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Generalization of the Laplace Transform of Powers of the sinc function

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Abstract. In this work, general expressions are found for the Laplace transform of $\mathcal{L}(\frac{\sin^m(at)}{t^n})$ or $\mathcal{L}(\frac{\sin^n(t)\sin^m(at)}{t^p})$. The results are expressed in terms of usual functions such as logarithms, arctangent, and rational functions. Thus, these expressions can be incorporated into the general literature to facilitate the calculation of Laplace transforms. In addition, the results of the article can be implemented in different mathematical software for a direct calculation of such transforms or to study the analytical properties of these transforms more directly.

1. Introduction

Let $f(t)$ be a function with exponential order defined for $t \geqslant 0$. Let us define its Laplace transform as the integral transform

$$\mathcal{L}(f(t))(s) = \int_0^\infty f(t)e^{-st}\,dt, \quad \Re(s) > \sigma_0,$$

for some σ_0 real. Throughout this work, we will denote the Laplace transform of $f(t)$ evaluated in s as $\mathcal{L}(f(t))$ or $F(s)$.

The Laplace transform has many applications in science and engineering, especially when solving linear differential equations, since it transforms them into algebraic equations, which are much easier to manipulate.

The cardinal sine function $\operatorname{sinc}(t)$ is defined as

$$\operatorname{sinc}(t) = \begin{cases} 1 & \text{if } t = 0, \\ \frac{\sin(t)}{t} & \text{if } t \neq 0. \end{cases}$$

This function is frequently used for processing digital signals and in information theory.

The initial purpose of this paper is to find the Laplace transform of powers of this function, i.e.,

$$\mathcal{L}(\operatorname{sinc}^n(t)), \quad n \geqslant 1, \tag{1.1}$$

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in terms of elementary functions.

Currently, results for the cases $n = 1$ and $n = 2$ are known. Indeed,

$$\mathcal{L}(\mathrm{sinc}(t)) = \arctan(1/s)$$

and

$$\mathcal{L}(\mathrm{sinc}^2(t)) = \arctan\left(\frac{2}{s}\right) - \frac{s}{4}\log\left(\frac{s^2+4}{s^2}\right),$$

see for example [**4**] or [**6**].

However, after a comprehensive search of the literature by the author, explicit results for $n \geqslant 3$ in terms of elementary functions are very difficult to obtain. Some software programs can find these results for small values of n through some available algorithms, but for larger values, these methods are very time-inefficient.

In [**3**], the author presents a general result for the Laplace transform of the absolute value of the sinc function:

$$\int_0^\infty \left|\frac{\sin(t)}{t}\right|^p e^{-st}\, dt,$$

for p a positive integer. The result is expressed in terms of other integrals that can be computed in terms of elementary functions for p even, but not so easily computable for p odd.

In [**2**], the author presents a result for the Laplace transform of a product function of the form

$$\mathcal{L}(f(t)G(t)),$$

where $f(t)$ is a function on t and $G(t)$ is the Laplace transform of some other function g. The author also shows various examples for the Laplace transform of products or ratios of functions, like $\mathcal{L}((1-\cos(t))/t^2)$ or $\mathcal{L}((\sin(t)-t-\cos(t))/t^3)$, without establishing general results for ratios of higher powers of sines or cosines and t.

In [**1**], the authors find a general result for the Fourier transform of powers of the sinc function. The authors also present a quite comprehensive historical review of the computation of integrals related to the sinc function and its powers, while indicating that such results rarely appear in the literature.

In this work, we present results more general than those initially established, eventually proving in Theorem 2.1 a result for

$$\mathcal{L}\Big(\frac{\sin^m(at)}{t^n}\Big),$$

for $m \geqslant n$ positive integers and $a > 0$, and another result in Theorem 3.3 for

$$\mathcal{L}\Big(\frac{\sin^n(t)\sin^m(at)}{t^p}\Big),$$

for $m + n \geqslant p$ integers and $a > 0$. Both results are expressed in terms of elementary functions and can be used/implemented directly for the computation of these transforms.

2. Powers of sines over powers of t

Let $n \geqslant 1$. We know that $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx$. If we make the change of variable $x = ut$ for $t > 0$ fixed, we obtain $\Gamma(n) = t^n \int_0^\infty u^{n-1}e^{-ut}\,du$, so

$$\frac{1}{t^n} = \frac{1}{(n-1)!}\int_0^\infty u^{n-1}e^{-ut}\,du; \quad t > 0, n \in \mathbb{N}.$$

Suppose we want to find $\mathcal{L}\left(\frac{f(t)}{t^n}\right)$, for some integer $n \geqslant 1$ and $t \geqslant 0$ such that $f(t)$ is a function with known Laplace transform $F(s)$. We have that

$$\begin{aligned}\mathcal{L}\left(\frac{f(t)}{t^n}\right) &= \int_0^\infty \frac{f(t)}{t^n}e^{-st}\,dt = \frac{1}{(n-1)!}\int_0^\infty f(t)e^{-st}\int_0^\infty u^{n-1}e^{-ut}\,du\,dt \\ &= \frac{1}{(n-1)!}\int_0^\infty u^{n-1}\int_0^\infty f(t)e^{-(s+u)t}\,dt\,du = \frac{1}{(n-1)!}\int_0^\infty u^{n-1}F(s+u)\,du.\end{aligned} \tag{2.1}$$

Therefore,

$$\mathcal{L}\left(\frac{f(t)}{t^n}\right) = \frac{1}{(n-1)!}\int_s^\infty (y-s)^{n-1}F(y)\,dy, \tag{2.2}$$

if and only if the integral on the right-hand side is well defined. Another way to justify (2.2) is found in [**1**], where, for some $f(t), g(t)$ with Laplace transforms $F(s), G(s)$ that satisfy certain integrability conditions, we have that

$$\mathcal{L}(f(t)G(t)) = \int_s^\infty F(y)g(y-s)\,dy. \tag{2.3}$$

In this case, it is enough to let $g(t) = \frac{t^{n-1}}{(n-1)!}$ to justify (2.2). Notice that (2.3) in a manner gives us a sort of result similar to the theorem of Convolution for the inverse Laplace transform of a product of functions.

The result given in (2.2) comes in handy when, for example, $F(y)$ is a rational function on y. This is the case when $f(t) = \sin^m(t)$ is a power of the sine function with $m \geqslant 1$ integer.

In this section, we want to compute

$$\mathcal{L}\left(\frac{\sin^m(at)}{t^n}\right) \tag{2.4}$$

with $a > 0, m \geqslant n$ positive integers. We start by computing (2.4) for the particular case when $a = 1$, after which we use the general fact that $\mathcal{L}(f(at)) = \frac{1}{a}F(\frac{s}{a})$.

Let $I_m(s) := \mathcal{L}(\sin^m(t)) = \int_0^\infty \sin^m(t)e^{-st}\,dt$ with $m \geqslant 0$. Integrating twice by parts, we obtain for $s > 0$

$$I_m(s) = \frac{m}{s^2}\Big((m-1)I_{m-2}(s) - mI_m(s)\Big),$$

and then

$$I_m(s) = \frac{m(m-1)}{m^2+s^2}I_{m-2}(s); \quad m \geqslant 2.$$

Additionally, we know that $I_0(s) = \frac{1}{s}; I_1(s) = \frac{1}{s^2+1}$. Therefore,

$$I_m(s) = \begin{cases} \frac{m!}{s\prod_{k=1}^{m/2}(s^2+(2k)^2)}, & \text{if } m \text{ is even} \\ \frac{m!}{s\prod_{k=1}^{(m+1)/2}(s^2+(2k-1)^2)}, & \text{if } m \text{ is odd.} \end{cases} \tag{2.5}$$

Thus, we obtain the following result for (2.4).

THEOREM 2.1. For $a > 0$, the Laplace transform $L := \mathcal{L}\left(\frac{\sin^m(at)}{t^n}\right)$ for $m \geqslant n$ positive integers, satisfies for $s > 0$

- if m is even, then

$$L = \frac{(-a)^{n-1}}{2^m(n-1)!}\left(\sum_{j=1}^{m/2}(-1)^j\binom{m}{m/2-j}\left(2\arctan\left(\frac{2ja}{s}\right)\sum_{l=0}^{\lfloor\frac{n-2}{2}\rfloor}(-1)^l\binom{n-1}{2l+1}(2j)^{2l+1}(s/a)^{n-2-2l}\right.\right.$$
$$\left.\left. -\log\left((s/a)^2+4j^2\right)\sum_{l=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^l\binom{n-1}{2l}(2j)^{2l}(s/a)^{n-1-2l}\right) - (s/a)^{n-1}\binom{m}{m/2}\log(s/a)\right);$$

- if m is odd, then

$$L = \frac{(-s)^{n-1}}{2^m(n-1)!}\left(\sum_{j=1}^{(m+1)/2}(-1)^{j-1}\binom{m}{(m+1)/2-j}\cdot\left(2\arctan\left(\frac{(2j-1)a}{s}\right)\sum_{l=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^l\binom{n-1}{2l}\left(\frac{(2j-1)a}{s}\right)^{2l}\right.\right.$$
$$\left.\left. +\log\left((s/a)^2+(2j-1)^2\right)\sum_{l=0}^{\lfloor\frac{n-2}{2}\rfloor}(-1)^l\binom{n-1}{2l+1}\left(\frac{(2j-1)a}{s}\right)^{2l+1}\right)\right)$$

PROOF. Let us prove the case m even and $a = 1$; for the general case, as noted before, we use

$$\mathcal{L}(f(at)) = \frac{1}{a}F(s/a)$$

Let $I_{m,n}(s) := \int_0^\infty \frac{sin^m(t)}{t^n}e^{-st}\,dt$ be the transform $\mathcal{L}\left(\frac{\sin^m(t)}{t^n}\right)$. From (2.2) and (2.5) we have that

$$I_{m,n}(s) = \frac{1}{(n-1)!}\int_s^\infty (y-s)^{n-1}I_m(y)\,dy, \quad (s>0).$$

By letting $y = 2z$ we obtain

$$I_{m,n}(s) = \frac{m!}{2^m(n-1)!}\int_{s/2}^\infty \frac{(2z-s)^{n-1}}{z(z^2+1^2)(z^2+2^2)\cdots(z^2+(m/2)^2)}\,dz. \tag{2.6}$$

We observe that the integrand in (2.6) is a rational function in z with the degree of the numerator $n-1$, which is less than the degree of the denominator $m+1$. By decomposing the integrand into partial fractions,

$$\frac{(2z-s)^{n-1}}{z(z^2+1^2)(z^2+2^2)\cdots(z^2+(m/2)^2)} = \frac{A_0}{z}+\frac{A_1z+B_1}{z^2+1^2}+\cdots+\frac{A_{m/2}z+B_{m/2}}{z^2+(m/2)^2}. \tag{2.7}$$

We have that

$$A_0 = \frac{(-s)^{n-1}}{(m/2)!^2}.$$

For $1 \leqslant j \leqslant m/2$ we have to solve the system

$$\begin{cases} (jA_j i + B_j)(ij) \prod_{\substack{k=1\\k\neq j}}^{m/2} (k^2 - j^2) = (2ij - s)^{n-1} \\ (-jA_j i + B_j)(-ij) \prod_{\substack{k=1\\k\neq j}}^{m/2} (k^2 - j^2) = (-2ij - s)^{n-1}, \end{cases} \tag{2.8}$$

where i is the pure imaginary number. By using the difference of squares and by completing squares, we have

$$\prod_{\substack{k=1\\k\neq j}}^{m/2} (k^2 - j^2) = \frac{(-1)^{j-1} m!}{2j^2 \binom{m}{m/2-j}}.$$

Solving the system (2.8), we obtain

$$\begin{cases} A_j = \frac{(-1)^j \binom{m}{m/2-j}}{m!} \Big((2ij - s)^{n-1} + (-2ij - s)^{n-1} \Big) \\ B_j = \frac{i(-1)^j j \binom{m}{m/2-j}}{m!} \Big((2ij - s)^{n-1} - (-2ij - s)^{n-1} \Big), \end{cases}$$

that is,

$$\begin{cases} A_j = \frac{2(-1)^j \binom{m}{m/2-j}}{m!} \Re((2ij - s)^{n-1}) \\ B_j = \frac{-2(-1)^j j \binom{m}{m/2-j}}{m!} \Im((2ij - s)^{n-1}). \end{cases} \tag{2.9}$$

Continuing the computations in (2.6) we have

(2.10)

$$\begin{aligned} I_{m,n}(s) &= \tfrac{m!}{2^m (n-1)!} \int_{s/2}^{\infty} \left(\tfrac{A_0}{z} + \sum_{j=1}^{m/2} \tfrac{A_j z + B_j}{z^2 + j^2} \right) dz \\ &= \tfrac{m!}{2^m (n-1)!} \lim_{M\to\infty} \left(\log \left(z^{A_0} (z^2+1)^{A_1/2} \cdots (z^2 + (m/2)^2)^{A_{m/2}} \right) + \sum_{j=1}^{m/2} \tfrac{B_j}{j} \arctan\left(\tfrac{z}{j}\right) \right) \Bigg|_{s/2}^{M}. \end{aligned}$$

Moreover, by comparing the coefficients of z^m in the numerators of decomposition (2.7) we see that

$$A_0 + A_1 + \cdots + A_{m/2} = 0,$$

so that the limit of the logarithm when M tends to ∞ in (2.10) vanishes. Then

(2.11)

$$I_{m,n}(s) = \frac{m!}{2^m (n-1)!} \left(\sum_{j=1}^{m/2} \frac{B_j}{j} \arctan\left(\frac{2j}{s}\right) - \Big(A_0 \log(s) + \frac{1}{2} \sum_{j=1}^{m/2} A_j \log(s^2 + 4j^2) \Big) \right).$$

Finally, using Newton's Binomial Theorem, we have that

$$(2ij-s)^{n-1}=\sum_{k=0}^{n-1}\binom{n-1}{k}(2j)^k(-s)^{n-1-k}i^k,$$

which implies that

$$\Re((2ij-s)^{n-1})=(-1)^{n-1}\sum_{l=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^l\binom{n-1}{2l}(2j)^{2l}s^{n-1-2l}$$
$$\Im((2ij-s)^{n-1})=(-1)^{n-2}\sum_{l=0}^{\lfloor\frac{n-2}{2}\rfloor}(-1)^l\binom{n-1}{2l+1}(2j)^{2l+1}s^{n-2-2l}.$$

Replacing all these results in (2.9) and (2.11), we have the result for $a=1$.

The proof for the case when m is odd is quite similar, we use that if $1\leqslant j\leqslant\frac{m+1}{2}$, then

$$\prod_{\substack{k=1\\k\neq j}}^{(m+1)/2}((2k-1)^2-(2j-1)^2)=\frac{(-1)^{j-1}2^m m!}{(4j-2)\binom{m}{(m+1)/2-j}}.$$

□

3. Product of powers of sines over powers of t

Let n,m,p be positive integers, with $p\leqslant n+m$, and let $a>0$. We want to find the transform

$$\mathcal{L}\Big(\frac{\sin^n t\sin^m(at)}{t^p}\Big).$$

Based on what we explained in (2.2), first we need to find

$$\mathcal{L}(\sin^n(t)\sin^m(at)).$$

The algebraic computations are as follows:

(3.1)

$$\sin^n(t)\sin^m(at)=\Big(\frac{e^{it}-e^{-it}}{2i}\Big)^n\Big(\frac{e^{iat}-e^{-iat}}{2i}\Big)^m=\frac{e^{it(n+am)}}{2^{n+m}i^{n+m}}(1-e^{-2it})^n(1-e^{-2ait})^m$$
$$=\frac{e^{it(n+am)}}{2^{n+m}i^{n+m}}\sum_{\substack{0\leqslant k\leqslant n\\0\leqslant l\leqslant m}}(-1)^{k+l}\binom{n}{k}\binom{m}{l}e^{-2it(k+al)}$$
$$=\frac{1}{2^{m+n}i^{m+n}}\sum_{\substack{0\leqslant k\leqslant n\\0\leqslant l\leqslant m}}(-1)^{k+l}\binom{n}{k}\binom{m}{l}e^{it((n-2k)+a(m-2l))}.$$

Expanding, for example, this last expression for the case when n, m are both odd,

$$
\begin{aligned}
&\sin^n(t)\sin^m(at) = \\
&= \frac{1}{2^{m+n} i^{m+n}} \sum_{\substack{0\leqslant k\leqslant \lfloor n/2\rfloor \\ 0\leqslant l\leqslant \lfloor m/2\rfloor}} \Bigg((-1)^{k+l}\binom{n}{k}\binom{m}{l} e^{it((n-2k)+a(m-2l))} \\
&+ (-1)^{n-k+m-l}\binom{n}{n-k}\binom{m}{m-l} e^{it((-n+2k)+a(-m+2l))} \\
&+ (-1)^{n-k+l}\binom{n}{n-k}\binom{m}{l} e^{it((-n+2k)+a(m-2l))} \\
&+ (-1)^{k+m-l}\binom{n}{k}\binom{m}{m-l} e^{it((n-2k)+a(-m+2l))} \Bigg) \\
&= \frac{1}{2^{m+n} i^{m+n}} \sum_{\substack{0\leqslant k\leqslant \lfloor n/2\rfloor \\ 0\leqslant l\leqslant \lfloor m/2\rfloor}} \Bigg((-1)^{k+l}\binom{n}{k}\binom{m}{l} \cos\big((n-2k+a(m-2l))t\big) \\
&- (-1)^{k-l}\binom{n}{k}\binom{m}{l} \cos\big((n-2k-a(m-2l))t\big) \Bigg) \\
&= \frac{(-1)^{(m+n)/2}}{2^{m+n-1}} \sum_{\substack{0\leqslant k\leqslant \lfloor n/2\rfloor \\ 0\leqslant l\leqslant \lfloor m/2\rfloor}} (-1)^{k+l}\binom{n}{k}\binom{m}{l} \Bigg(\cos\big((n-2k+a(m-2l))t\big) \\
&- \cos\big((n-2k-a(m-2l))t\big) \Bigg).
\end{aligned}
$$

We can proceed similarly for the cases when n and m change parity. Since Laplace's transform satisfies linearity and since $\mathcal{L}(\sin(wt)) = \frac{w}{s^2+w^2}, \mathcal{L}(\cos(wt)) = \frac{s}{s^2+w^2}$, we have the following result.

LEMMA 3.1. Let n, m be positive integers, and let $a > 0$, for $s > 0$

i) If n, m are even, then

$$
\begin{aligned}
\mathcal{L}(\sin^n t \sin^m(at)) = \frac{(-1)^{(m+n)/2}}{2^{m+n-1}} &\sum_{\substack{0\leqslant k\leqslant (n-1)/2 \\ 0\leqslant l\leqslant (m-1)/2}} (-1)^{k+l}\binom{n}{k}\binom{m}{l} \\
\cdot \left(\frac{s}{s^2+(n-2k+a(m-2l))^2} - \frac{s}{s^2+(n-2k-a(m-2l))^2} \right)
\end{aligned}
$$

ii) If n is odd and m is even, then

$$\mathcal{L}(\sin^n(t)\sin^m(at)) = \frac{(-1)^{(m+n-1)/2}}{2^{m+n-1}}\left(\sum_{\substack{0\leqslant k\leqslant(n-1)/2\\0\leqslant l\leqslant(m-2)/2}}(-1)^{k+l}\binom{n}{k}\binom{m}{l}\right.$$
$$\cdot\left(\frac{(n-2k)+a(m-2l)}{s^2+(n-2k+a(m-2l))^2}+\frac{(n-2k)-a(m-2l)}{s^2+(n-2k-a(m-2l))^2}\right)$$
$$\left.+(-1)^{m/2}\binom{m}{m/2}\sum_{0\leqslant k\leqslant(n-1)/2}(-1)^k\binom{n}{k}\frac{n-2k}{s^2+(n-2k)^2}\right)$$

iii) If n is even and m is odd, then

$$\mathcal{L}(\sin^n(t)\sin^m(at)) = \frac{(-1)^{(m+n-1)/2}}{2^{m+n-1}}\left(\sum_{\substack{0\leqslant k\leqslant(n-2)/2\\0\leqslant l\leqslant(m-1)/2}}(-1)^{k+l}\binom{n}{k}\binom{m}{l}\right.$$
$$\cdot\left(\frac{(n-2k)+a(m-2l)}{s^2+(n-2k+a(m-2l))^2}-\frac{(n-2k)-a(m-2l)}{s^2+(n-2k-a(m-2l))^2}\right)$$
$$\left.+(-1)^{n/2}\binom{n}{n/2}\sum_{0\leqslant k\leqslant(m-1)/2}(-1)^l\binom{m}{l}\frac{a(m-2l)}{s^2+(a(m-2l))^2}\right)$$

iv) If n, m are even, then

$$\mathcal{L}(\sin^n(t)\sin^m(at)) = \frac{(-1)^{(m+n)/2}}{2^{m+n-1}}\left(\sum_{\substack{0\leqslant k\leqslant(n-2)/2\\0\leqslant l\leqslant(m-2)/2}}(-1)^{k+l}\tbinom{n}{k}\tbinom{m}{l}\cdot\left(\tfrac{s}{s^2+(n-2k+a(m-2l))^2}\right.\right.$$
$$\left.+\frac{s}{s^2+(n-2k-a(m-2l))^2}\right)+(-1)^{n/2}\binom{n}{n/2}\sum_{0\leqslant l\leqslant(m-2)/2}(-1)^l\binom{m}{l}\frac{s}{s^2+(a(m-2l))^2}$$
$$\left.+(-1)^{m/2}\binom{m}{m/2}\sum_{0\leqslant k\leqslant(n-2)/2}(-1)^k\binom{n}{k}\frac{s}{s^2+(n-2k)^2}\right)+\frac{1}{2^{m+n}}\frac{\binom{n}{n/2}\binom{m}{m/2}}{s}.$$

In Lemma 3.1 we can see that $\mathcal{L}(\sin^n(t)\sin^m(at))$ is a rational function in s. However, it is not clear how large the degree of the denominator is relative to that of the numerator. The following result clarifies this situation.

LEMMA 3.2. For $n, m \geqslant 0$ integers, let $\mathcal{L}(\sin^n(t)\sin^m(at)) = \frac{P_{n,m}(s)}{Q_{n,m}(s)}$, where the numerator and denominator are polynomials of degrees N and D, respectively. Then $D - N = n + m - 1$.

PROOF. By induction on n. If $n = 0$, the result follows from (2.5). Suppose the result holds for n and prove it for $n+1$. We have that

$$\begin{aligned}\mathcal{L}\Big(\sin^{n+1}(t)\sin^m(at)\Big) &= \mathcal{L}\Big(\Big(\frac{e^{it}-e^{-it}}{2i}\Big)\sin^n(t)\sin^m(at)\Big)\\ &= \frac{1}{2i}\left(\frac{P_{n,m}(s-i)}{Q_{n,m}(s-i)} - \frac{P_{n,m}(s+i)}{Q_{n,m}(s+i)}\right) = \frac{P_{n+1,m}(s)}{Q_{n+1,m}(s)}\end{aligned}$$

where the degree of $Q_{n+1,m}(s)$ equals $2D - r$ and the degree of $P_{n+1,m}(s)$ equals $N + D - r - 1$, where -1 is because the final dominant terms of the numerator cancel each other out, and the -r term comes from the possibility of simplification of the whole quotient. Therefore,

$$\deg(Q_{n+1,m}(s)) - \deg(P_{n+1,m}(s)) = D - N + 1,$$

which, by induction hypothesis, equals $n + m + 2$. □

Let n, m, p be positive integers, with $p \leqslant n + m$, and let $a > 0$. Let us find

$$\mathcal{L}\Big(\frac{\sin^n(t)\sin^m(at)}{t^p}\Big).$$

For this, we use (2.2), taking into account that $\mathcal{L}(\sin^n(t)\sin^m(at))$ has been computed in Lemma 3.1.

Let us do the computations for n, m odd. We have that

$$\begin{aligned}L =& \frac{(-1)^{(m+n)/2}}{2^{m+n-1}(p-1)!}\cdot\int_s^\infty (y-s)^{p-1}\sum_{\substack{0\leqslant k\leqslant(n-1)/2\\0\leqslant l\leqslant(m-1)/2}}(-1)^{k+l}\binom{n}{k}\binom{m}{l}\\ &\cdot\left(\frac{y}{y^2+(n-2k+a(m-2l))^2} - \frac{y}{y^2+(n-2k-a(m-2l))^2}\right)dy.\end{aligned}$$

The integrand of this last expression is a rational function in the variable y, so by Lemma 3.2, the difference between the degrees of numerator and denominator equals

$$(n+m+1)-(p-1) = n+m-p+2 \geqslant 2. \tag{3.2}$$

Therefore, the integrand can be decomposed into partial fractions. Then we have

$$\begin{aligned}L =& \frac{(-1)^{(m+n)/2}}{2^{m+n-1}(p-1)!}\cdot\int_s^\infty \sum_{\substack{0\leqslant k\leqslant(n-1)/2\\0\leqslant l\leqslant(m-1)/2}}(-1)^{k+l}\binom{n}{k}\binom{m}{l}\\ &\cdot\left(\frac{a_{kl}y+b_{kl}}{y^2+(n-2k+a(m-2l))^2} - \frac{c_{kl}y+d_{kl}}{y^2+(n-2k-a(m-2l))^2}\right)dy,\end{aligned} \tag{3.3}$$

where

$$
\begin{aligned}
a_{kl} &= \Re\Big(\big((n-2k+a(m-2l))i-s\big)^{p-1}\Big)\\
b_{kl} &= -(n-2k+a(m-2l))\Im\Big(\big((n-2k+a(m-2l))i-s\big)^{p-1}\Big)\\
c_{kl} &= \Re\Big(\big((n-2k-a(m-2l))i-s\big)^{p-1}\Big)\\
d_{kl} &= -(n-2k-a(m-2l))\Im\Big(\big((n-2k-a(m-2l))i-s\big)^{p-1}\Big)
\end{aligned}
$$

By (3.2), the dominant term in the numerator of the integrand in (3.3) is zero, that is,

$$
\sum_{\substack{0\leqslant k\leqslant (n-1)/2\\ 0\leqslant l\leqslant (m-1)/2}} (-1)^{k+l}\binom{n}{k}\binom{m}{l}(a_{kl}-c_{kl}) = 0. \tag{3.4}
$$

Let us now expand

$$
\begin{aligned}
\Big((n-2k+a(m-2l))i-s\Big)^{p-1} &= \sum_{j=0}^{p-1}\binom{p-1}{j}\Big(((n-2k)+a(m-2l))i\Big)^{j}(-s)^{p-1-j}\\
= (-s)^{p-1}&\sum_{j=0}^{\lfloor (p-1)/2\rfloor}(-1)^j\binom{p-1}{2j}\Big(\frac{n-2k+a(m-2l)}{s}\Big)^{2j}\\
+\, i(-s)^{p-2}(n-2k+a(m-2l))&\sum_{j=0}^{\lfloor (p-2)/2\rfloor}(-1)^j\binom{p-1}{2j+1}\Big(\frac{n-2k+a(m-2l)}{s}\Big)^{2j}.
\end{aligned}
$$

We then have that

$$
\begin{aligned}
a_{kl} &= (-s)^{p-1}\sum_{j=0}^{\lfloor (p-1)/2\rfloor}(-1)^j\binom{p-1}{2j}\Big(\frac{n-2k+a(m-2l)}{s}\Big)^{2j}\\
b_{kl} &= -(-s)^{p-2}(n-2k+a(m-2l))^2\sum_{j=0}^{\lfloor (p-2)/2\rfloor}(-1)^j\binom{p-1}{2j+1}\Big(\frac{n-2k+a(m-2l)}{s}\Big)^{2j-1},
\end{aligned}
$$

and the coefficients c_{kl} y d_{kl} similar to a_{kl} and b_{kl}, but with $(n-2k)-a(m-2l)$ instead of $(n-2k)+a(m-2l)$, respectively.

We integrate (3.3) term by term similarly as done in (2.10), taking into account (3.4) so that the terms with logarithms disappear at infinity. We thus obtain the following result for n, m odd, which can be proved similarly for the cases of different parity.

THEOREM 3.3. Let m, n, p be positive integers, with $p \leqslant m+n$, let $a>0$, let $L := \mathcal{L}\Big(\frac{\sin^n(t)\sin^m(at)}{t^p}\Big)$, and let

$$A_{kl}(s) := s^{p-1}\sum_{j=0}^{\lfloor (p-1)/2\rfloor}(-1)^j\binom{p-1}{2j}\Big(\frac{n-2k+a(m-2l)}{s}\Big)^{2j}$$

$$C_{kl}(s) := s^{p-1}\sum_{j=0}^{\lfloor (p-1)/2\rfloor}(-1)^j\binom{p-1}{2j}\Big(\frac{n-2k-a(m-2l)}{s}\Big)^{2j}$$

$$E_{kl}(s) := s^{p-1}\sum_{j=0}^{\lfloor (p-1)/2\rfloor}(-1)^j\binom{p-1}{2j}\Big(\frac{n-2k}{s}\Big)^{2j}$$

$$\overline{E}_{kl}(s) := s^{p-1}\sum_{j=0}^{\lfloor (p-1)/2\rfloor}(-1)^j\binom{p-1}{2j}\Big(\frac{a(m-2l)}{s}\Big)^{2j}$$

$$B_{kl}(s) := s^{p-2}\sum_{j=0}^{\lfloor (p-2)/2\rfloor}(-1)^j\binom{p-1}{2j+1}\Big(\frac{n-2k+a(m-2l)}{s}\Big)^{2j}$$

$$D_{kl}(s) := s^{p-2}\sum_{j=0}^{\lfloor (p-2)/2\rfloor}(-1)^j\binom{p-1}{2j+1}\Big(\frac{n-2k-a(m-2l)}{s}\Big)^{2j}$$

$$F_{kl}(s) := s^{p-2}\sum_{j=0}^{\lfloor (p-2)/2\rfloor}(-1)^j\binom{p-1}{2j+1}\Big(\frac{n-2k}{s}\Big)^{2j}$$

$$\overline{F}_{kl}(s) := s^{p-2}\sum_{j=0}^{\lfloor (p-2)/2\rfloor}(-1)^j\binom{p-1}{2j+1}\Big(\frac{a(m-2l)}{s}\Big)^{2j}$$

i) If n, m are odd, then

$$L = \frac{(-1)^{(m+n)/2+p-1}}{2^{m+n-1}(p-1)!}\sum_{\substack{0\leqslant k\leqslant (n-1)/2\\ 0\leqslant l\leqslant (m-1)/2}}(-1)^{k+l}\binom{n}{k}\binom{m}{l}\Bigg(-\frac{A_{kl}(s)}{2}\log\Big(s^2+\big(n-2k+a(m-2l)\big)^2\Big)+$$
$$\frac{C_{kl}(s)}{2}\log\Big(s^2+\big(n-2k-a(m-2l)\big)^2\Big)+B_{kl}(s)\big(n-2k+a(m-2l)\big)\arctan\Big(\frac{n-2k+a(m-2l)}{s}\Big)$$
$$-D_{kl}(s)\big(n-2k-a(m-2l)\big)\arctan\Big(\frac{n-2k-a(m-2l)}{s}\Big)\Bigg)$$

ii) If n is odd and m is even, then

$$L = \frac{(-1)^{(m+n-1)/2+p-1}}{2^{m+n-1}(p-1)!} \cdot$$

$$\left((-1)^{m/2} \binom{m}{m/2} \sum_{k=0}^{(n-1)/2} (-1)^k \binom{n}{k} \left(\frac{n-2k}{2} F_{kl}(s) \log \left(s^2 + (n-2k)^2 \right) + E_{kl}(s) \arctan \left(\frac{n-2k}{s} \right) \right) \right.$$

$$+ \sum_{\substack{0 \leqslant k \leqslant (n-1)/2 \\ 0 \leqslant l \leqslant (m-2)/2}} (-1)^{k+l} \binom{n}{k} \binom{m}{l} \left(\frac{B_{kl}(s)}{2} \big(n-2k+a(m-2l)\big) \log \left(s^2 + \big(n-2k+a(m-2l)\big)^2 \right) + \right.$$

$$A_{kl}(s) \arctan \left(\frac{n-2k+a(m-2l)}{s} \right) + \frac{D_{kl}(s)}{2} \big(n-2k-a(m-2l)\big) \log \left(s^2 + \big(n-2k-a(m-2l)\big)^2 \right)$$

$$\left. \left. + C_{kl}(s) \arctan \left(\frac{n-2k-a(m-2l)}{s} \right) \right) \right)$$

iii) If n is even and m is odd, then

$$L = \frac{(-1)^{(m+n-1)/2+p-1}}{2^{m+n-1}(p-1)!} \cdot$$

$$\left((-1)^{n/2} \binom{n}{n/2} \sum_{l=0}^{(m-1)/2} (-1)^k \binom{m}{l} \left(\frac{a(m-2l)}{2} \overline{F}_{kl}(s) \log \left(s^2 + (a(m-2l))^2 \right) + \overline{E}_{kl}(s) \arctan \left(\frac{a(m-2l)}{s} \right) \right) \right.$$

$$+ \sum_{\substack{0 \leqslant k \leqslant (n-2)/2 \\ 0 \leqslant l \leqslant (m-1)/2}} (-1)^{k+l} \binom{n}{k} \binom{m}{l} \left(\frac{B_{kl}(s)}{2} \big(n-2k+a(m-2l)\big) \log \left(s^2 + \big(n-2k+a(m-2l)\big)^2 \right) + \right.$$

$$A_{kl}(s) \arctan \left(\frac{n-2k+a(m-2l)}{s} \right) - \frac{D_{kl}(s)}{2} \big(n-2k-a(m-2l)\big) \log \left(s^2 + \big(n-2k-a(m-2l)\big)^2 \right)$$

$$\left. \left. - C_{kl}(s) \arctan \left(\frac{n-2k-a(m-2l)}{s} \right) \right) \right)$$

iv) If n, m are even, then

$$L = -\frac{\binom{n}{n/2}\binom{m}{m/2}}{2^{m+n}(p-1)!}(-s)^{p-1}\log(s) + \frac{(-1)^{(m+n)/2+p-1}}{2^{m+n-1}(p-1)!}\cdot$$
$$\Bigg(\sum_{\substack{0\leqslant k\leqslant (n-2)/2\\ 0\leqslant l\leqslant (m-2)/2}} (-1)^{k+l}\binom{n}{k}\binom{m}{l}\Big(-\frac{A_{kl}(s)}{2}\log\Big(s^2+\big(n-2k+a(m-2l)\big)^2\Big)-$$
$$\frac{C_{kl}(s)}{2}\log\Big(s^2+\big(n-2k-a(m-2l)\big)^2\Big)+B_{kl}(s)\big(n-2k+a(m-2l)\big)\arctan\Big(\frac{n-2k+a(m-2l)}{s}\Big)$$
$$+D_{kl}(s)\big(n-2k-a(m-2l)\big)\arctan\Big(\frac{n-2k-a(m-2l)}{s}\Big)\Big)+$$
$$(-1)^{n/2}\binom{n}{n/2}\sum_{l=0}^{(m-2)/2}(-1)^l\binom{m}{l}\Big(\frac{-\overline{E}_{kl}(s)}{2}\log\Big(s^2+(a(m-2l))^2\Big)$$
$$+\overline{F}_{kl}(s)(a(m-2l))\arctan\Big(\frac{a(m-2l)}{s}\Big)\Big)$$
$$+(-1)^{m/2}\binom{m}{m/2}\sum_{k=0}^{(n-2)/2}(-1)^k\binom{n}{k}\Big(\frac{-E_{kl}(s)}{2}\log\Big(s^2+(n-2k)^2\Big)+F_{kl}(s)(n-2k)\arctan\Big(\frac{n-2k}{s}\Big)\Big)\Bigg)$$

4. Applications

1. In Theorem 2.1, if we let $a = 1$ and $m = n$ we obtain the Laplace Transform for the powers of sinc. We then have that $L = \mathcal{L}(\operatorname{sinc}^n(t))$ satisfies the following:

a) If n is even, then

$$L = \frac{s^{n-2}}{2^n(n-1)!}\Bigg(\sum_{j=1}^{n/2}(-1)^j\binom{n}{n/2-j}\Big(-4j\arctan(2j/s)\sum_{l=0}^{n/2-1}(-1)^l\binom{n-1}{2l+1}(2j/s)^{2l}$$
$$+s\log(s^2+4j^2)\sum_{l=0}^{n/2-1}(-1)^l\binom{n-1}{2l}(2j/s)^{2l}\Big)+\binom{n}{n/2}s\log(s)\Bigg)$$

b) If n is odd, then

$$L = \frac{(-s)^{n-1}}{2^n(n-1)!}$$
$$\cdot\Bigg(\sum_{j=1}^{(n+1)/2}(-1)^{j-1}\binom{n}{(n+1)/2-j}\Big(2\arctan((2j-1)/s)\sum_{l=0}^{(n-1)/2}(-1)^l\binom{n-1}{2l}((2j-1)/s)^{2l}$$
$$+\log(s^2+(2j-1)^2)\sum_{l=0}^{(n-3)/2}(-1)^l\binom{n-1}{2l+1}((2j-1)/s)^{2l+1}\Big)\Bigg)$$

Thus, the Laplace Transform of the first five powers of sinc are:

$$\mathcal{L}(\mathrm{sinc}(t)) = \arctan\left(\frac{1}{s}\right)$$

$$\mathcal{L}(\mathrm{sinc}^2(t)) = \frac{1}{4}\left(-s\log\left(s^2+4\right) + 2s\log(s) + 4\arctan\left(\frac{2}{s}\right)\right)$$

$$\mathcal{L}(\mathrm{sinc}^3(t)) = \frac{1}{16}s^2\Bigg(-\frac{6\log\left(s^2+9\right)}{s} - 2\left(1-\frac{9}{s^2}\right)\arctan\left(\frac{3}{s}\right)$$
$$+3\left(\frac{2\log\left(s^2+1\right)}{s} + 2\left(1-\frac{1}{s^2}\right)\arctan\left(\frac{1}{s}\right)\right)\Bigg)$$

$$\mathcal{L}(\mathrm{sinc}^4(t)) = \frac{1}{96}s^2\Bigg(\left(1-\frac{48}{s^2}\right)s\log\left(s^2+16\right) - 8\left(3-\frac{16}{s^2}\right)\arctan\left(\frac{4}{s}\right)$$
$$-4\left(\left(1-\frac{12}{s^2}\right)s\log\left(s^2+4\right) - 4\left(3-\frac{4}{s^2}\right)\arctan\left(\frac{2}{s}\right)\right) + 6s\log(s)\Bigg)$$

$$\mathcal{L}(\mathrm{sinc}^5(t)) = \frac{1}{768}s^4\Bigg(2\left(\frac{625}{s^4}-\frac{150}{s^2}+1\right)\arctan\left(\frac{5}{s}\right) + \left(\frac{20}{s}-\frac{500}{s^3}\right)\log\left(s^2+25\right)$$
$$+10\bigg(2\left(\frac{1}{s^4}-\frac{6}{s^2}+1\right)\arctan\left(\frac{1}{s}\right) + \left(\frac{4}{s}-\frac{4}{s^3}\right)\log\left(s^2+1\right)\bigg)$$
$$-5\bigg(2\left(\frac{81}{s^4}-\frac{54}{s^2}+1\right)\arctan\left(\frac{3}{s}\right) + \left(\frac{12}{s}-\frac{108}{s^3}\right)\log\left(s^2+9\right)\bigg)\Bigg)$$

Besides using the result of Theorem 2.1, another way of computing $\mathcal{L}\Big(\frac{\sin^m(at)}{t^n}\Big)$ is using the general properties that $\mathcal{L}(f(at)) = \frac{1}{a}F(s/a)$ and $\mathcal{L}(t^k f(t)) = (-1)^k \frac{d^k}{ds^k}F(s)$. Thus, we have that $\mathcal{L}\Big(\frac{\sin^m(at)}{t^n}\Big) = a^{m-1}\frac{d^{m-n}}{ds^{m-n}}(\mathcal{L}(\mathrm{sinc}^m(t))(s/a))$.

2. Suppose we want to compute $L := \mathcal{L}\Big(\frac{(\cos(at)-\cos(bt))^m}{t^n}\Big)$, with $m \geqslant 2n$ positive integer and $a \geqslant b$. By using elementary properties of the Laplace Transform, we have

$$L = (-1)^m 2^{m+1-n}(a+b)^{n-1}\mathcal{L}\Big(\frac{\sin^m(t)\sin^m((a-b)t/(a+b))}{t^n}\Big)\Big(\frac{2s}{a+b}\Big).$$

Thus, we can use the result from Theorem 3.3. For example, for $a \geqslant b$ we obtain

$$\mathcal{L}\Big(\frac{\cos(at)-\cos(bt)}{t}\Big) = \frac{1}{2}\log\Big(\frac{s^2+b^2}{s^2+a^2}\Big)$$

$$\mathcal{L}\Big(\frac{\cos(at)-\cos(bt)}{t^2}\Big) = -a\arctan(a/s)+b\arctan(b/s)-\frac{s}{2}\log\Big(\frac{s^2+b^2}{s^2+a^2}\Big)$$

$$\mathcal{L}\Big(\frac{(\cos(at)-\cos(bt))^2}{t}\Big) = \frac{1}{4}\Bigg(-\log\Big(\frac{4a^2+s^2}{(a+b)^2}\Big)-\log\Big(\frac{4b^2+s^2}{(a+b)^2}\Big)+2\log\Big(\frac{(a-b)^2+s^2}{(a+b)^2}\Big)$$

$$+2\log\Big(\frac{4s^2}{(a+b)^2}+4\Big)-4\log\Big(\frac{2s}{a+b}\Big)\Bigg)$$

3. In Theorem 2.1, if we make $a = 1$ and let s tend to 0^+, for any $m \geqslant n$ positive integers, we obtain the integral

$$I := \int_0^\infty \frac{\sin^m(t)}{t^n}\,dt$$

satisfies the following:

i) If m and n are even, then

$$I = \frac{(-1)^{n/2}\pi}{2^m(n-1)!}\sum_{j=1}^{m/2}(-1)^j\binom{m}{m/2-j}(2j)^{n-1}.$$

ii) If m is even and n is odd, then

$$I = \frac{(-1)^{(n+1)/2}}{2^{m-1}(n-1)!}\sum_{j=1}^{m/2}(-1)^j\log(2j)\binom{m}{m/2-j}(2j)^{n-1}.$$

iii) If m is odd and n is even, then

$$I = \frac{(-1)^{n/2}}{2^{m-1}(n-1)!}\sum_{j=1}^{(m+1)/2}(-1)^{j-1}\log(2j-1)\binom{m}{(m+1)/2-j}(2j-1)^{n-1}.$$

iv) If m and n are odd, then

$$I = \frac{(-1)^{(n-1)/2}\pi}{2^m(n-1)!}\sum_{j=1}^{(m+1)/2}(-1)^{j-1}\binom{m}{(m+1)/2-j}(2j-1)^{n-1}.$$

These results are known (see, for example, [**5**]).

4. For $a, b > 0$ and small values of m, n, p, we have by theorem 3.3

$$\mathcal{L}\Big(\frac{\sin(at)\sin(bt)}{t^2}\Big) = \frac{1}{2}\Big((a+b)\arctan\Big(\frac{a+b}{s}\Big) - (a-b)\arctan\Big(\frac{a-b}{s}\Big)\Big) + \frac{s}{4}\log\Big(\frac{s^2+(a+b)^2}{s^2+(a-b)^2}\Big).$$

$$\mathcal{L}\Big(\frac{\sin(at)\sin^2(bt)}{t^3}\Big) = \frac{1}{8}\Big(2(s^2-a^2)\arctan\Big(\frac{a}{s}\Big) + ((a-2b)^2-s^2)\arctan\Big(\frac{a-2b}{s}\Big)$$

$$+((a+2b)^2+s^2)\arctan\Big(\frac{a+2b}{s}\Big) + as\log\Big(\frac{(a^2+s^2)^2}{((a-2b)^2+s^2)((a+2b)^2+s^2)}\Big)$$

$$+ 2bs\log\Big(\frac{(a-2b)^2+s^2}{(a+2b)^2+s^2}\Big).$$

$$\mathcal{L}\Big(\frac{\sin^2(at)\sin^2(bt)}{t^4}\Big) = \frac{1}{96}\left(-2\left(s\left(s^2-12b^2\right)\log\left(\frac{4b^2+s^2}{a^2}\right) + 4\left(4b^3-3bs^2\right)\tan^{-1}\left(\frac{2b}{s}\right)\right)\right.$$

$$- s\left(12(a-b)^2-s^2\right)\log\left(\frac{4(a-b)^2+s^2}{a^2}\right) - s\left(12(a+b)^2-s^2\right)\log\left(\frac{4(a+b)^2+s^2}{a^2}\right)$$

$$- 2\left(4\left(4a^3-3as^2\right)\tan^{-1}\left(\frac{2a}{s}\right) + s\left(s^2-12a^2\right)\log\left(\frac{s^2}{a^2}+4\right)\right)$$

$$+ 4(a-b)\left(4(a-b)^2-3s^2\right)\tan^{-1}\left(\frac{2(a-b)}{s}\right)$$

$$\left. + 4(a+b)\left(4(a+b)^2-3s^2\right)\tan^{-1}\left(\frac{2(a+b)}{s}\right) + 4s^3\log\left(\frac{s}{a}\right)\right)$$

$$\mathcal{L}\Big(\frac{\sin(at)\sin^3(bt)}{t^4}\Big) = \frac{1}{96}\left(s\left(3(a-3b)^2-s^2\right)\log\left(\frac{(a-3b)^2+s^2}{a^2}\right)\right.$$

$$- s\left(3(a+3b)^2-s^2\right)\log\left(\frac{(a+3b)^2+s^2}{a^2}\right)$$

$$+ 3\left(s\left(s^2-3(a-b)^2\right)\log\left(\frac{(a-b)^2+s^2}{a^2}\right) + s\left(3(a+b)^2-s^2\right)\log\left(\frac{(a+b)^2+s^2}{a^2}\right)\right.$$

$$\left. + 2(a-b)\left((a-b)^2-3s^2\right)\tan^{-1}\left(\frac{a-b}{s}\right) - 2(a+b)\left((a+b)^2-3s^2\right)\tan^{-1}\left(\frac{a+b}{s}\right)\right)$$

$$\left. - 2(a-3b)\left((a-3b)^2-3s^2\right)\tan^{-1}\left(\frac{a-3b}{s}\right) + 2(a+3b)\left((a+3b)^2-3s^2\right)\tan^{-1}\left(\frac{a+3b}{s}\right)\right)$$

For similar results with smaller powers in the denominator, one can use the same theorem 3.3 or derive the preceding results using the general property $\mathcal{L}(tf(t)) = -\frac{dF(s)}{ds}$.

5. In theorem 3.3, if we let s tend to 0^+, for any $m + n \geqslant p$ positive integers and $a > 0$, we have that the integral

$$I := \int_0^\infty \frac{\sin^n(t)\sin^m(at)}{t^p}\,dt$$

satisfies the following:

i) If m and n are odd and

a) if p is even, then

$$I=\frac{(-1)^{(m+n+p)/2}\pi}{2^{m+n}(p-1)!}\cdot$$

$$\sum_{\substack{0\leqslant k\leqslant (n-1)/2\\ 0\leqslant l\leqslant (m-1)/2}}(-1)^{k+l}\binom{n}{k}\binom{m}{l}\Big((n-2k+a(m-2l))^{p-1}-|n-2k-a(m-2l)|^{p-1}\Big)$$

b) if p is odd, then

$$I=\frac{(-1)^{(m+n+p-1)/2}}{2^{m+n}(p-1)!}\cdot$$

$$\sum_{\substack{0\leqslant k\leqslant (n-1)/2\\ 0\leqslant l\leqslant (m-1)/2}}(-1)^{k+l}\binom{n}{k}\binom{m}{l}\Big(-\log\left((n-2k+a(m-2l))^2\right)(n-2k+a(m-2l))^{p-1}$$

$$+\log\left((n-2k-a(m-2l))^2\right)(n-2k-a(m-2l))^{p-1}\Big)$$

ii) If n is odd and m is even, then

a) if p is even, then

$$I=\frac{(-1)^{(m+n+p-1)/2}}{2^{m+n}(p-1)!}\Bigg((-1)^{m/2}\binom{m}{m/2}\sum_{k=0}^{(n-1)/2}(-1)^k\binom{n}{k}(n-2k)^{p-1}\log(n-2k)^2+$$

$$\sum_{\substack{0\leqslant k\leqslant (n-1)/2\\ 0\leqslant l\leqslant (m-2)/2}}(-1)^{k+l}\binom{n}{k}\binom{m}{l}\Big((n-2k+a(m-2l))^{p-1}\log((n-2k+a(m-2l))^2)$$

$$+(n-2k-a(m-2l))^{p-1}\log((n-2k-a(m-2l))^2)\Big)\Bigg)$$

b) if p is odd, then

$$I=-\frac{(-1)^{(m+n+p)/2}\pi}{2^{m+n}(p-1)!}\Bigg((-1)^{m/2}\binom{m}{m/2}\sum_{k=0}^{(n-1)/2}(-1)^k\binom{n}{k}(n-2k)^{p-1}+$$

$$\sum_{\substack{0\leqslant k\leqslant (n-1)/2\\ 0\leqslant l\leqslant (m-2)/2}}(-1)^{k+l}\binom{n}{k}\binom{m}{l}\Big((n-2k+a(m-2l))^{p-1}$$

$$+(n-2k-a(m-2l))^{p-1}\mathrm{sign}(n-2k-a(m-2l))\Big)\Bigg)$$

In results i) and ii), certain indeterminate forms could appear for particular cases, this is given when for example for certain values $a>0$ and $0\leqslant k\leqslant (n-1)/2, 0\leqslant l\leqslant (m-1)/2$ it is satisfied that $n-2k-a(m-2l)=0$ and when $p=1$. In these cases, it must be considered that: 0^0 is considered equal to 1, $0\cdot\infty$ is considered 0, and $\log(0)$ is replaced by ∞.

iii) If m, n are even, then

a) If p is even

$$I = \frac{(-1)^{(m+n+p)/2}\pi}{2^{m+n}(p-1)!}\left(\sum_{\substack{0\leqslant k\leqslant (n-2)/2\\ 0\leqslant l\leqslant (m-2)/2}} (-1)^{k+l}\binom{n}{k}\binom{m}{l}\Big((n-2k+a(m-2l))^{p-1}\right.$$

$$+|n-2k-a(m-2l)|^{p-1}\Big) + (-1)^{n/2}\binom{n}{n/2}\sum_{l=0}^{(m-2)/2}(-1)^l\binom{m}{l}(a(m-2l))^{p-1}$$

$$\left. + (-1)^{m/2}\binom{m}{m/2}\sum_{k=0}^{(n-2)/2}(-1)^k\binom{n}{k}(n-2k)^{p-1}\right).$$

b) If $p > 1$ is odd

$$I = \frac{(-1)^{(m+n+p+1)/2}}{2^{m+n}(p-1)!}\left(\sum_{\substack{0\leqslant k\leqslant (n-2)/2\\ 0\leqslant l\leqslant (m-2)/2}} (-1)^{k+l}\binom{n}{k}\binom{m}{l}\Big((n-2k+a(m-2l))^{p-1}\right.$$

$$\cdot \log\left((n-2k+a(m-2l))^2\right) + (n-2k-a(m-2l))^{p-1}\log\left((n-2k-a(m-2l))^2\right)\Big)$$

$$+ (-1)^{n/2}\binom{n}{n/2}\sum_{l=0}^{(m-2)/2}(-1)^l\binom{m}{l}\left(\left(a(m-2l)\right)^{p-1}\log\left((a(m-2l))^2\right)\right)$$

$$\left. + (-1)^{m/2}\binom{m}{m/2}\sum_{k=0}^{(n-2)/2}(-1)^k\binom{n}{k}\left(\left(n-2k\right)^{p-1}\log\left((n-2k)^2\right)\right)\right).$$

If $p = 1$ the integral I diverges.

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