_{k_2 - k_1} \odot v_j$ are adjacent.
(b) We may assume v_1 and v_2 are adjacent in $\mathcal{G}$. If v_i is adjacent to v_j in $\mathcal{G}^{\odot k_1}$, then $\underbrace{v_1 \odot v_2 \odot \cdots \odot v_1 \odot v_2}_{\frac{k_2 - k_1}{2}} \odot v_i$ and $\underbrace{v_1 \odot v_2 \odot \cdots \odot v_1 \odot v_2}_{\frac{k_2 - k_1}{2}} \odot v_j$ are adjacent. □

4. Compatibility with graph spectra

The *spectrum* of a graph is the spectrum (multiset of eigenvalues) of its adjacency matrix. This is well defined since isomorphic graphs correspond to permutation-similar adjacency matrices, and similar matrices have the same spectrum. Symmetric tensor powers respect graph spectra. Although the next result is familiar to representation theorists and mathematical physicists, we prove it for the sake of graph theorists to whom it may be novel.

THEOREM 4.1. Let $\mathcal{G}$ be a graph on n vertices whose adjacency matrix A has spectrum $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the spectrum of $\mathcal{G}^{\odot k}$ consists of the $\binom{n+k-1}{k}$ products $\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$ with $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant n$; these are all of the eigenvalues of $A^{\odot k}$.

PROOF. Let $\mathcal{G}$ have vertices $v_1, v_2, \ldots, v_n$. For $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant n$,

$$
\begin{aligned}
A^{\odot k}(v_{i_1} \odot v_{i_2} \odot \cdots \odot v_{i_k}) &= \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (A v_{i_{\sigma(1)}}) \otimes (A v_{i_{\sigma(2)}}) \otimes \cdots \otimes (A v_{i_{\sigma(k)}}) \\
&= \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (\lambda_{i_1} v_{i_{\sigma(1)}}) \otimes (\lambda_{i_2} v_{i_{\sigma(2)}}) \otimes \cdots \otimes (\lambda_{i_k} v_{i_{\sigma(k)}}) \\
&= \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} \left(\frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}} \otimes \cdots \otimes v_{i_{\sigma(k)}} \right) \\
&= \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} (v_{i_1} \odot v_{i_2} \odot \cdots \odot v_{i_k}).
\end{aligned}
$$

Thus, $v_{i_1} \odot v_{i_2} \odot \cdots \odot v_{i_k}$ is an eigenvector of $A^{\odot k}$ with eigenvalue $\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$. There are no others since the $v_{i_1} \odot v_{i_2} \odot \cdots \odot v_{i_k}$ form a basis for $\mathcal{V}^{\odot k}$. $\square$

EXAMPLE 4.1. Figure 4 illustrates the graphs $\mathcal{G}^{\odot k}$ for $k = 2, 3, 4, 5$, in which $\mathcal{G}$ is the scepter graph (Figure 4a). The corresponding adjacency matrices are

$$
A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad A^{\odot 2} = \begin{bmatrix} 1 & 1 & \sqrt{2} \\ 1 & 0 & 0 \\ \sqrt{2} & 0 & 1 \end{bmatrix}, \quad A^{\odot 3} = \begin{bmatrix} 1 & 1 & \sqrt{3} & \sqrt{3} \\ 1 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 2 & 1 \\ \sqrt{3} & 0 & 1 & 0 \end{bmatrix},
$$

$$
A^{\odot 4} = \begin{bmatrix} 1 & 1 & 2 & \sqrt{6} & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 3 & \sqrt{6} & 1 \\ \sqrt{6} & 0 & \sqrt{6} & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad A^{\odot 5} = \begin{bmatrix} 1 & 1 & \sqrt{5} & \sqrt{10} & \sqrt{10} & \sqrt{5} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{5} & 0 & 4 & 3\sqrt{2} & 2\sqrt{2} & 1 \\ \sqrt{10} & 0 & 3\sqrt{2} & 3 & 1 & 0 \\ \sqrt{10} & 0 & 2\sqrt{2} & 1 & 0 & 0 \\ \sqrt{5} & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.
$$

The respective graph spectra are

$$
\begin{aligned}
\mathcal{G}: &\quad \tfrac{1}{2}(1 - \sqrt{5}), \quad \tfrac{1}{2}(1 + \sqrt{5}), \\
\mathcal{G}^{\odot 2}: &\quad -1, \quad \tfrac{1}{2}(3 - \sqrt{5}), \quad \tfrac{1}{2}(\sqrt{5} + 3), \\
\mathcal{G}^{\odot 3}: &\quad \tfrac{1}{2}(-1 - \sqrt{5}), \quad 2 - \sqrt{5}, \quad \tfrac{1}{2}(-1 + \sqrt{5}), \quad \sqrt{5} + 2, \\
\mathcal{G}^{\odot 4}: &\quad 1, \quad \tfrac{1}{2}(7 - 3\sqrt{5}), \quad \tfrac{1}{2}(-\sqrt{5} - 3), \quad \tfrac{1}{2}(\sqrt{5} - 3), \quad \tfrac{1}{2}(3\sqrt{5} + 7), \\
\mathcal{G}^{\odot 5}: &\quad \tfrac{1}{2}(11 - 5\sqrt{5}), \quad -\sqrt{5} - 2, \quad \tfrac{1}{2}(1 - \sqrt{5}), \quad \sqrt{5} - 2, \quad \tfrac{1}{2}(\sqrt{5} + 1), \quad \tfrac{1}{2}(5\sqrt{5} + 11).
\end{aligned}
$$

These agree with the results of Theorem 4.1. The eigenvalues $\lambda_1 = \frac{1}{2}(1 - \sqrt{5})$ and $\lambda_2 = \frac{1}{2}(1 + \sqrt{5})$ of A give rise to the eigenvalues of $A^{\odot 2}$:

$$\lambda_1^2 = \tfrac{1}{2}(3 - \sqrt{5}), \qquad \lambda_1 \lambda_2 = -1, \qquad \text{and} \qquad \lambda_2^2 = \tfrac{1}{2}(3 + \sqrt{5}).$$

A similar calculation shows that the eigenvalues of $A^{\odot 3}$ are

$$\lambda_1^3 = 2 - \sqrt{5}, \quad \lambda_1^2 \lambda_2 = \tfrac{1}{2}(-1 + \sqrt{5}), \quad \lambda_1 \lambda_2^2 = \tfrac{1}{2}(-1 - \sqrt{5}), \quad \text{and} \quad \lambda_2^3 = \sqrt{5} + 2.$$

Theorem 4.1 provides convenient formulas for the trace and determinant of a graph, that is, the trace and determinant of its adjacency matrix.

COROLLARY 4.1. Let $\mathcal{G}$ be a graph with spectrum $\lambda_1, \lambda_2, \ldots, \lambda_n$ and let $k \in \mathbb{N}$. Then $\det(\mathcal{G}^{\odot k}) = (\det \mathcal{G})^{\binom{n+k-1}{n}}$ and $\operatorname{tr}(\mathcal{G}^{\odot k}) = \sum_{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$, the complete homogeneous symmetric polynomial of degree k in the eigenvalues of $\mathcal{G}$.

PROOF. The trace of a graph $\mathcal{G}$ is the sum of the eigenvalues of its adjacency matrix A, so the desired trace formula follows from Theorem 4.1. The determinant is the product of the eigenvalues, so

$$\det(A^{\odot k}) = \prod_{\substack{\mathbf{i} = (i_1, i_2, \ldots, i_k) \\ 1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant n}} \prod_{j=1}^{n} \lambda_j^{\mathbf{m(i)}_j} = \prod_{i=1}^{n} \lambda_i^{\beta_i},$$

in which β_i is the number of times λ_i appears in the whole product. This is equivalent to adding the contribution of each individual product, so

$$\beta_i = \sum_{\ell=0}^{k} \ell \cdot \binom{k - \ell + n - 2}{k - \ell} = \binom{k + n - 1}{n},$$

in which the final equality arises as follows: let $(x_1, x_2, \cdots, x_n)$ be a composition of k into n nonnegative parts. Fix $i \in [n]$ such that $x_i = \ell$ and take out $0 \leqslant r < \ell$ from the part x_i to a new part x_{n+1}. There are ℓ choices for r to take to the new part x_{n+1} and it cannot be 0 so $x_{n+1} > 0$, so we have taken one out, which adds up to $k - 1$, and so these are compositions of $k - 1$ into $n + 1$ nonnegative parts which are counted by $\binom{(k-1)+n+1-1}{n+1-1} = \binom{n+k-1}{n}$. $\qquad\square$

5. Combinatorial properties

The number $|E_{\mathcal{G}^{\otimes k}}|$ of edges of a tensor (Kronecker) power $\mathcal{G}^{\otimes k}$ of an undirected simple graph $\mathcal{G}$ is related to the original number of edges $|E_{\mathcal{G}}|$ by $|E_{\mathcal{G}^{\otimes k}}| = 2^{k-1}|E_{\mathcal{G}}|^k$ [11]. This yields the upper bound in the next result. The lower bound is obtained by noticing that an edge in $\mathcal{G}^{\otimes k}$ can be seen as an edge in $\mathcal{G}^{\otimes k}$ when one permute the indices of one of the vertices, so $|E_{\mathcal{G}^{\otimes k}}| \leqslant |E_{\mathcal{G}^{\odot k}}| \cdot k!$.

PROPOSITION 5.1. For a simple undirected graph, $\mathcal{G}$ we have

$$\frac{2^{k-1}|E_{\mathcal{G}}|^k}{k!} \leqslant |E_{\mathcal{G}^{\odot k}}| \leqslant 2^{k-1}|E_{\mathcal{G}}|^k.$$

The number of edges adjacent to a vertex v of a graph $\mathcal{G}$ is the *degree* of v in $\mathcal{G}$, denoted $\deg_\mathcal{G} v$. This is the size of the set $N_\mathcal{G}(v) = \{u \in V_\mathcal{G} : \{u, v\} \in E_\mathcal{G}\}$ of neighboring vertices. The next proposition characterizes the neighbor set of a vertex and gives a bound on its cardinality. The Cartesian product is to generate sequences from the neighbours of elements in the vertex.

PROPOSITION 5.2. Let $\mathcal{G}$ be an undirected graph with vertex set $V_\mathcal{G} = \{v_1, v_2, \ldots, v_n\}$ and let $k \in \mathbb{N}$. For $\mathbf{i} \in V_{\mathcal{G}\odot k}$ (that is, $\mathbf{i}$ is a nondecreasing element of $[n]^k$),

$$N_{\mathcal{G}\odot k}(\mathbf{i}) = \left\{ \mathbf{j} \in V_{\mathcal{G}\odot k} : \mathrm{Orb}(\mathbf{j}) \cap \left(N(v_1)_\mathcal{G}^{\mathbf{m(i)}_1} \times N(v_2)_\mathcal{G}^{\mathbf{m(i)}_2} \times \cdots \times N(v_n)_\mathcal{G}^{\mathbf{m(i)}_n} \right) \neq \emptyset \right\}.$$

Its cardinality is bounded by

$$\deg_{\mathcal{G}\odot k}(\mathbf{i}) \leqslant \binom{|\bigcup_{\mathbf{m(x)}_i > 0} N_\mathcal{G}(v_i)| + k - 1}{k} \leqslant \prod_{\ell=1}^n \deg_\mathcal{G}(v_\ell)^{\mathbf{m(x)}_\ell}.$$

PROOF. It follows from Proposition 2.2 that the positivity of an entry requires the existence of $(\mathbf{q}, \mathbf{t}) \in \mathrm{Orb}(\mathbf{x}) \times \mathrm{Orb}(\mathbf{y})$ such that $\mathbf{t}_i \in N_\mathcal{G}(\mathbf{q}_\ell)$ for every ℓ. Therefore, the elements must belong to the union of the elements in $\mathbf{x}$. The second part follows from the first since there is a single neighboring set. □

If $\mathbf{i} = (v_\ell, v_\ell, \ldots, v_\ell)$, with v_ℓ repeated k times, then $\deg_{\mathcal{G}\odot k}(\mathbf{i}) = \binom{\deg_\mathcal{G}(v_\ell)+k-1}{k}$ since the neighbors of $\mathbf{i}$ are k-sequences of nondecreasing neighbors of v_ℓ. We can completely characterize the number of loops, edges, and the degrees of vertices in the second symmetric tensor power.

PROPOSITION 5.3. Let $\mathcal{G}$ be an undirected graph with vertex set $V_\mathcal{G} = \{v_1, v_2, \ldots, v_n\}$.
(1) If $\mathbf{i} = (v_a, v_b)$, then

$$\deg_{\mathcal{G}\odot 2}(\mathbf{i}) = \begin{cases} \binom{\deg_\mathcal{G}(v_a)+1}{2} & \text{if } a = b, \\ \deg_\mathcal{G}(v_a)\deg_\mathcal{G}(v_b) - \binom{|N_\mathcal{G}(v_a) \cap N_\mathcal{G}(v_b)|}{2} & \text{if } a < b. \end{cases}$$

(2) The number of loops on $\mathcal{G}^{\odot 2}$ is $E_\mathcal{G} + \ell_d$, where

$$\ell_d = \left| \left\{ \{v_1, v_2\} \notin E_\mathcal{G} : \{v_1, v_1\}, \{v_2, v_2\} \in E_\mathcal{G} \right\} \right|,$$

is the number of non-adjacent pairs of loops in $\mathcal{G}$.
(3) The number of edges in $\mathcal{G}^{\odot 2}$ is

$$\frac{1}{2} \left(E_\mathcal{G} + \ell_d + \sum_{i \in [n]} \binom{\deg_\mathcal{G}(v_i)+1}{2} + \sum_{1 \leqslant i < j \leqslant n} \deg_\mathcal{G}(v_i)\deg_\mathcal{G}(v_j) - \sum_{1 \leqslant i < j \leqslant n} \binom{|N_\mathcal{G}(v_i) \cap N_\mathcal{G}(v_j)|}{2} \right).$$

PROOF. (a) If $a = b$, Proposition 5.2 gives the result. If $a \neq b$, then every element in $\deg_{\mathcal{G}\odot 2}(\mathbf{i})$ belongs to $N(v_a) \times N(v_b)$. This overcounts the choices for which the chosen vertices can be in either position of the tuple. Then take out the number of ways to choose two elements in the intersection of the neighbours to obtain the result.
(b) There are two kinds of loops: those for which (a, b) goes to (b, a) in $\mathcal{G}^{\odot 2}$ when $\{a, b\} \in E_\mathcal{G}$ or (a, b) goes to (a, b) when a and b have loops. This last case is counted by ℓ_d when $\{a, b\} \notin E_\mathcal{G}$.

(c) The result is now a consequence of the handshake lemma adding over the degrees with the formula established above.   □

COROLLARY 5.1. *If $\mathcal{G}$ is 1-regular, then $\mathcal{G}^{\odot k}$ is 1-regular.*

PROOF. Using the expression for the neighbourhood set, there is just one element that connects for $\mathbf{x} \in V_{\mathcal{G}^{\odot k}}$, namely $\mathbf{y}$ such that the only element in $N(v_1)^{x_1} \times N(v_2)^{x_2} \times \cdots \times N(v_n)^{x_n}$ is contained in $\mathrm{Orb}(\mathbf{y})$.   □

THEOREM 5.1 (Theorem 1 of [**16**]). *(a) If $\mathcal{G}$ is connected and non-bipartite, then $\mathcal{G}^{\otimes 2}$ is connected. (b) If $\mathcal{G}$ is connected and bipartite, then $\mathcal{G}^{\otimes 2}$ has two connected components.*

COROLLARY 5.2. *(a) If $\mathcal{G}$ is connected and non-bipartite, then $\mathcal{G}^{\odot 2}$ is connected. (b) If $\mathcal{G}$ is connected and bipartite, then $\mathcal{G}^{\odot 2}$ has two components.*

PROOF. (a) This follows from Theorem 5.1 and the graph-theoretic construction of the symmetric tensor product.

(b) If $(v_i, v_j) \in V_{\mathcal{G}^{\odot 2}}$, the connectivity of $\mathcal{G}$ provides a path from v_i to v_j. This path lifts to a path in $\mathcal{G}^{\otimes 2}$ from (v_i, v_j) to (v_j, v_i), so identifying points in $\mathfrak{S}_2$ orbits does not decrease the number of connected components.   □

COROLLARY 5.3. *If $\mathcal{G}$ is a connected, non-bipartite, loopless graph, then $\mathcal{G}^{\odot k}$ is connected. In general, the number of connected components on $\mathcal{G}^{\odot k}$ is at most 2^{k-1}.*

6. Properties of $\mathcal{G}^{\odot k}$ for particular graphs $\mathcal{G}$

A number of curious features arise when one considers symmetric tensor powers of familiar graphs. In this section, we describe a variety of such phenomena.

6.1. Complete graphs with loops. The complete graph with loops on n vertices, denoted $\mathcal{J}_n$, is the graph with vertex set $[n]$ and edge set $[n] \times [n]$. Thus, all pairs of vertices are adjacent and each vertex has a loop. Therefore, $\mathcal{J}_n$ has $n + \binom{n}{2} = \binom{n+1}{2}$ edges and its adjacency matrix is the all-ones matrix; see Figure 5a.

PROPOSITION 6.1. *The weight of the edge connecting $\mathbf{i}$ and $\mathbf{j}$ in $\mathcal{J}_n^{\odot k}$ is $\sqrt{\binom{k}{\mathbf{m}(\mathbf{i})}\binom{k}{\mathbf{m}(\mathbf{j})}}$. As unweighted graphs, $\mathcal{J}_n^{\odot k} = \mathcal{J}_{\binom{n+k-1}{k}}$.*

PROOF. All vertices are connected, so this follows from Proposition 2.2.   □

The weights in the loops of the second symmetric tensor power of $\mathcal{J}_n$ determine the size of the orbit of the corresponding vertex; see Figure 5b.

COROLLARY 6.1. *The number of edges in $\mathcal{J}_n^{\odot 2}$ is $\binom{\binom{n+1}{2}+1}{2}$.*

This agrees with Proposition 5.3. The resulting sequence $0, 1, 6, 21, 55, 120, 231, \ldots$ is A002817 in OEIS.

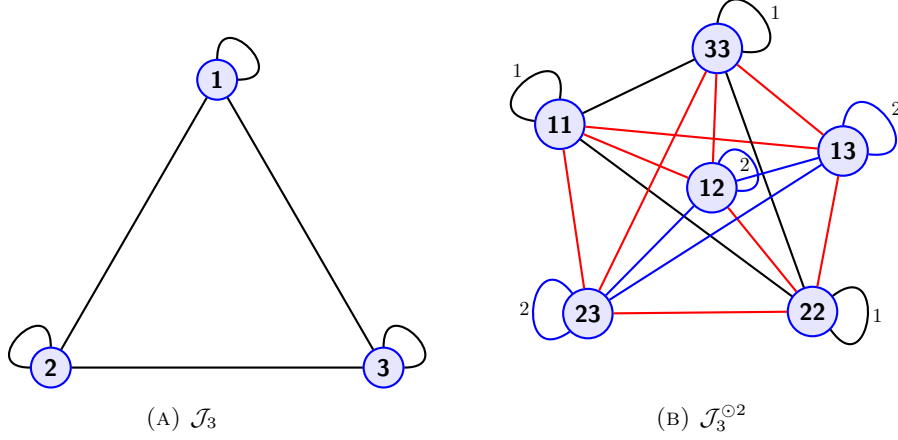(A) $\mathcal{J}_3$ (B) $\mathcal{J}_3^{\odot 2}$

FIGURE 5. $\mathcal{J}_3$, the complete graph with loops on three vertices, and its second symmetric tensor power. In (b) red edges have weight $\sqrt{2}$, blue edges have weight 2, and black edges have weight 1.

6.2. Path graphs. The *path graph* $\mathcal{P}_n$ is the graph with adjacency matrix $[a_{ij}] \in M_n$ with $a_{ij} = 1$ if $|i - j| = 1$ and $a_{ij} = 0$ otherwise; see Figures 2a and 2c, which suggest the connectivity of symmetric tensor powers of path graphs is of interest.

THEOREM 6.1. $\mathcal{P}_n^{\odot k}$ has $\lceil \frac{k+1}{2} \rceil$ connected components. (a) If $k = 2\ell$, then one of those components contains $\binom{n+\ell-2}{\ell}$ loops. (b) If k is odd, then $\mathcal{P}_n^{\odot k}$ contains no loops.

PROOF. If $\mathbf{i}$ and $\mathbf{j}$ share an edge in $\mathcal{P}_n^{\odot k}$, then

$$\mathbf{m}(\mathbf{i})_{\text{odd}} = \sum_{\ell \text{ odd}} \mathbf{m}(\mathbf{i})_\ell = \sum_{\ell \text{ even}} \mathbf{m}(\mathbf{j})_\ell = \mathbf{m}(\mathbf{j})_{\text{even}},$$

$$\mathbf{m}(\mathbf{i})_{\text{even}} = \sum_{\ell \text{ even}} \mathbf{m}(\mathbf{i})_\ell = \sum_{\ell \text{ odd}} \mathbf{m}(\mathbf{j})_\ell = \mathbf{m}(\mathbf{j})_{\text{odd}},$$

because for every $r \in [n]$ $\mathbf{m}(\mathbf{i})_r$ is splitted and added to $\mathbf{m}(\mathbf{j})_{r+1}$ and $\mathbf{m}(\mathbf{j})_{r-1}$ if possible.

Consider vertices $\mathbf{i}_{a,b}$ with $a + b = k$ and $a \geqslant b$ such that $\mathbf{m}(\mathbf{i}_{a,b}) = (a, b, 0, \ldots, 0)$. All of them lie in different connected components, otherwise if $\mathbf{i}_{a_1,b_1}$ and $\mathbf{i}_{a_2,b_2}$ with $a_1 > a_2$ are in the same component, then $a_1 = a_2$ or $a_1 = b_2$, which is a contradiction.

Furthermore, every vertex $\mathbf{i} = (v_1, v_2, \cdots, v_k)$ is in the same connected component with a vertex of the form $\mathbf{i}_{a,b}$. To see this, consider moving from $\mathbf{m}(\mathbf{i}) = (v_1, v_2, \cdots, v_n)$ to $\mathbf{m}(\mathbf{i}') = (v_2, v_1 + v_3, v_4, \ldots, v_n, 0)$ until the only nonzero entries are the first two, if needed swap the first two coordinates. This implies that the number of connected components is the same as the number of different vertices of the form $\mathbf{i}_{a,b}$. There are $\lceil \frac{k+1}{2} \rceil$ such vertices, because there are $k + 1$ ways to choose a, b such that $a + b = k$ and $b \leqslant a$ gives the correct count.
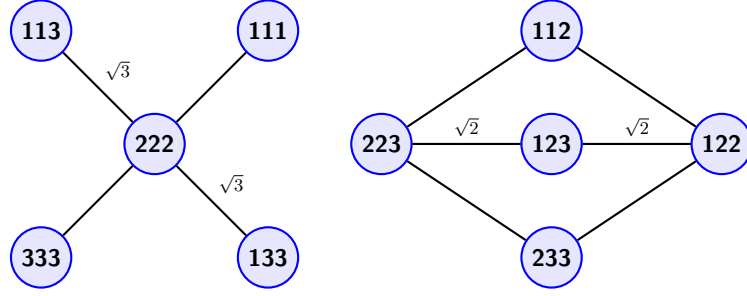
(A) $\mathcal{P}_3^{\odot 3}$ has $\lceil \frac{3+1}{2} \rceil = 2$ connected components and no loops, as predicted by Theorem 6.1.b.
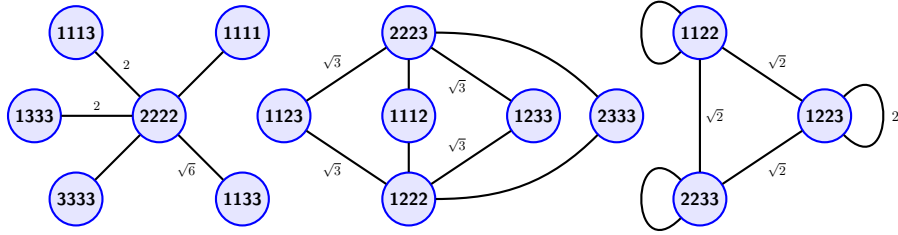
(B) $\mathcal{P}_3^{\odot 4}$ has $\lceil \frac{4+1}{2} \rceil = 3$ connected components and $\binom{3+2-2}{2} = 3$ loops, as predicted by Theorem 6.1.a.

FIGURE 6. Symmetric tensor powers $\mathcal{P}_3^{\odot 3}$ and $\mathcal{P}_3^{\odot 4}$ of the path graph $\mathcal{P}_3$ on three vertices. Only nonzero edge weights unequal to 1 are included.

There are no loops for k odd. Otherwise a loop in $\mathbf{j}$ implies $\mathbf{m(j)}_{\text{even}} = \mathbf{m(j)}_{\text{odd}}$. This contradicts the parity of k. For k even, write $k = 2\ell$ and let $a_{n,\ell}$ the number of loops in $\mathcal{P}_n^{\odot k}$. They satisfy the recursion

$$a_{n,\ell} = a_{n-1,\ell} + a_{n,\ell-1},$$

with initial conditions $a_{1,\ell} = 0$ and $a_{n,1} = n - 1$. The argument is by induction. Note that if $\mathbf{i}$ contains a loop, then either $\mathbf{m(i)}_n = 0$ or not: if it is 0 remove it, getting a vertex in $\mathcal{P}_{n-1}^{\odot k}$ that contain a loop. If $\mathbf{m(i)}_n \neq 0$, notice that $\mathbf{m(i)}_{n-1} \neq 0$ because otherwise it was not a loop and so consider $\mathbf{x}'$ given by $\mathbf{m(x')} = (\mathbf{m(i)}_1, \ldots, \mathbf{m(i)}_{n-1} - 1, \mathbf{m(i)}_n - 1)$, this is a loop in $\mathcal{P}_n^{\odot(k-2)}$. This implies the result by the usual recursions for binomial coefficients. $\qquad\square$

Figures 6 and 7 illustrate the previous theorem.

6.3. Cycle graphs. The *cycle graph* $\mathcal{C}_n$ is the graph with adjacency matrix $[a_{ij}] \in \mathrm{M}_n$, in which $a_{ij} = 1$ if $|i - j| \in \{1, n - 1\}$ and $a_{ij} = 0$ otherwise; see Figure 8.

PROPOSITION 6.2. (a) $\mathcal{C}_n^{\odot k}$ has $\lceil \frac{k+1}{2} \rceil$ connected components if n is even. (b) $\mathcal{C}_n^{\odot k}$ is connected if n is odd.
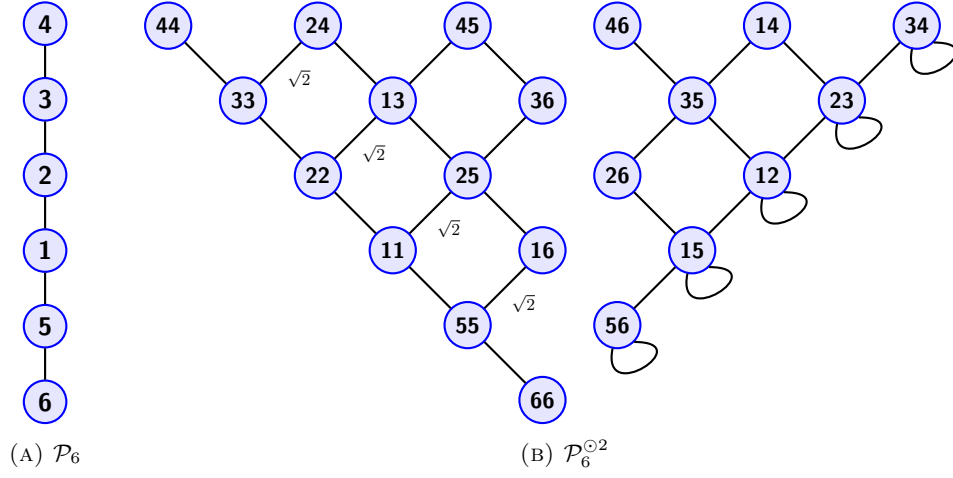
(A) $\mathcal{P}_6$

(B) $\mathcal{P}_6^{\odot 2}$

FIGURE 7. Path graph on six vertices and its second symmetric tensor power. Only edge weights unequal to 1 are included.
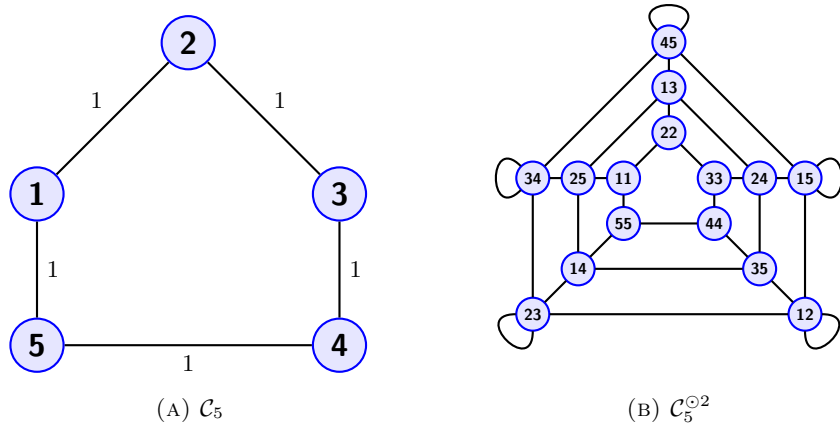

(A) $\mathcal{C}_5$

(B) $\mathcal{C}_5^{\odot 2}$

FIGURE 8. The cycle graph $\mathcal{C}_5$ and its second symmetric tensor power.

PROOF. (a) The case in which n is even can be done in a similar fashion as in the proof of Proposition 6.1.

(b) If n is odd and $a_1 > a_2$, there is a path from $\mathbf{i}_{a_1,b_1}$ to $\mathbf{i}_{a_2,b_2}$, so $\mathcal{C}_n^{\odot k}$ is connected:

$$\mathbf{m}(\mathbf{i}_{a_1,b_1}) = (a_1, b_1, \underbrace{0, \ldots, 0}_{\text{odd}}) \to (b_1, 0, \underbrace{0, \ldots, 0}_{\text{even}}, a_1) \to (a_2, b_1, \underbrace{0, \ldots, 0}_{\text{odd}}, a_2 - a_1, 0)$$

$$\to (0, 0, b_1, \underbrace{0, \ldots, 0}_{\text{odd}}, a_2 - a_1, 0, a_2) \to (a_2, b_1, 0, \underbrace{0, \ldots, 0}_{\text{even}}, a_2 - a_1, 0, 0, 0)$$

$$\to \cdots \to (0, 0, b_1, 0, a_2 - a_1, 0, \ldots, 0, a_2) \to (a_2, b_1, 0, a_2 - a_1, 0, \ldots, 0)$$

$$\to (0, 0, b_1 + a_2 - a_1, 0, \ldots, a_2) \to (a_2, b_1 + a_2 - a_1, 0, \ldots, 0) = \mathbf{m}(\mathbf{i}_{a_2,b_2}). \quad \square$$

6.4. Complete bipartite graphs. The *complete bipartite graph* $\mathcal{K}_{n,m}$ has vertex set $[n+m]$ and edges connecting every vertex in $[n]$ with every vertex in $[n+m] \smallsetminus [n]$. A vertex in $\mathcal{K}_{n,m}^{\odot k}$ is formed as follows: choose $0 \leqslant i \leqslant n$ and compose i elements from $[n]$ and $k - i$ elements from $[n + m] \smallsetminus [n]$. Observe that if $i < k - i$ this connection is possible by taking i elements from $[n + m] \smallsetminus [n]$ and $k - i$ elements from $[n]$. For every $2i < k$, this forms a new bipartite graph

$$\mathcal{K}_{\binom{i+n-1}{i}\binom{k-i+m-1}{k-i}, \binom{i+m-1}{i}\binom{k-i+n-1}{k-i}}.$$

For k even, write $2i = k$. Then any vertex having i elements from each block is connected to every other one, including itself. This yields a copy of the graph $\mathcal{J}_{\binom{k/2+n-1}{k/2}\binom{k/2+m-1}{k/2}}$. This proves the next result.

THEOREM 6.2. For $k, n, m \in \mathbb{N}$, as unweighted graphs we have

$$\mathcal{K}_{n,m}^{\odot k} = \begin{cases} \displaystyle\bigcup_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \mathcal{K}_{\binom{i+n-1}{i}\binom{k-i+m-1}{k-i}, \binom{i+m-1}{i}\binom{k-i+n-1}{k-i}} & \text{if } k \text{ is odd,} \\[2em] \mathcal{J}_{\binom{k/2+n-1}{k/2}\binom{k/2+m-1}{k/2}} \cup \displaystyle\bigcup_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \mathcal{K}_{\binom{i+n-1}{i}\binom{k-i+m-1}{k-i}, \binom{i+m-1}{i}\binom{k-i+n-1}{k-i}} & \text{otherwise.} \end{cases}$$

This graph has $1 + \lfloor \frac{k-1}{2} \rfloor + \frac{1+(-1)^k}{2} = \lceil \frac{k+1}{2} \rceil$ connected components.

We illustrate the previous result as follows. Figure 9 presents the third and fourth symmetric power of $\mathcal{K}_{2,1}$; the second symmetric power is illustrated in Figure 2. A complete bipartite graph is a *star graph* if $n = 1$; see Figure 10.

7. Wiener index

Let $\mathcal{G}$ be an undirected graph with $V_{\mathcal{G}} = \{v_1, v_2, \ldots, v_n\}$. The *Wiener index*

$$W(\mathcal{G}) = \sum_{\substack{\{v_i, v_j\} \subseteq V_{\mathcal{G}} \\ v_i \neq v_j}} d_{\mathcal{G}}(v_i, v_j),$$

where $d_{\mathcal{G}}(v_i, v_j)$ is the minimum distance between v_i and v_j, measures the complexity of $\mathcal{G}$. In this section, we compute the Wiener index of the symmetric tensor product for several types of graphs.

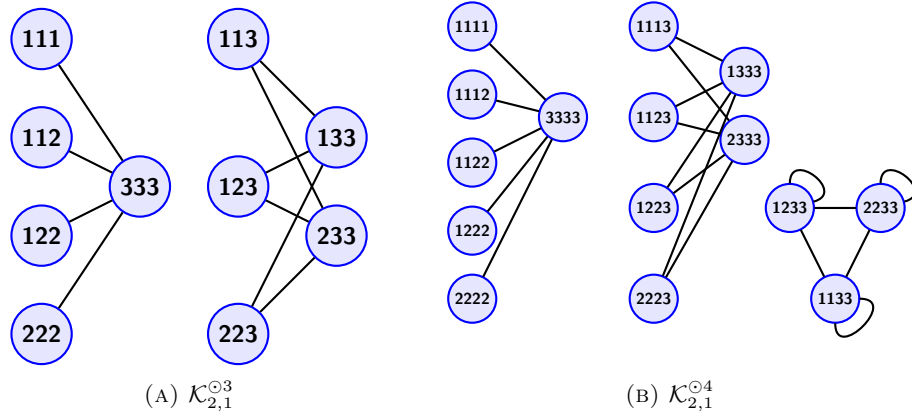PROPOSITION 7.1. (1) $W(\mathcal{J}_n^{\odot k}) = \left(\binom{\binom{k+n-1}{k}}{2} \right)$.

(A) $\mathcal{K}_{2,1}^{\odot 3}$

(B) $\mathcal{K}_{2,1}^{\odot 4}$

FIGURE 9. The graphs $\mathcal{K}_{2,1}^{\odot 3}$ and $\mathcal{K}_{2,1}^{\odot 4}$ illustrate Theorem 6.2.

(A) $\mathcal{S}_3$

(B) $\mathcal{S}_3^{\odot 2}$
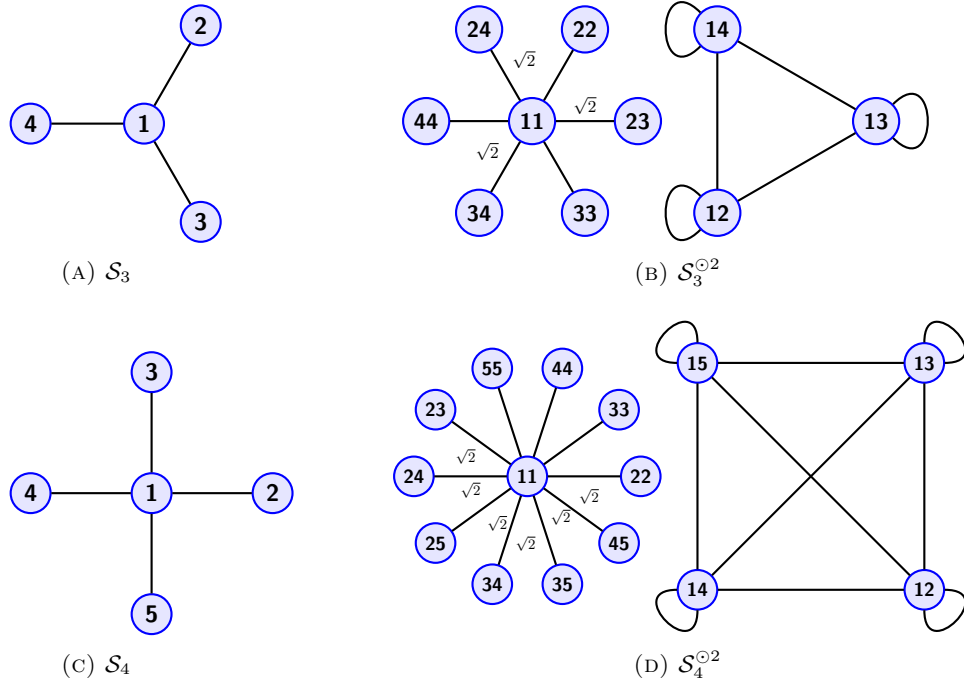
(C) $\mathcal{S}_4$

(D) $\mathcal{S}_4^{\odot 2}$

FIGURE 10. Star graphs are bipartite graphs. The images above illustrate Theorem 6.2.

$$(2)\ W(\mathcal{K}_n^{\odot k}) = \binom{\binom{k+n-1}{k}}{2} + \frac{n}{2}\left(\binom{k+2(n-2)+1}{2(n-1)} - \binom{k - \lceil\frac{k}{2}\rceil - \left(\frac{1+(-1)^k}{2}\right)+n-1}{n-1}\right).$$

PROOF. It is clear that the graph $\mathcal{J}_n^{\odot k}$ has every possible edge and so, every vertex is at distance one of every other vertex, implying that the sum is given by choosing the two vertices. For the complete graph, every two vertices are at distance no more than two. Call D_k the number of pairs of vertices, without order, that are not connected by an edge. This happens only if there is one element appearing more times than the rest of the elements. In this situation, using the pigeonhole principle, we will have to assign one of the copies of this vertex to itself, but the complete graph contains no loops, hence it is impossible to go by an edge. Let $\{\mathbf{i}, \mathbf{j}\}$ be one of such pairs, then there is a $r \in [n]$ such that $\mathbf{i}_s > \sum_{r\neq s}\mathbf{j}_r = k - \mathbf{j}_s$, and so the condition becomes $\mathbf{i}_s + \mathbf{j}_s > k$. Notice also that this can only happen in one index s, and without loss of generality, we can say that it happens in the first component of the compositions. An expression for the Wiener index of the complete graph becomes

$$\begin{aligned} W\left(\mathcal{K}_n^{\odot k}\right) &= 1\cdot\left(\binom{\binom{k+n-1}{k}}{2} - n\cdot D_k\right) + 2\cdot n\cdot D_k \\ &= \binom{\binom{k+n-1}{k}}{2} + 2\cdot n\cdot D_k, \end{aligned}$$

but $T_k = 2\cdot D_k + F_k$, where $T_k = |\{(\mathbf{i},\mathbf{j}) : \mathbf{i}_1 + \mathbf{j}_1 > k\}|$ denotes the number of pairs of compositions of k of size n for which the sum of their first parts exceeds k and $F_k = |\{\mathbf{i} : 2\mathbf{i}_1 > k\}|$ corresponds to those pairs for which the two compositions are the same. Using stars and bars one gets $F_k = \binom{k-\lceil\frac{k}{2}\rceil-(\frac{1+(-1)^k}{2})+n-1}{n-1}$. On the other hand,

$$\begin{aligned} T_k &= \sum_{s=1}^{k}\sum_{t=k+1-s}^{k}\binom{k-s+n-2}{n-2}\binom{k-t+n-2}{n-2} \\ &= \sum_{s=1}^{k}\sum_{t=0}^{s-1}\binom{k-s+n-2}{n-2}\binom{t+n-2}{n-2} \\ &= \sum_{s=1}^{k}\binom{k-s+n-2}{n-2}\binom{s+n-2}{n-1} = \binom{k+2(n-2)+1}{2(n-2)}. \end{aligned}$$

The proof is complete. $\qquad\square$

Recall that $\mathcal{C}_n$ is the cycle graph on N vertices. Corollary 5.2 shows that if n is odd, then $\mathcal{C}_n^{\odot 2}$ is connected. We compute its Wiener index.

PROPOSITION 7.2. Let $n \geqslant 3$ be odd, then $\mathcal{C}_n^{\odot 2}$ is connected and

$$W(\mathcal{C}_n^{\odot 2}) = \binom{n+2}{2}W(\mathcal{C}_n) + \left(n\binom{n}{2} - 2W(C_n)\right)\binom{\frac{n+3}{2}}{2} - 2n^2\binom{\frac{n+3}{2}}{3}.$$

PROOF. The set of vertices of $\mathcal{C}_n^{\odot 2}$ can be identified with $\{v_i \odot v_j : i \leqslant j\}$. Partition this set into $\frac{n+1}{2}$ blocks of n vertices such that the graph induced by each of the parts

is isomorphic to $\mathcal{C}_n$. Consider the partition of the vertices of $\mathcal{C}_n^{\odot 2}$ given by

$$V_{\mathcal{C}_n^{\odot 2}} = \bigcup_{\ell=0}^{(n-1)/2} V_\ell,$$

in which $V_\ell = \{v_i \odot v_j : j = i + \ell\}$. Each block of this partition is connected to other two (except the blocks $v_i \odot v_i$ and $v_i \odot v_{i+1}$) by the edges $v_i \odot v_j$ to $v_{i+1} \odot v_{j-1}$ and $v_{i-1} \odot v_{j+1}$ as seen in Figure 8b. Thus, $\mathcal{C}_n^{\odot 2}$ is connected and its Wiener index depends on which of the $\frac{n+1}{2}$ blocks we are taking the vertices. If d represents the distance from one block to the other, then the distance in between the elements of each block is d. There are n^2 choices of vertices and $\frac{n+1}{2} - d$ choices for the two blocks with distance d, so

$$W(\mathcal{C}_n^{\odot 2}) = \sum_{i=0}^{\frac{n+1}{2}} \left(2W(\mathcal{C}_n) + i \cdot n^2\right) \left(\frac{n+1}{2} - i\right) - \frac{n+1}{2} W(\mathcal{C}_n).$$

The result follows by expanding the product and using binomial identities. $\qquad\square$

8. Open questions

Although one can consider symmetric tensor products of distinct graphs, we have restricted our attention to symmetric powers to avoid the necessary notational hurdles. Obviously, the investigation of symmetric tensor products of distinct graphs is wide open territory that warrants exploration.

PROBLEM 8.1. Develop analogous results and observations for symmetric tensor products of distinct graphs.

PROBLEM 8.2. Figure 4 suggests that $\mathcal{G}^{\odot k}$ is always contained in $\mathcal{G}^{\odot(k+1)}$ in the sense of weighted graphs (edge weights can increase as one moves to a higher power). It looks like one can prepend 1 to each vertex label to pass to the next power. But this fails in Figure 3, even though nesting occurs. What can be said about nesting in successive symmetric tensor powers?

PROBLEM 8.3. If $k \leqslant r$, is $\mathcal{G}^{\odot k}$ a subgraph of $\mathcal{G}^{\odot r}$ in the sense of weighted graphs. That is, are the weights in $\mathcal{G}^{\odot k}$ at most the corresponding weights in $\mathcal{G}^{\odot r}$?

Both problems have a positive answer if one of the vertices has a loop. Just call that vertex 1 and prepend it to the sequence. If not, as in the example of $\mathcal{P}_3$, one can think of picking an edge and alternating it.

PROBLEM 8.4. In Appendix A, we show how the symmetric tensor power of a permutation matrix is a permutation matrix. How are the cycle decompositions of a permutation and its symmetric tensor powers related?

Appendix A. The Symmetric tensor power of a graph is well-defined

We now show that isomorphic graphs, that is, graphs whose adjacency matrices are permutation similar, have isomorphic symmetric tensor powers.

THEOREM A.1. *The symmetric tensor power operation on graphs is well defined; that is, isomorphic graphs have isomorphic symmetric powers.*

LEMMA A.1. *Let $\Gamma_\sigma \in M_n$ represent the permutation $\sigma \in \mathfrak{S}_n$, that is,*

$$[\Gamma_\sigma]_{i,j} = \begin{cases} 1 & \text{if } j = \sigma(i), \\ 0 & \text{if otherwise,} \end{cases}$$

and let $N = \binom{n+k-1}{k}$. Then $\Gamma_\sigma^{\odot k} \in M_N$ is the permutation matrix associated to $\sigma^{\odot k} \in \mathfrak{S}_N$, in which $\sigma^{\odot k}(\mathbf{x})$ is the only nondecreasing element of $[n]^k$ in $\mathrm{Orb}(\sigma(\mathbf{x}))$.

PROOF. Proposition 2.2 gives

$$\left[\Gamma_\sigma^{\odot k}\right]_{\mathbf{x},\mathbf{y}} = \frac{1}{\sqrt{\binom{k}{\mathbf{m}(\mathbf{x})}\binom{k}{\mathbf{m}(\mathbf{y})}}} \sum_{\substack{\mathbf{t}\in\mathrm{Orb}(\mathbf{x}) \\ \mathbf{q}\in\mathrm{Orb}(\mathbf{y})}} \prod_{\ell=1}^{k} [\Gamma_\sigma]_{\mathbf{t}_\ell,\mathbf{q}_\ell}.$$

If $\mathbf{z} \in \mathrm{Orb}(\sigma(\mathbf{x}))$ and $\mathbf{y} \neq \mathbf{z}$, then $[\Gamma_\sigma]_{\mathbf{t}_\ell,\mathbf{q}_\ell} = 0$. This happens exactly $|\mathrm{Orb}(\mathbf{x})| = \binom{k}{\mathbf{m}(\mathbf{x})}$ times. Since $\binom{k}{\mathbf{m}(\mathbf{x})} = \binom{k}{\mathbf{m}(\sigma(\mathbf{x}))}$, it follows that $[\Gamma_\sigma^{\odot k}]_{\mathbf{x},\sigma(\mathbf{x})} = 1$. Thus, $\Gamma_\sigma^{\odot k}$ is the permutation matrix corresponding to $\sigma^{\odot k}$ in the group of permutations of the N nondecreasing elements of $[n]^k$ (we identify this group with $\mathfrak{S}_N$). $\qquad\square$

EXAMPLE A.1. Let $n = k = 2$ and let $P = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Then (2.4) and (2.5) ensure that

$$P^{\odot 2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P^{\odot 3} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Let $\mathbf{h} = (1,1), \mathbf{i} = (2,2)$ and $\mathbf{j} = (1,2)$ be the three posible indices when $n = k = 2$. Consider $\sigma = (21)$ a permutation on two elements, then $\sigma^{\odot 2}(\mathbf{h}) = \mathbf{i}$, $\sigma^{\odot 2}(\mathbf{i}) = \mathbf{h}$ and $\sigma^{\odot 2}(\mathbf{j}) = \mathbf{j}$. Order the basis elements as $(1,1),(2,2),(1,2)$ in the nondecreasing elements of $[2]^2$.

EXAMPLE A.2. Let $n = 3$, $k = 2$, and let $P = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right]$. Then (2.6) ensures that

$$P^{\odot 2} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

By imposing a linear order on the basis of the symmetric tensor product using the lexicographic, one can check that the list of permutations obtained for $n = 3, k = 2$ is $(123)^{\odot 2} = (123456)$, $(132)^{\odot 2} = (132546)$, $(213)^{\odot 2} = (213465)$, $(231)^{\odot 2} = (231645)$, $(312)^{\odot 2} = (312564)$, and $(321)^{\odot 2} = (321654)$. Theorem 3.1 ensures that each permutation is a prefix of its second symmetric power.

PROOF OF THEOREM A.1. Suppose that $\mathcal{G}$ and $\mathcal{H}$ are isomorphic graphs with adjacency matrices A and B, respectively. Then there is a permutation $\sigma \in \mathfrak{S}_n$ and corresponding permutation matrix $\Gamma \in M_n$ such that $B = \Gamma A \Gamma^\top$. By Lemma A.1 ,

$\Gamma^{\odot k}$ is the permutation matrix such that $[\Gamma_\sigma^{\odot k}]_{\mathbf{i},\mathbf{j}} = 1$ if and only if $\mathbf{j} \in \mathrm{Orb}(\sigma(\mathbf{i}))$. For $A \in \mathrm{M}_N$, we have $[\Gamma A \Gamma^\top]_{i,j} = [A]_{\sigma(i),\sigma(j)}$. Lemma 2.2 shows that

$$
\begin{aligned}
[(\Gamma A \Gamma^\top)^{\odot k}]_{\mathbf{i},\mathbf{j}} &= \frac{1}{\sqrt{\binom{k}{\mathbf{m}(\mathbf{i})}\binom{k}{\mathbf{m}(\mathbf{j})}}} \sum_{\substack{\mathbf{t} \in \mathrm{Orb}(\mathbf{i}) \\ \mathbf{q} \in \mathrm{Orb}(\mathbf{j})}} \prod_{\ell=1}^{k} A_{\sigma(\mathbf{t}_\ell),\sigma(\mathbf{q}_\ell)} \\
&= \frac{1}{\sqrt{\binom{k}{\mathbf{m}(\mathbf{i})}\binom{k}{\mathbf{m}(\mathbf{j})}}} \sum_{\substack{\mathbf{t} \in \mathrm{Orb}(\sigma(\mathbf{i})) \\ \mathbf{q} \in \mathrm{Orb}(\sigma(\mathbf{j}))}} \prod_{\ell=1}^{k} A_{\mathbf{t}_\ell,\mathbf{q}_\ell} \\
&= \frac{1}{\sqrt{\binom{k}{\mathbf{m}(\sigma(\mathbf{i}))}\binom{k}{\mathbf{m}(\sigma(\mathbf{j}))}}} \sum_{\substack{\mathbf{t} \in \mathrm{Orb}(\sigma(\mathbf{i})) \\ \mathbf{q} \in \mathrm{Orb}(\sigma(\mathbf{j}))}} \prod_{\ell=1}^{k} A_{\mathbf{t}_\ell,\mathbf{q}_\ell} \\
&= [A^{\odot k}]_{\sigma(\mathbf{i}),\sigma(\mathbf{j})} \\
&= [\Gamma^{\odot k} A^{\odot k} (\Gamma^T)^{\odot k}]_{\mathbf{i},\mathbf{j}}.
\end{aligned}
$$

Thus, $A^{\odot k}$ and $B^{\odot k}$ are permutation similar and hence $\mathcal{G}^{\odot k}$ and $\mathcal{H}^{\odot k}$ are isomorphic graphs. □

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