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Using exponential generating functions to evaluate definite integrals

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Abstract. Numerous examples where various types of definite integrals evaluated using a technique exploiting exponential generating functions are presented. We show how some classically famous but tricky integrals can be readily found using such an approach. Brief historical remarks on some of the more notable integrals considered are made. Application of some of the integrals found to a class of classically important series containing the central binomial coefficients are given.

1. Introduction

The evaluation of definite integrals in closed form has continued to occupy the attention of mathematicians ever since the advent of the integral calculus in the late seventeenth century. Their evaluation has been said to be one of the most intriguing topics of elementary mathematics [**2**, p. 161] and described as a subject full of interconnections to many other parts of mathematics [**1**, p. 535]. Of the many methods that have been developed for evaluating definite integrals the purpose of the present paper is to provide an expository introduction to an often overlooked and underappreciated technique of evaluating definite integrals that makes use of exponential generating functions.

Given a sequence $\{a_n\}_{n\geqslant 0}$ where n is a non-negative integer, the *exponential generating function* of the sequence is defined by

$$a(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}. \tag{1.1}$$

Here the formal power series in z given in (1.1) forms a ring. As we are interested in applying exponential generating function to the evaluation of definite integrals which are analytic objects, we will be concerned with the analytic theory of generating functions. Here we assume z is a complex number while issues of convergence are

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of paramount importance. The notation $[z^n/n!]\, a(z)$ will be used to represent the coefficient a_n in the exponential generating function for $a(z)$.

As a technique, the use of exponential generating functions are often well suited to the evaluation of definite integrals that contain logarithmic terms raised to an integral positive power. The most famous example of an integral of this type is perhaps

$$\int_0^{\frac{\pi}{2}} \log^n(\sin x)\, dx. \tag{1.2}$$

Here n is a non-negative integer. Of course when $n = 1$ we have the famous log-sine integral of Euler [**16**]

$$\int_0^{\frac{\pi}{2}} \log(\sin x)\, dx = -\frac{\pi}{2}\log(2). \tag{1.3}$$

It represents an example of the evaluation of a definite integral whose indefinite form is unknown. A simple proof of the result can be found in [**47**, p. 229]. For some interesting history on the evaluation of this integral see [**34**] while for a number of alternative approaches to its evaluation see, for example, [**50**], [**42**, pp. 160–161].

As an example of a technique for the evaluation of definite integrals which is rarely found in texts, numerous examples illustrating the method will be given. Our intention in writing this expository paper is to act as a source of examples showcasing such a technique. By highlighting the types of definite integrals where applying such a method to their evaluation is well suited, we hope to bring greater attention to this often undervalued technique.

2. Preliminaries

In this section we give those special functions and various other results we are going to have a need for in later sections.

The *digamma function* is defined by

$$\psi(x) = \frac{d}{dx}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{2.1}$$

where $\Gamma(x)$ is the classical *gamma function* defined by the Eulerian integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt. \tag{2.2}$$

These are respectively entries **5.2.2** and **5.2.1** in [**44**]. Closely connected to the gamma function is the *beta function* defined by

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \quad x, y > 0, \tag{2.3}$$

which is related to the gamma function by the identity

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.4}$$

These are respectively entries **8.380.1** and **8.384.1** in [**19**]. Alternative integral representations for the beta function can be given. Substituting $t = \sin^2\theta$ into (2.3) gives

$$\mathrm{B}(x,y) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta \cos^{2y-1}\theta \, d\theta, \tag{2.5}$$

while substituting $t = u/(1+u)$ into (2.3) gives

$$\mathrm{B}(x,y) = \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}}\, du, \quad x, y > 0. \tag{2.6}$$

These are respectively entries **8.380.2** and **8.380.3** in [**19**]. Euler's reflexion formula for the gamma function is given by

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \quad z \neq 0, -1, -2, \ldots \tag{2.7}$$

while Legendre's duplication formula for the gamma function is

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \quad 2z \neq 0, -1, -2, \ldots \tag{2.8}$$

These are respectively entries **5.5.1** and **5.5.5** in [**44**].

The *functional relation* for the digamma function is

$$\psi(x+1) = \psi(x) + \frac{1}{x}. \tag{2.9}$$

This is entry **5.5.2** in [**44**]. If $n \in \mathbb{N}$, repeated application of the functional relation for the digamma function gives the *difference equation* of

$$\psi(x+n) - \psi(x) = \sum_{k=0}^{n-1} \frac{1}{x+k}. \tag{2.10}$$

This is entry **8.365.3** in [**19**]. A series representation for the digamma function is

$$\psi(x+1) = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+x}\right), \quad x \neq -1, -2, -3, \ldots \tag{2.11}$$

Here γ is the *Euler–Mascheroni constant*. This is entry **5.7.6** in [**44**].

The Maclaurin series expansion for the secant function is

$$\sec(x) = \sum_{n=0}^{\infty} \frac{|E_n| x^n}{n!}, \qquad |x| < \frac{\pi}{2}, \tag{2.12}$$

where E_n denote the *Euler numbers*. This is entry **1.411.9** in [**19**]. The first few of these numbers are:

$$E_0 = 1, E_2 = -1, E_4 = 5, E_6 = -61, E_8 = 1385, \tag{2.13}$$

while all odd-indexed Euler numbers are equal to zero. The Maclaurin series expansion for the tangent function is

$$\tan(x) = \sum_{n=1}^{\infty} \frac{2^{n+1}(2^{n+1}-1)|B_{n+1}|}{(n+1)!} x^n, \qquad |x| < \frac{\pi}{2}, \tag{2.14}$$

where B_n denote the *Bernoulli numbers*. This is entry **1.411.5** in [**19**]. The first few of these numbers are:

$$B_0 = 1, B_1 = \frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}. \tag{2.15}$$

In fact, $B_{2n+1} = 0$ for all $n \in \mathbb{N}$. Finally, a series expansion for the cotangent function is

$$\cot(x) = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{n+1}|B_{n+1}|x^n}{(n+1)!}, \qquad |x| < \pi. \tag{2.16}$$

This is entry **1.411.7** in [**19**].

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{2.17}$$

for $s > 1$ while the Maclaurin series expansion for the digamma function is

$$\psi(x+1) = -\gamma - \sum_{n=1}^{\infty} (-1)^n \zeta(n+1) t^n, \qquad |x| < 1. \tag{2.18}$$

This is entry **5.7.4** in [**44**].

Suppose $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The nth *generalised harmonic number* of order $p \in \mathbb{N}$ is defined by

$$H_n^{(p)} = \sum_{k=1}^{n} \frac{1}{k^p}, \tag{2.19}$$

such that $H_0^{(p)} \equiv 0$. When $p = 1$ we have $H_n \equiv H_n^{(1)}$, the nth *harmonic number*. The nth *generalised skew-harmonic number* of order $p \in \mathbb{N}$ is defined by

$$\overline{H}_n^{(p)} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k^p}, \tag{2.20}$$

such that $\overline{H}_0^{(p)} \equiv 0$. When $p = 1$ we have $\overline{H}_n \equiv \overline{H}_n^{(1)}$, the nth *skew-harmonic number* [**9**]. Note that

$$\sum_{k=1}^{n} \frac{1}{(2k-1)^p} = H_{2n}^{(p)} - \frac{1}{2^p} H_n^{(p)}, \tag{2.21}$$

and

$$\overline{H}_{2n}^{(p)} = \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = H_{2n}^{(p)} - \frac{1}{2^{p-1}} H_n^{(p)}. \tag{2.22}$$

When $p = 1$ in this last result it is known as the *Botez–Catalan identity* [**48**, p. 89].

The digamma function is related to the harmonic numbers by

$$\psi(x+1) = H_x - \gamma, \tag{2.23}$$

and is entry **5.4.14** in [**44**]. This result allows the harmonic numbers to be analytically continued to all $x \in \mathbb{R}$, $x \neq -1, -2, -3, \ldots$. If $n \in \mathbb{N}$, at half-integer order arguments for the digamma function one has

$$\psi\left(n + \frac{1}{2}\right) = -\gamma - 2\log(2) + 2\sum_{k=1}^{n} \frac{1}{2k-1} = -\gamma - 2\log(2) + 2H_{2n} - H_n, \tag{2.24}$$

where we have made use of the result given in (2.21) with $p = 1$. This is entry **5.4.15** in [**44**].

THEOREM 2.1 (The Cauchy Product of Power Series). *Consider the power series* $\sum_{n=0}^{\infty} a_n x^n$ *with a radius of convergence* R_1 *and the power series* $\sum_{n=0}^{\infty} b_n x^n$ *with a radius of convergence* R_2. *Then whenever both of these power series converge, for their product we have*

$$\sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} x^n. \tag{2.25}$$

This power series has a radius of convergence R *such that* $R = \min\{R_1, R_2\}$.

A proof of this result can be found in [**22**, pp. 136–137]. Often it is convenient to give this result when both summation indices start at one rather than zero. In this case we have

$$\sum_{n=1}^{\infty} a_n x^n \cdot \sum_{n=1}^{\infty} b_n x^n = \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_k b_{n-k+1} x^{n+1}. \tag{2.26}$$

We conclude this section with an important result concerning the power series expansion involving a difference between two digamma functions with differing arguments. This we give in the following lemma.

LEMMA 2.1. *For* $|t| < 1$

$$\psi\left(\frac{t}{2} + 1\right) - \psi(t+1) = \sum_{k=1}^{\infty} (-1)^k \left(1 - \frac{1}{2^k}\right) \zeta(k+1) t^k. \tag{2.27}$$

PROOF. From (2.11) we see that

$$\psi\left(\frac{t}{2} + 1\right) - \psi(t+1) = \sum_{n=1}^{\infty} \left(\frac{1}{n+t} - \frac{1}{n+t/2}\right). \tag{2.28}$$

Also for $|t| < 1$, as

$$\frac{1}{n+t} = \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{n^{k+1}} = \frac{1}{n} + \sum_{k=1}^{\infty} \frac{(-1)^k t^k}{n^{k+1}},$$

and

$$\frac{1}{n+t/2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{n^{k+1}} \frac{t^k}{2^k} = \frac{1}{n} + \sum_{k=1}^{\infty} \frac{(-1)^k}{n^{k+1}} \frac{t^k}{2^k},$$

and are nothing more than infinite geometric series expansions, the term appearing in the bracket on the right of (2.28) can be rewritten as

$$\psi\left(\frac{t}{2}+1\right)-\psi(t+1)=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{(-1)^k}{n^{k+1}}\left(1-\frac{1}{2^k}\right)t^k\right)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}a_{nk}(t),$$

or

$$\psi\left(\frac{t}{2}+1\right)-\psi(t+1)=\sum_{k=1}^{\infty}(-1)^k\left(1-\frac{1}{2^k}\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^{k+1}}\right)t^k,$$

after the order of the double summation has been interchanged and is permissible since for $|t|<1$ and each $n\in\mathbb{N}$

$$\sum_{k=1}^{\infty}|a_{nk}(t)|<\frac{t}{n^2-tn}=M_n(t)<\infty \quad \text{and} \quad \sum_{n=1}^{\infty}M_n(t)<\infty.$$

Recognising the right most term within brackets as the Riemann zeta function given in (2.17) completes the proof. □

3. Some examples

In this section we give some examples that use the method of exponential generating functions to evaluate definite integrals. We begin with two very simple examples that could be readily found using other methods. And while using an exponential generating function approach to evaluate these first two integrals may seem like wielding a sledgehammer to crush a small pea, they are given in order to help gently introduce and establish the main ideas in applying the technique.

Example 3.1. Consider the integral

$$\int_0^{\infty}x^n e^{-ax}\,dx,$$

where $n\in\mathbb{N}_0$ and $a>0$. Of course the integral is nothing more than the gamma function given in (2.2) after a transformation of $x\mapsto x/a$ is enforced. Denoting the integral to be found by $I_n(a)$ and consider the exponential generating function given by

$$G(t)=\sum_{n=0}^{\infty}\frac{I_n(a)t^n}{n!}.$$

So for $0<|t|<a$ we have

$$G(t)=\int_0^{\infty}e^{-ax}\sum_{n=0}^{\infty}\frac{(tx)^n}{n!}\,dx=\int_0^{\infty}e^{-(a-t)x}\,dx=\left[-\frac{e^{-(a-t)x}}{a-t}\right]_0^{\infty}=\frac{1}{a-t}$$
$$=\frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{t}{a}\right)^n=\sum_{n=0}^{\infty}\left(\frac{n!}{a^{n+1}}\right)\frac{t^n}{n!}.$$

Thus we see that

$$\left[\frac{t^n}{n!}\right]G(t)=I_n(a)=\frac{n!}{a^{n+1}},$$

yielding

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}, \quad a > 0, n \in \mathbb{N}_0,$$

as expected.

What this first example shows is how the positive integral power n for the logarithmic term is made the basis of a summation index when the exponential generating function is introduced before disappearing when the resultant power series expansion is recognised as being that for the exponential function. It is this basic idea behind the technique that makes using an exponential generating function so useful.

EXAMPLE 3.2. As a second simple example consider the integral

$$\int_0^1 x^p \log^n(x)\,dx,$$

where $p \geqslant 0$ and $n \in \mathbb{N}_0$. Denoting the integral to be found by $J_n(p)$ and consider the exponential generating function given by

$$G(t) = \sum_{n=0}^\infty \frac{J_n(p)t^n}{n!}.$$

Thus

$$G(t) = \int_0^1 x^p \sum_{n=0}^\infty \frac{(t\log x)^n}{n!}\,dx = \int_0^1 x^{p+t}\,dx = \frac{1}{t+p+1} = \sum_{n=0}^\infty \left(\frac{(-1)^n n!}{(p+1)^{n+1}}\right)\frac{t^n}{n!}.$$

So we see that

$$\left[\frac{t^n}{n!}\right] G(t) = J_n(p) = \frac{(-1)^n n!}{(p+1)^{n+1}},$$

yielding

$$\int_0^1 x^p \log^n(x)\,dx = \frac{(-1)^n n!}{(p+1)^{n+1}}, \quad p \geqslant 0, n \in \mathbb{N}_0. \tag{3.1}$$

For an alternative derivation of (3.1) using a reduction formula see [**13**]. The result is very old, dating back at least to the time of Legendre [**27**, p. 459], if not before. Note the integral can be found by either employing integration by parts n times or by transforming the integral to a gamma function using a substitution of $x = e^{-u}$. When $p = 0$ we have the elementary result of

$$\int_0^1 \log^n(x)\,dx = (-1)^n n!. \tag{3.2}$$

Our next example in terms of difficulty is a slight step up from the previous two examples.

EXAMPLE 3.3. In this example we wish to evaluate

$$\int_0^{\frac{\pi}{2}} \log^n(\tan x)\,dx,$$

where $n \in \mathbb{N}_0$. The integral has an interesting history which we shall briefly touch upon in a remark at the conclusion of its evaluation. Denoting the integral to be found by T_n and consider the exponential generating function given by $G(t) = \sum_{n=0}^{\infty} \frac{T_n t^n}{n!}$. Thus

(3.3)
$$G(t) = \int_0^{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{(t \log \tan x)^n}{n!} \, dx = \int_0^{\frac{\pi}{2}} \tan^t x \, dx = \frac{1}{2} \mathrm{B}\left(\frac{1+t}{2}, \frac{1-t}{2}\right) = \frac{\pi}{2} \sec\left(\frac{t\pi}{2}\right),$$

where we have made use of the integral representation for the beta function given in (2.5) and Euler's reflexion formula (2.7). From the power series expansion for the secant function, namely (2.12), we can write (3.3) as

$$G(t) = \sum_{n=0}^{\infty} \left(\frac{|E_n| \pi^{n+1}}{2^{n+1}}\right) \frac{t^n}{n!}.$$

Thus

$$\left[\frac{t^n}{n!}\right] G(t) = T_n = \left(\frac{\pi}{2}\right)^{n+1} |E_n|,$$

yielding

$$\int_0^{\frac{\pi}{2}} \log^n(\tan x) \, dx = \left(\frac{\pi}{2}\right)^{n+1} |E_n|. \tag{3.4}$$

Since the Euler numbers are equal to zero for all odd integers, according to parity we can immediately write (3.4) as

$$\int_0^{\frac{\pi}{2}} \log^{2n+1}(\tan x) \, dx = 0, \tag{3.5}$$

and

$$\int_0^{\frac{\pi}{2}} \log^{2n}(\tan x) \, dx = \left(\frac{\pi}{2}\right)^{2n+1} |E_{2n}|. \tag{3.6}$$

These are respectively entries **4.227.6** and **4.227.5** in [**19**]. From the values given in (2.13) the first four non-zero log-tangent integrals are:

$$\int_0^{\frac{\pi}{2}} \log^2(\tan x) \, dx = \frac{\pi^3}{8}, \qquad \int_0^{\frac{\pi}{2}} \log^4(\tan x) \, dx = \frac{5\pi^5}{32},$$
$$\int_0^{\frac{\pi}{2}} \log^6(\tan x) \, dx = \frac{61\pi^7}{128}, \qquad \int_0^{\frac{\pi}{2}} \log^8(\tan x) \, dx = \frac{1385\pi^9}{512}.$$

Remark 3.1. Equivalent integrals to the log-tangent integral just found are

$$\int_0^{\infty} \frac{\log^{2n}(t)}{1+t^2} \, dt = 2\int_0^1 \frac{\log^{2n} t}{1+t^2} \, dt = \left(\frac{\pi}{2}\right)^{2n+1} |E_{2n}|, \tag{3.7}$$

obtained on enforcing a substitution of $t = \tan x$ in (3.6) and

$$\int_0^{\frac{\pi}{2}} \log^{2n}(\cot x) \, dx = \left(\frac{\pi}{2}\right)^{2n+1} |E_{2n}|, \tag{3.8}$$

obtained on enforcing a substitution of $x \mapsto \frac{\pi}{2} - x$ in (3.6). Each of these integrals appear in [**15**] under articles §1084 and §1095 respectively. Alternative evaluations for them can also be found in [**42**, pp. 138–139].

REMARK 3.2. An evaluation of the log-tangent integral in the form of (3.7) seems to have been first made by Bidone [**5**, p. 100]. Though the expression Bidone gave was in terms of what today we call the *Dirichlet beta function* $\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$, valid for $s > 0$ (see entry **3.6.4** in [**43**]), namely

$$\int_0^{\infty} \frac{\log^{2n}(x)}{1+x^2} = 2(2n)!\,\beta(2n+1). \tag{3.9}$$

As

$$\beta(2n+1) = \frac{|E_{2n}|}{2(2n)!}\left(\frac{\pi}{2}\right)^{2n+1},$$

which is entry **3.13.3** in [**43**], we see (3.9) is obviously equivalent to (3.7). Later the evaluation of (3.6) appeared as a problem where its proposer, Joseph Wolstenholme, suggested that even though its general form appeared to be

$$\int_0^{\frac{\pi}{2}} \log^{2n}(\tan x)\,dx = \left(\frac{\pi}{2}\right)^{2n+1} \times \text{(an odd number)},$$

he thought finding an exact expression for this generalisation would be difficult [**52**]. The best Wolstenholme could do was express the answer in terms of the $2n$-th order derivative of $\sec x$ evaluated at $x = 0$ [**51**, p. 334, Problem 1920], it corresponding to the coefficient of $\frac{x^{2n}}{(2n)!}$ in the power series expansion for $\sec x$. The connection to the Euler numbers was found by several of its solvers a few months later [**21**]. Much later evaluations for the log-tangent integral in the form of (3.7) that made use of contour integration and the method of residues can be found in [**18, 3**].

For our next example we give what is perhaps the most famous example to make use of the exponential generating function approach in its evaluation. It is the log-sine integral given in (1.2) with the final result for the integral being expressed in terms of a recurrence relation.

EXAMPLE 3.4. We shall obtain a recurrence relation for the following log-sine integral raised to a positive integral power

$$S_n = \int_0^{\frac{\pi}{2}} \log^n(\sin x)\,dx. \tag{3.10}$$

Here $n \in \mathbb{N}_0$. Consider the exponential generating function $G(t) = \sum_{n=0}^{\infty} \frac{S_n t^n}{n!}$. We therefore have

$$G(t) = \int_0^{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{(t \log \sin x)^n}{n!} dx = \int_0^{\frac{\pi}{2}} \sin^t x\,dx = \frac{1}{2}\,\mathrm{B}\left(\frac{t+1}{2}, \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{t+1}{2}\right)}{\Gamma\left(\frac{t}{2}+1\right)}. \tag{3.11}$$

Here we have made use of the integral representation for the beta function given in (2.5). If in the duplication formula for the gamma function given in (2.8) we set

$z = (t+1)/2$ one finds

$$\Gamma\left(\frac{t+1}{2}\right) = \frac{\sqrt{\pi}}{2^t}\frac{\Gamma(t+1)}{\Gamma\left(\frac{t}{2}+1\right)}. \tag{3.12}$$

Substituting this result into (3.11) one obtains

$$G(t) = \frac{\pi}{2^{t+1}}\frac{\Gamma(t+1)}{\Gamma^2\left(\frac{t}{2}+1\right)}.$$

Taking the logarithmic derivative with respect to t yields

$$\frac{G'(t)}{G(t)} = -\log(2) + \psi(t+1) - \psi\left(\frac{t}{2}+1\right) = -\log(2) - \sum_{n=1}^{\infty}(-1)^n\left(1-\frac{1}{2^n}\right)\zeta(n+1)t^n, \tag{3.13}$$

where we have made use of the power series expansion appearing in (2.27). Noting that

$$G'(t) = \sum_{n=1}^{\infty}\frac{S_n t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty}\frac{S_{n+1}t^n}{n!}, \tag{3.14}$$

after a reindexing of $n \mapsto n+1$ has been made, (3.13) becomes

$$\sum_{n=0}^{\infty}\frac{S_{n+1}t^n}{n!} = -\log(2)\sum_{n=0}^{\infty}\frac{S_n t^n}{n!} - \left(\sum_{n=0}^{\infty}\frac{S_n t^n}{n!}\right)\cdot\left(\sum_{n=1}^{\infty}(-1)^n\left(1-\frac{1}{2^n}\right)\zeta(n+1)t^n\right).$$

Adjusting all sums so they start at $n=1$ we have

$$\begin{aligned} S_1 + \sum_{n=1}^{\infty}\frac{S_{n+1}t^n}{n!} &= -\log(2)S_0 - \log(2)\sum_{n=1}^{\infty}\frac{S_n t^2}{n!} \\ &\quad - \frac{1}{t}\left(\sum_{n=1}^{\infty}\frac{S_{n-1}t^n}{(n-1)!}\right)\cdot\left(\sum_{n=1}^{\infty}(-1)^n\left(1-\frac{1}{2^n}\right)\zeta(n+1)t^n\right). \end{aligned} \tag{3.15}$$

Now $S_0 = \int_0^{\frac{\pi}{2}} dx = \pi/2$ and S_1 is just Euler's log-sine integral given in (1.3). Employing the Cauchy product of power series given in (2.26), (3.15) becomes

$$\sum_{n=1}^{\infty}\frac{S_{n+1}t^n}{n!} = \sum_{n=1}^{\infty}\left(-\log(2)S_n - \sum_{k=1}^{n}(-1)^k\left(1-\frac{1}{2^k}\right)\frac{n!\zeta(k+1)}{(n-k)!}S_{n-k}\right)\frac{t^n}{n!}.$$

Comparing coefficients of $t^n/n!$ on both sides yields the required recurrence relation of

$$S_{n+1} = -\log(2)S_n - \sum_{k=1}^{n}(-1)^k\left(1-\frac{1}{2^k}\right)\frac{n!\zeta(k+1)}{(n-k)!}S_{n-k}, \tag{3.16}$$

subject to the initial conditions of $S_0 = \frac{\pi}{2}$ and $S_1 = -\frac{\pi}{2}\log(2)$. Applying (3.16), values for the first five log-sine integrals are:

$$\int_0^{\frac{\pi}{2}}\log(\sin x)\,dx = -\frac{\pi}{2}\log(2), \tag{3.17}$$

$$\int_0^{\frac{\pi}{2}} \log^2(\sin x)\,dx = \frac{\pi^3}{24} + \frac{\pi}{2}\log(2), \tag{3.18}$$

$$\int_0^{\frac{\pi}{2}} \log^3(\sin x)\,dx = -\frac{\pi}{2}\log^3(2) - \frac{\pi^3}{8}\log(2) - \frac{3\pi}{4}\zeta(3), \tag{3.19}$$

$$\int_0^{\frac{\pi}{2}} \log^4(\sin x)\,dx = \frac{19\pi^5}{480} + \frac{\pi}{2}\log^4(2) + \frac{\pi^3}{4}\log^2(2) + 3\pi\zeta(3)\log(2), \tag{3.20}$$

$$\int_0^{\frac{\pi}{2}} \log^5(\sin x)\,dx = -\frac{\pi}{2}\log^5(2) - \frac{5\pi^3}{12}\log^3(2) - \frac{15\pi}{2}\zeta(3)\log^2(2) - \frac{19\pi^5}{96}\log(2) - \frac{5\pi^3}{8}\zeta(3) - \frac{45\pi}{4}\zeta(5). \tag{3.21}$$

REMARK 3.3. Other integrals related to S_n can be readily found. Making the change of variable $x \mapsto \frac{\pi}{2} - x$ in (3.10) we see that

$$S_n = \int_0^{\frac{\pi}{2}} \log^n(\sin x)\,dx = \int_0^{\frac{\pi}{2}} \log^n(\cos x)\,dx,$$

while the change of variable $t = \sin x$ yields

$$S_n = \int_0^1 \frac{\log^n(t)}{\sqrt{1-t^2}}\,dt. \tag{3.22}$$

A related integral that often appears in the literature is

$$\int_0^{\pi} \log^n(\sin x)\,dx = 2\int_0^{\frac{\pi}{2}} \log^n(\sin x)\,dx = 2S_n, \quad n \in \mathbb{N}_0, \tag{3.23}$$

it following from the symmetry of the integrand on the interval $(0, \pi)$ about the point $x = \pi/2$. Values for the first five of these related log-sine integrals are

$$\int_0^{\pi} \log(\sin x)\,dx = -\pi\log(2), \tag{3.24}$$

$$\int_0^{\pi} \log^2(\sin x)\,dx = \frac{\pi^3}{12} + \pi\log(2), \tag{3.25}$$

$$\int_0^{\pi} \log^3(\sin x)\,dx = -\pi\log^3(2) - \frac{\pi^3}{4}\log(2) - \frac{3\pi}{2}\zeta(3), \tag{3.26}$$

$$\int_0^{\pi} \log^4(\sin x)\,dx = \frac{19\pi^5}{240} + \pi\log^4(2) + \frac{\pi^3}{2}\log^2(2) + 6\pi\zeta(3)\log(2), \tag{3.27}$$

$$\int_0^{\pi} \log^5(\sin x)\,dx = -\pi\log^5(2) - \frac{5\pi^3}{6}\log^3(2) - 15\pi\zeta(3)\log^2(2) - \frac{19\pi^5}{48}\log(2) - \frac{5\pi^3}{4}\zeta(3) - \frac{45\pi}{2}\zeta(5). \tag{3.28}$$

REMARK 3.4. Another integral closely related to (1.2) is

$$\int_0^{\pi} x\log^n(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} \log^n(\sin x)\,dx = \pi\int_0^{\frac{\pi}{2}} \log^n(\sin x)\,dx = \pi S_n, \tag{3.29}$$

for $n \in \mathbb{N}_0$, the result being obtained on enforcing a substitution of $x \mapsto \pi - x$. Values for the first five of these closely related log-sine integrals are:

$$\int_0^\pi x \log(\sin x)\, dx = -\frac{\pi^2}{2}\log(2), \tag{3.30}$$

$$\int_0^\pi x \log^2(\sin x)\, dx = \frac{\pi^4}{24} + \frac{\pi^2}{2}\log(2), \tag{3.31}$$

$$\int_0^\pi x \log^3(\sin x)\, dx = -\frac{\pi^2}{2}\log^3(2) - \frac{\pi^4}{8}\log(2) - \frac{3\pi^2}{4}\zeta(3), \tag{3.32}$$

$$\int_0^\pi x \log^4(\sin x)\, dx = \frac{19\pi^6}{480} + \frac{\pi^2}{2}\log^4(2) + \frac{\pi^4}{4}\log^2(2) + 3\pi^2\zeta(3)\log(2), \tag{3.33}$$

$$\begin{aligned}\int_0^\pi x \log^5(\sin x)\, dx = &-\frac{\pi^2}{2}\log^5(2) - \frac{5\pi^4}{12}\log^3(2) - \frac{15\pi^2}{2}\zeta(3)\log^2(2) \\ &- \frac{19\pi^6}{96}\log(2) - \frac{5\pi^4}{8}\zeta(3) - \frac{45\pi^2}{4}\zeta(5).\end{aligned} \tag{3.34}$$

Remark 3.5. Recently the following integral appeared as a problem [**23**]

$$\int_0^1 \frac{\log^n(x)}{\sqrt{x(1-x)}}\, dx. \tag{3.35}$$

Here $n \in \mathbb{N}_0$. Enforcing a substitution of $x \mapsto \sin^2 x$, we immediately see that

$$\int_0^1 \frac{\log^n(x)}{\sqrt{x(1-x)}}\, dx = 2^{n+1} S_n.$$

Remark 3.6. A generalisation of the integral given in (3.10) is

$$S_n(a) = \int_0^{\frac{\pi}{2}} \log^n\left(\frac{2\sin x}{a}\right) dx. \tag{3.36}$$

Here $a > 0$ and $n \in \mathbb{N}_0$. In a similar manner to how the integral was found in Example 3.4 it can be shown that

$$S_{n+1}(a) = -\log(a) S_n(a) - \sum_{k=1}^n (-1)^k \left(1 - \frac{1}{2^k}\right) \frac{n!\zeta(k+1)}{(n-k)!} S_{n-k}(a), \tag{3.37}$$

subject to the initial conditions of $S_0(a) = \frac{\pi}{2}$ and $S_1(a) = -\frac{\pi}{2}\log(a)$. When $a = 2$, (3.37) reduces to the integral given in Example 3.4. Setting $a = 1$ in (3.37) the logarithmic constant of integration term disappears. For this reason this form for the log-sine integral is often preferred. Here the first five log-sine integrals in this case are given by

$$\int_0^{\frac{\pi}{2}} \log(2\sin x)\, dx = 0, \tag{3.38}$$

$$\int_0^{\frac{\pi}{2}} \log^2(2\sin x)\, dx = \frac{\pi^3}{24}, \tag{3.39}$$

$$\int_0^{\frac{\pi}{2}} \log^3(2\sin x)\, dx = -\frac{3\pi}{4}\zeta(3), \tag{3.40}$$

$$\int_0^{\frac{\pi}{2}} \log^4(2\sin x)\,dx = \frac{19\pi^5}{480}, \tag{3.41}$$

$$\int_0^{\frac{\pi}{2}} \log^5(2\sin x)\,dx = -\frac{5\pi^3}{8}\zeta(3) - \frac{45\pi}{4}\zeta(5). \tag{3.42}$$

The simplification in the final values for the log-sine integrals for the case when $a = 1$ compared to when $a = 2$ is obvious. Indeed, the $a = 1$ case for the log-sine integral is usually singled out and defined as 'the' log-sine integral. For $n \in \mathbb{N}$ and $k \geqslant 0$ the *generalised log-sine integral* is defined as

$$\mathrm{Ls}_n^{(k)}(\theta) = -\int_0^{\theta} x^k \log^{n-1-k}\left|2\sin\frac{x}{2}\right|\,dx.$$

Note the modulus sign is not required if $0 \leqslant \theta \leqslant 2\pi$. For $k = 0$ one has the (basic) log-sine integrals $\mathrm{Ls}_n(\theta) = \mathrm{Ls}_n^{(0)}(\theta)$. If in the basic log-sine integral a substitution of $x \mapsto 2x$ is enforced one sees that

$$\mathrm{Ls}_n(\theta) = -2\int_0^{\frac{\theta}{2}} \log^{n-1}(2\sin x)\,dx.$$

The first six values for $\mathrm{Ls}_n(\pi)$ therefore immediately follow from (3.38) through to (3.42). They are:

$$\mathrm{Ls}_1(\pi) = -\pi,\ \mathrm{Ls}_2(\pi) = 0,\ \mathrm{Ls}_3(\pi) = -\frac{\pi^3}{12},\ \mathrm{Ls}_4(\pi) = \frac{3\pi}{2}\zeta(3),$$

$$\mathrm{Ls}_5(\pi) = -\frac{19\pi^5}{240}, \quad \text{and} \quad \mathrm{Ls}_6(\pi) = \frac{5\pi^3}{4}\zeta(3) + \frac{45\pi}{2}\zeta(5).$$

Such values can be found, for example, in [**7**].

Remark 3.7. The first attempt to evaluate the log-sine integral given by (3.10) seems to be that of Nielsen [**41**, Eq. (22.6)] though earlier he had given explicit expressions for the log-sine integrals (3.18) and (3.19) [**39**, Eqs (6) and (7)], [**40**, Eqs (17) and (27)]. The log-sine integral of (3.18) appears as an exercise in [**51**, p. 332, Problem 1919(30)] and in [**11**, p. 476, Ex. 46] while two alternative evaluations for the integral can be found in [**42**, pp. 162–163]. Some years later an evaluation for the generalised log-sine integral of (3.36) was given by Bowman [**8**] who expresses the final result in the form of an $n \times n$ determinant. He also gave explicit expressions for log-sine integrals (3.38), (3.39), (3.40), and (3.41). Some years after Bowman, Kölbig gave an expression for the log-sine integral (3.10) which was another form for Bowman's determinant [**24**, Eq. (10)]. He also gives explicit expressions for (3.17), (3.18), (3.19), and (3.20), and some years later an explicit expression for (3.21) [**25**, p. 569]. More recently, an evaluation of (3.36) when $n = 2$ and $a \mapsto \frac{2}{a}$ can be found in [**37**, pp. 234–236].

The first to apply an exponential generating function approach to the evaluation of any of the log-sine type integrals whose final result was then expressed in terms of a recurrence relation appears to be that of Lewin [**30, 31**]. Lewin introduced the notation for the log-sine integral $\mathrm{Ls}_n(\theta)$ and gives explicit results for $\mathrm{Ls}_2(\pi)$ through to $\mathrm{Ls}_7(\pi)$ in [**30**, Eq. (9)],[**31**, p. 198, Eq. (7.113)] and later in [**32**, p. 219 Eq. (7.113)].

Shortly after Lewin, and obviously unaware of his work, Beumer also evaluated the log-sine integral (3.10) in terms of a recurrence relation [**4**] while Narasimhan gave a recurrence relation for the evaluation of the equivalent log-sine integral of (3.22) [**38**]. Later evaluations of log-sine integrals using an exponential generating function approach can be found in [**6**, pp. 245–248] and [**7**]. In the form of (3.22) Legendre gave values for (3.17), (3.18), and (3.19) [**27**, §44, p. 461]. From time to time several log-sine type integrals have appeared as problems in the problem sections of various journals. For example, (3.25) in [**10**] and (3.39) in [**26, 20**]. Finally, (3.17) is entry **4.224.3** in [**19**] while (3.18) is entry **4.224.7** in [**19**].

Example 3.5. For our next example consider the sine-log-tangent integral of

$$б_n = \int_0^{\frac{\pi}{2}} \sin x \log^n(\tan x)\, dx,$$

where $n \in \mathbb{N}_0$. Thus we have

$$G(t) = \sum_{n=0}^{\infty} \frac{б_n t^n}{n!} = \int_0^{\frac{\pi}{2}} \sin x \sum_{n=0}^{\infty} \frac{(t \log \tan x)^n}{n!}\, dx = \int_0^{\frac{\pi}{2}} \sin^{t-1} x \cos^{-t} x\, dx$$
$$= \frac{1}{2} \mathrm{B}\left(\frac{t}{2}+1, \frac{1-t}{2}\right) = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{t}{2}+1\right) \Gamma\left(\frac{1-t}{2}\right). \tag{3.43}$$

Note the integral representation for the beta function given in (2.5) has been used here. If in the duplication formula for the gamma function given in (2.8) we set $z = (1-t)/2$ one finds

$$\Gamma\left(\frac{1-t}{2}\right) = \frac{\sqrt{\pi} 2^t \Gamma(1-t)}{\Gamma\left(1-\frac{t}{2}\right)}. \tag{3.44}$$

From Euler's reflexion formula, as

$$\Gamma(1-t) = \frac{\pi}{\sin(\pi t)\Gamma(t)} \quad \text{and} \quad \Gamma\left(1-\frac{t}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi t}{2}\right)\Gamma\left(\frac{t}{2}\right)},$$

combining these results with (3.44) allows us to rewrite (3.43) as

$$G(t) = \frac{2^{t-1}\Gamma\left(\frac{t}{2}\right)\Gamma\left(\frac{t}{2}+1\right)}{\Gamma(t)\cos\left(\frac{\pi t}{2}\right)}. \tag{3.45}$$

Taking the logarithmic derivative of (3.45) with respect to t we find

$$\frac{G'(t)}{G(t)} = \psi\left(\frac{t}{2}+1\right) - \psi(t+1) + \log(2) + \frac{\pi}{2}\tan\left(\frac{\pi t}{2}\right). \tag{3.46}$$

Here a simplification leading to this final expression using the functional relation for the digamma function of (2.9) has been made. Making use of the Maclaurin series expansions for the tangent function given in (2.14) and the power series expansion for the difference between the two digamma functions given in (2.27) allows one to rewrite

(3.46) as

$$
\begin{aligned}
б_1 + \sum_{n=1}^{\infty} \frac{б_n t^n}{n!} &= б_0 \log(2) + \log(2) \sum_{n=1}^{\infty} \frac{б_n t^n}{n!} \\
&+ \frac{1}{t} \left(\sum_{n=1}^{\infty} \frac{б_{n-1} t^n}{(n-1)!} \right) \cdot \left(\sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{1}{2^n}\right) \zeta(n+1) t^n \right) \\
&+ \frac{1}{t} \left(\sum_{n=1}^{\infty} \frac{б_{n-1} t^n}{(n-1)!} \right) \cdot \left(\sum_{n=1}^{\infty} \frac{2^{n+1}(2^{n+1}-1)|B_{n+1}|}{(n+1)!} \left(\frac{\pi}{2}\right)^{n+1} t^n \right),
\end{aligned} \tag{3.47}
$$

where all sums have been adjusted to start at one. Since

$$
б_0 = \int_0^{\frac{\pi}{2}} \sin x \, dx = 1 \quad \text{and} \quad б_1 = \int_0^{\frac{\pi}{2}} \sin x \log(\tan x) \, dx = \log(2), \tag{3.48}
$$

both these integrals being elementary, after applying the Cauchy product of power series where needed, (3.47) becomes

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{б_{n+1} t^n}{n!} &= \log(2) \sum_{n=1}^{\infty} \frac{б_n t^n}{n!} + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} (-1)^k \left(1 - \frac{1}{2^k}\right) \frac{n!\, \zeta(k+1)}{(n-k)!} б_{n-k} \right) \frac{t^n}{n!} \\
&+ \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{2^{k+1}(2^{k+1}-1)|B_{k+1}| n! б_{n-k}}{(k+1)!(n-k)!} \left(\frac{\pi}{2}\right)^{k+1} \right) \frac{t^n}{n!}.
\end{aligned} \tag{3.49}
$$

On comparing equal coefficients for $t^n/n!$ in (3.49) yields the required recurrence relation of

$$
\begin{aligned}
б_{n+1} &= \log(2) б_n + \sum_{k=1}^{n} (-1)^k \left(1 - \frac{1}{2^k}\right) \frac{n!\, \zeta(k+1)}{(n-k)!} б_{n-k} \\
&+ \sum_{k=1}^{n} \frac{2^{k+1}(2^{k+1}-1)|B_{k+1}| n!}{(k+1)!(n-k)!} \left(\frac{\pi}{2}\right)^{k+1} б_{n-k},
\end{aligned} \tag{3.50}
$$

subject to the initial conditions given in (3.48). Applying (3.50), values for the first five sine-log-tangent integrals are:

$$\int_0^{\frac{\pi}{2}} \sin x \log(\tan x)\, dx = \log(2), \tag{3.51}$$

$$\int_0^{\frac{\pi}{2}} \sin x \log^2(\tan x)\, dx = \log^2(2) + \frac{\pi^2}{6}, \tag{3.52}$$

$$\int_0^{\frac{\pi}{2}} \sin x \log^3(\tan x)\, dx = \log^3(2) + \frac{3}{2}\zeta(3) + \frac{\pi^2}{2}\log(2), \tag{3.53}$$

$$\int_0^{\frac{\pi}{2}} \sin x \log^4(\tan x)\, dx = \log^4(2) + 6\log(2)\zeta(3) + \pi^2\log^2(2) + \frac{3\pi^4}{20}, \tag{3.54}$$

$$\int_0^{\frac{\pi}{2}} \sin x \log^5(\tan x)\, dx = \log^5(2) + \frac{45}{2}\zeta(5) + \frac{5\pi^2}{2}\zeta(3) + 15\zeta(3)\log^2(2) + \frac{5\pi^2}{3}\log^3(2) + \frac{3\pi^4}{4}\log(2). \tag{3.55}$$

REMARK 3.8. Enforcing a substitution of $x \mapsto \frac{\pi}{2} - x$ one has the equivalent integral of

$$\int_0^{\frac{\pi}{2}} \sin x \log^n(\tan x)\, dx = \int_0^{\frac{\pi}{2}} \cos x \log^n(\cot x)\, dx.$$

REMARK 3.9. If $n \in \mathbb{N}_0$, for odd and even parity in the index n one has

$$\int_0^{\frac{\pi}{2}} \cos x \log^{2n}(\tan x)\, dx = \int_0^{\frac{\pi}{2}} \sin x \log^{2n}(\tan x)\, dx,$$

and

$$\int_0^{\frac{\pi}{2}} \cos x \log^{2n+1}(\tan x)\, dx = -\int_0^{\frac{\pi}{2}} \sin x \log^{2n+1}(\tan x)\, dx,$$

In showing these two results, let

$$\mathfrak{S}_n^{(s)} = \int_0^{\frac{\pi}{2}} \sin x \log^n(\tan x)\, dx, \tag{3.56}$$

and

$$\mathfrak{S}_n^{(c)} = \int_0^{\frac{\pi}{2}} \cos x \log^n(\sin x)\, dx. \tag{3.57}$$

For even parity if we let $t = \tan x$, for the difference between (3.56) and (3.57) one has

$$\mathfrak{S}_{2n}^{(s)} - \mathfrak{S}_{2n}^{(c)} = \int_0^{\infty} \frac{t-1}{(1+t^2)^{3/2}} \log^{2n}(t)\, dt. \tag{3.58}$$

Enforcing a substitution of $t \mapsto \frac{1}{t}$ in (3.58) we find $\mathfrak{S}_{2n}^{(s)} - \mathfrak{S}_{2n}^{(c)} = -(\mathfrak{S}_{2n}^{(s)} - \mathfrak{S}_{2n}^{(c)})$ or $\mathfrak{S}_{2n}^{(s)} = \mathfrak{S}_{2n}^{(c)}$ as required to show. A similar thing can be done for the odd parity case if we again let $t = \tan x$ and show the sum between $\mathfrak{S}_{2n+1}^{(s)}$ and $\mathfrak{S}_{2n+1}^{(c)}$ is zero.

For our next example, an integral that in its evaluation follows that of the previous example is given.

EXAMPLE 3.6. Consider the integral

$$\int_0^\infty e^{-x} \log^n x \, dx, \tag{3.59}$$

where $n \in \mathbb{N}_0$. Denoting the integral to be found by $\mathcal{E}_n$ and considering the exponential generating function $G(t) = \sum_{n=0}^\infty \frac{\mathcal{E}_n t^n}{n!}$, we have

$$G(t) = \int_0^\infty e^{-x} \sum_{n=0}^\infty \frac{(t \log x)^n}{n!} \, dx = \int_0^\infty x^t e^{-x} \, dx = \Gamma(t+1),$$

or

$$G'(t) = G(t) \cdot \psi(t+1), \tag{3.60}$$

after taking the logarithmic derivative with respect to t. Substituting (3.14) for $G'(t)$ and (2.18) for the Maclaurin series expansion for the digamma function into (3.60) yields

$$\begin{aligned} \mathcal{E}_1 + \sum_{n=1}^\infty \frac{\mathcal{E}_{n+1} t^n}{n!} &= -\gamma \mathcal{E}_0 - \gamma \sum_{n=1}^\infty \frac{\mathcal{E}_n t^n}{n!} \\ &\quad - \frac{1}{t} \left(\sum_{n=1}^\infty \frac{\mathcal{E}_{n-1} t^n}{(n-1)!} \right) \cdot \left(\sum_{n=1}^\infty (-1)^n \zeta(n+1) t^n \right), \end{aligned} \tag{3.61}$$

after all the sums have been adjusted to start at one. Since $\mathcal{E}_0 = \int_0^\infty e^{-x} \, dx = 1$ and $\mathcal{E}_1 = \int_0^\infty e^{-x} \log(x) \, dx = -\gamma$, the latter being entry **4.331.1** in [**19**], employing the Cauchy product of power series given in (2.26), (3.61) becomes

$$\sum_{n=1}^\infty \frac{\mathcal{E}_{n+1} t^n}{n!} = \sum_{n=1}^\infty \left(-\gamma \mathcal{E}_n - \sum_{k=1}^n (-1)^k \frac{n! \zeta(k+1)}{(n-k)!} \mathcal{E}_{n-k} \right) \frac{t^n}{n!}. \tag{3.62}$$

Comparing the coefficient of $t^n/n!$ on both sides of (3.62) yields the required recurrence relation of

$$\mathcal{E}_{n+1} = -\gamma \mathcal{E}_n - \sum_{k=1}^n (-1)^k \frac{n! \zeta(k+1)}{(n-k)!} \mathcal{E}_{n-k}, \tag{3.63}$$

subject to the initial conditions of $\mathcal{E}_0 = 1$ and $\mathcal{E}_1 = -\gamma$. Applying (3.63), values for the first five log-exponential integrals are:

$$\int_0^\infty e^{-x} \log(x) \, dx = -\gamma, \tag{3.64}$$

$$\int_0^\infty e^{-x} \log^2(x) \, dx = \gamma^2 + \frac{\pi^2}{6}, \tag{3.65}$$

$$\int_0^\infty e^{-x} \log^3(x) \, dx = -\gamma^3 - \frac{\pi^2}{2} \gamma - 2\zeta(3), \tag{3.66}$$

$$\int_0^\infty e^{-x} \log^4(x) \, dx = \gamma^4 + \pi^2 \gamma^2 + 8\gamma \zeta(3) + \frac{3\pi^4}{20}, \tag{3.67}$$

$$\int_0^\infty e^{-x} \log^5(x) \, dx = -\gamma^5 - \frac{5\pi^2}{3} \gamma^3 - 20\gamma^2 \zeta(3) - \frac{3\pi^4}{4} \gamma \tag{3.68}$$

$$- \frac{10\pi^2}{3}\zeta(3) - 24\zeta(5).$$

An evaluation for (3.65) was given as a problem in [**28**]. In the editorial comment provided to its solution [**33**], it was noted that one of the solvers, a certain G. E. Raynor, observed the integral could be evaluated for other positive integral exponents of $\log x$ other than two. Twenty-two years later the solution to integral (3.59) in terms of the recurrence relation (3.63) we just gave was given in a short note by Levenson, the proposer of the original problem [**29**]. The values for integrals (3.64), (3.65), and (3.66) appear in [**19**] under entries **4.331.1**, **4.335.1**, and **4.335.3** respectively while the values for (3.67) and (3.68) do not appear in [**19**].

Example 3.7. An integral closely related to the one just given in Example 3.6 can also be found. Consider

$$\ni_n = \int_0^\infty e^{-x^2} \log^n(x)\, dx,$$

where $n \in \mathbb{N}_0$. Thus we have

$$G(t) = \sum_{n=0}^\infty \frac{\ni_n t^n}{n!} = \int_0^\infty e^{-x^2} \sum_{n=0}^\infty \frac{(t\log x)^n}{n!}\, dx = \int_0^\infty x^t e^{-x^2}\, dx.$$

Enforcing a substitution of $x \mapsto \sqrt{x}$ produces

$$G(t) = \frac{1}{2}\int_0^\infty e^{-x} x^{\frac{t-1}{2}}\, dx = \frac{1}{2}\Gamma\left(\frac{t+1}{2}\right) = \frac{\sqrt{\pi}}{2^t}\frac{\Gamma(t+1)}{\Gamma\left(\frac{t}{2}+1\right)},$$

where the result for the gamma function given in (3.12) has been used. Taking the logarithmic derivative with respect to t we find

$$\begin{aligned}\frac{G'(t)}{G(t)} &= -\log(2) + \psi(t+1) - \frac{1}{2}\psi\left(\frac{t}{2}+1\right)\\ &= -\log(2) + \frac{1}{2}\left[\psi(t+1) - \psi\left(\frac{t}{2}+1\right)\right] + \frac{1}{2}\psi(t+1).\end{aligned} \tag{3.69}$$

From (2.27) and (2.18), writing all sums to start at one and employing the Cauchy product of power series where needed, we can rewrite (3.69) as

$$\begin{aligned}\ni_1 + \sum_{n=1}^\infty \frac{\ni_{n+1}t^n}{n!} &= -\log(2)\ni_0 - \log(2)\sum_{n=1}^\infty \frac{\ni_n t^n}{n!} - \frac{\gamma}{2}\ni_0 - \frac{\gamma}{2}\sum_{n=1}^\infty \frac{\ni_n t^n}{n!}\\ &\quad - \sum_{n=1}^\infty \left(\sum_{k=1}^n (-1)^k \left(2 - \frac{1}{2^k}\right)\frac{n!\zeta(k+1)}{(n-k)!}\ni_{n-k}\right)\frac{t^n}{n!}.\end{aligned} \tag{3.70}$$

Now

$$\ni_0 = \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2} \quad \text{and} \quad \ni_1 = \int_0^\infty e^{-x^2}\log(x)\, dx = -\frac{\sqrt{\pi}}{4}\left(\gamma + 2\log(2)\right). \tag{3.71}$$

Here the former integral is just the Gaussian integral while the latter integral is entry **4.333.1** in [**19**]. Substituting these values into (3.70) before equating equal coefficients of $t^n/n!$ we find

$$\text{Э}_{n+1} = -\left(\log(2) + \frac{\gamma}{2}\right) \text{Э}_n - \frac{1}{2} \sum_{k=1}^{n} (-1)^k \left(2 - \frac{1}{2^k}\right) \frac{n!\zeta(k+1)}{(n-k)!} \text{Э}_{n-k}, \tag{3.72}$$

subject to the initial conditions given in (3.71). Applying (3.72), values for the first four log-exponential-squared integrals are:

$$\int_0^\infty e^{-x^2} \log(x)\, dx = -\frac{\sqrt{\pi}}{4} (\gamma + 2\log(2)), \tag{3.73}$$

$$\int_0^\infty e^{-x^2} \log^2(x)\, dx = \frac{\sqrt{\pi}}{8} \left[(\gamma + 2\log(2))^2 + \frac{\pi^2}{2}\right], \tag{3.74}$$

$$\begin{aligned}\int_0^\infty e^{-x^2} \log^3(x)\, dx = -\frac{\sqrt{\pi}}{32} \big[&24\gamma \log^2(2) + 16\log^3(2) + 6\pi^2 \log(2) \\ &+12\gamma^2 \log(2) + 3\gamma\pi^2 + 2\gamma^3 + 28\zeta(3)\big],\end{aligned} \tag{3.75}$$

$$\begin{aligned}\int_0^\infty e^{-x^2} \log^4(x)\, dx = \sqrt{\pi} \bigg[&\frac{7}{4}\gamma\zeta(3) + \frac{7}{2}\zeta(3)\log(2) + \frac{\gamma^4}{32} + \frac{7\pi^4}{128} + \frac{1}{2}\log^4(2) \\ &+ \gamma \log^3(2) + \frac{3\pi^2}{8}\log^2(2) + \frac{3\pi^2\gamma^2}{32} + \frac{3\gamma^2}{4}\log^2(2) \\ &+ \frac{\gamma^3}{4}\log(2) + \frac{3\pi^2\gamma}{8}\log(2)\bigg].\end{aligned} \tag{3.76}$$

The values for integrals (3.73) and (3.74) appear in [**19**] under entries **4.333.1** and **4.335.2** respectively while the values for (3.75) and (3.76) do not appear in [**19**].

The next example we give is for a quite general integral from which many interesting and well-known integrals follow as special cases.

EXAMPLE 3.8. Consider the integral

$$\text{Ч}_{m,n} = \int_0^1 x^{m-1} \log^n(1-x)\, dx, \tag{3.77}$$

where $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Thus

$$\begin{aligned} G(t) &= \sum_{n=0}^{\infty} \frac{\text{Ч}_{m,n} t^n}{n!} = \int_0^1 x^{m-1} \sum_{n=0}^{\infty} \frac{(t\log(1-x))^n}{n!}\, dx = \int_0^1 x^{m-1}(1-x)^t\, dx \\ &= \mathrm{B}(m, t+1) = \frac{\Gamma(m)\Gamma(t+1)}{\Gamma(t+m+1)}. \end{aligned}$$

Taking the logarithmic derivative with respect to t we find

$$\frac{G'(t)}{G(t)} = \psi(t+1) - \psi(t+m+1) = -\sum_{k=1}^{m} \frac{1}{t+k}, \tag{3.78}$$

where for the equality on the right the difference equation for the digamma function given in (2.10) has been used. Expanding the summand found in (3.78) in terms of an infinite geometric expansion yields

$$\frac{G'(t)}{G(t)} = -\sum_{k=1}^{m} \frac{1}{k} \sum_{i=0}^{\infty} (-1)^i \left(\frac{t}{k}\right)^i = -\sum_{i=0}^{\infty} (-1)^i t^i \sum_{k=1}^{m} \frac{1}{k^{i+1}},$$

after the order of the finite and infinite summations have been interchanged and is permissible since for $|t| < 1$ the sequence $\{a_{ik}\} = \left\{(-1)^i t^i / k^{i+1}\right\}$ converges as $i \to \infty$. Thus

$$\frac{G'(t)}{G(t)} = -\sum_{n=1}^{\infty} (-1)^n H_m^{(n+1)} t^m, \tag{3.79}$$

where we have changed the summation index i back to n and made use of the definition for the nth generalised harmonic number given in (2.19). From (3.79) we may write

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{Ч_{m,n+1} t^n}{n!} &= -\left(\sum_{n=0}^{\infty} \frac{Ч_{m,n} t^n}{n!}\right) \cdot \left(\sum_{n=0}^{\infty} (-1)^n H_m^{(n+1)} t^n\right) \\ &= -\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^k \frac{n! H_m^{(k+1)}}{(n-k)!} Ч_{m,n-k}\right) \frac{t^n}{n!}.\end{aligned}$$

In the second line we have used the Cauchy product for power series given in (2.25). On comparing the coefficient for $t^n/n!$ we immediately find

$$Ч_{m,n+1} = \sum_{k=0}^{n} (-1)^{k+1} \frac{n! H_m^{(k+1)}}{(n-k)!} Ч_{m,n-k}, \tag{3.80}$$

subject to the initial condition of $Ч_{m,0} = \int_0^1 x^{m-1}\, dx = \frac{1}{m}$.

Remark 3.10. A number of well-known and interesting cases follow from (3.80) as special cases. Setting $m = 1$, since $H_1^{(k+1)} = 1$ for all $k \in \mathbb{N}_0$ we have $Ч_{1,0} = 1$ and

$$Ч_{1,n+1} = \sum_{k=0}^{n} (-1)^{k+1} \frac{n!}{(n-k)!} Ч_{1,n-k}, \tag{3.81}$$

or $Ч_{1,n+1} = (-1)^{n+1}(n+1)!$, a result that can be proved using induction on n. It corresponds to the elementary result of

$$\int_0^1 \log^n(1-x)\, dx = (-1)^n n!.$$

Enforcing a substitution of $x \mapsto 1 - x$ immediately yields (3.2).

The case when n is a non-negative integer gives the interesting case where the integrals requiring evaluation can be obtained in terms of the (generalised) harmonic numbers. When $n = 0$ we have

$$Ч_{m,1} = \int_0^1 x^{m-1} \log(1-x)\, dx = -\frac{H_m}{m}. \tag{3.82}$$

This is a well known classical result [**49**, p. 214, Problem 1031]. It also appears in thinly disguised form as entry **4.293.8** in [**19**].

For higher positive integral powers of the logarithmic term in (3.77), setting n equal to 1 through to 4 in (3.80) yields

$$\int_0^1 x^{m-1}\log^2(1-x)\,dx = \frac{H_m^2 + H_m^{(2)}}{m}, \tag{3.83}$$

$$\int_0^1 x^{m-1}\log^3(1-x)\,dx = -\frac{H_m^3 + 3H_m H_m^{(2)} + 2H_m^{(3)}}{m}, \tag{3.84}$$

$$\int_0^1 x^{m-1}\log^4(1-x)\,dx = \frac{1}{m}\Big(H_m^4 + 6H_m^2 H_m^{(2)} + 8H_m H_m^{(3)} + 3(H_m^{(2)})^2 + 6H_m^{(4)}\Big), \tag{3.85}$$

$$\int_0^1 x^{m-1}\log^5(1-x)\,dx = -\frac{1}{m}\Big(H_m^5 + 10H_m^3 H_m^{(2)} + 20H_m^2 H_m^{(3)} + 30H_m H_m^{(4)} + 15H_m (H_m^{(2)})^2 + 20H_m^{(2)} H_m^{(3)} + 24H_m^{(5)}\Big). \tag{3.86}$$

The integrals (3.83), (3.84), and (3.85) appear as Eqs (1.5), (1.6), and (1.7) in [**48**] and as Eqs (2.71), (2.72), and (2.73) in [**42**]. For the general case, Vălean gives the expression for integral (3.77) in terms of the complete homogeneous symmetric polynomials [**48**, pp. 63–64] rather than in terms of a recurrence relation as we have done here.

Example 3.9. As our final example in this section we consider the quite general logarithmic integral of

$$\Lambda_{m,n} = \int_0^\infty \frac{\log^n(x)}{(1+x)^m}\,dx.$$

Here $m = 2, 3, 4, \ldots$ while $n \in \mathbb{N}_0$. Thus we have

$$G(t) = \sum_{n=0}^\infty \frac{\Lambda_{m,n} t^n}{n!} = \int_0^\infty \frac{1}{(1+x)^m} \sum_{n=0}^\infty \frac{(t\log x)^n}{n!}\,dx = \int_0^\infty \frac{x^t}{(1+x)^m}\,dx$$

$$= \mathrm{B}(t+1, m-t-1) = \frac{\Gamma(t+1)\Gamma(m-t-1)}{\Gamma(m)} = \frac{t\Gamma(t)\Gamma(m-t-1)}{\Gamma(m)}. \tag{3.87}$$

Note here the integral representation for the beta function given in (2.6) has been used. As m is a positive integer greater than one, from repeated application of the functional relation for the gamma function we see that

$$\Gamma(m-1-t) = \Gamma(1-t)\prod_{k=1}^{m-2}(k-t) = \frac{\pi}{\sin(\pi t)\Gamma(t)}\prod_{k=1}^{m-2}(k-t),$$

where in the equality on the right we have applied Euler's reflexion formula. Thus (3.87) becomes

$$G(t) = \frac{t\pi}{\Gamma(m)\sin(\pi t)}\prod_{k=1}^{m-2}(k-t).$$

Taking the logarithmic derivative with respect to t we have

$$\frac{G'(t)}{G(t)} = \frac{1}{t} - \pi\cot(\pi t) - \sum_{k=1}^{m-2} \frac{1}{k-t}. \tag{3.88}$$

Expanding the cotangent term into a power series using (2.16) with x replaced with πt and expanding the summand for the finite sum found in (3.88) in terms of an infinite geometric series, for $|t| < 1$ (3.88) may be written as

$$\frac{G'(t)}{G(t)} = \sum_{n=1}^{\infty} \frac{2^{n+1}|B_{n+1}|\pi^{n+1}}{(n+1)!} t^n - \sum_{k=1}^{m-2} \sum_{n=0}^{\infty} \frac{t^n}{k^{n+1}} \tag{3.89}$$

The order of the finite and infinite summations appearing in (3.89) can be interchanged since for $|t| < 1$ the sequence $\{a_{nk}\} = \left\{t^n/k^{n+1}\right\}$ converges as $n \to \infty$. Adjusting all sums in the summation index n so they start at one and employing the Cauchy product of power series where needed, we can rewrite (3.89) as

$$\begin{aligned} \Lambda_{m,1} + \sum_{n=1}^{\infty} \frac{\Lambda_{m,n+1} t^n}{n!} &= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{2^{k+1}|B_{k+1}|\pi^{k+1} n! \Lambda_{m,n-k}}{(n-k)!(k+1)!} \right) \frac{t^n}{n!} \\ &\quad - H_{m-2}\Lambda_{m,0} - \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} \frac{H_{m-2}^{(k+1)} n! \Lambda_{m,n-k}}{(n-k)!} \right) \frac{t^n}{n!}. \end{aligned} \tag{3.90}$$

Here we have made use of the definition for the nth generalised harmonic number given in (2.19). Now

$$\Lambda_{m,0} = \int_0^{\infty} \frac{dx}{(1+x)^m} = \mathrm{B}(1, m-1) = \frac{\Gamma(m-1)}{\Gamma(m)} = \frac{1}{m-1}. \tag{3.91}$$

Also

$$\begin{aligned} \Lambda_{m,1} &= \int_0^{\infty} \frac{\log(x)}{(1+x)^m} = \frac{\partial}{\partial s} \int_0^{\infty} \frac{x^s}{(1+x)^m}\, dx \Big|_{s=0} = \frac{\partial}{\partial s} \mathrm{B}(s+1, m-s-1)\big|_{s=1} \\ &= \frac{\Gamma(m-1)}{\Gamma(m)} \left(\psi(1) - \psi(m-1)\right) = -\frac{H_{m-2}}{m-1}, \end{aligned} \tag{3.92}$$

where the result in (2.23) has been used together with the known value of $\psi(1) = -\gamma$ which is entry **8.366.1** in [**19**]. Substituting this value into (3.90) before equating equal coefficients of $t^n/n!$ we find

$$\Lambda_{m,n+1} = \sum_{k=1}^{n} \frac{2^{k+1}|B_{k+1}|\pi^{k+1} n! \Lambda_{m,n-k}}{(n-k)!(k+1)!} - \sum_{k=0}^{n} \frac{H_{m-2}^{(k+1)} n! \Lambda_{m,n-k}}{(n-k)!}, \tag{3.93}$$

subject to the initial conditions given in (3.91) and (3.92).

Remark 3.11. Setting $m = 2$ in (3.93), we are interested in the special case

$$\Lambda_{2,n} = \int_0^{\infty} \frac{\log^n(x)}{(1+x)^2}\, dx.$$

Since $H_0^{(k+1)} = 0$ for all k and as $\Lambda_{2,0} = 1$ and $\Lambda_{2,1} = 0$ the result given in (3.93) reduces to

$$\Lambda_{2,n+1} = \sum_{k=1}^{n} \frac{2^{k+1}|B_{k+1}|\pi^{k+1}n!\Lambda_{2,n-k}}{(n-k)!(k+1)!}. \tag{3.94}$$

Indeed, when the exponent of the logarithmic term is odd we have

$$\int_0^\infty \frac{\log^{2n+1}(x)}{(1+x)^2}\,dx = 0,$$

where $n \in \mathbb{N}_0$ and is a result that can be readily confirmed either by enforcing a substitution of $x \mapsto \frac{1}{x}$ or by recalling $B_{2k+1} = 0$ for all $k \in \mathbb{N}$.

For this specialisation, from the values given in (2.15) the first four non-zero integrals are:

$$\int_0^\infty \frac{\log^2(x)}{(1+x)^2}\,dx = \frac{\pi^2}{3}, \qquad \int_0^\infty \frac{\log^4(x)}{(1+x)^2}\,dx = \frac{7\pi^4}{15},$$
$$\int_0^\infty \frac{\log^6(x)}{(1+x)^2}\,dx = \frac{31\pi^6}{21}, \qquad \int_0^\infty \frac{\log^8(x)}{(1+x)^2}\,dx = \frac{127\pi^8}{15}.$$

Remark 3.12. For the specialisation $m = 3$ the first four integrals are:

$$\int_0^\infty \frac{\log^2(x)}{(1+x)^3}\,dx = \frac{\pi^2}{6}, \qquad \int_0^\infty \frac{\log^3(x)}{(1+x)^3}\,dx = -\frac{\pi^2}{2},$$
$$\int_0^\infty \frac{\log^4(x)}{(1+x)^3}\,dx = \frac{7\pi^4}{30}, \qquad \int_0^\infty \frac{\log^5(x)}{(1+x)^3}\,dx = -\frac{7\pi^4}{6}.$$

And for the specialisation $m = 4$ the first four integrals are:

$$\int_0^\infty \frac{\log^2(x)}{(1+x)^4}\,dx = \frac{\pi^2}{9} + \frac{1}{3}, \qquad \int_0^\infty \frac{\log^3(x)}{(1+x)^4}\,dx = -\frac{\pi^2}{2},$$
$$\int_0^\infty \frac{\log^4(x)}{(1+x)^4}\,dx = \frac{2\pi^2}{3} + \frac{7\pi^4}{45}, \qquad \int_0^\infty \frac{\log^5(x)}{(1+x)^4}\,dx = -\frac{7\pi^4}{6}.$$

Observation of the values when the power of the logarithmic term is odd for the cases when $m = 3$ and 4 suggests

$$\int_0^\infty \frac{\log^{2n+1}(x)}{(1+x)^3}\,dx = \int_0^\infty \frac{\log^{2n+1}(x)}{(1+x)^4}\,dx,$$

for all $n \in \mathbb{N}_0$. That this is indeed the case can be readily shown. Consider

$$\Lambda_{3,2n+1} - \Lambda_{4,2n+1} = \int_0^\infty \frac{x\log^{2n+1}(x)}{(1+x)^4}\,dx.$$

Enforcing a substitution of $x \mapsto \frac{1}{x}$ one readily sees that $\Lambda_{3,2n+1} - \Lambda_{4,2n+1} = -(\Lambda_{3,2n+1} - \Lambda_{4,2n+1})$ or $\Lambda_{3,2n+1} = \Lambda_{4,2n+1}$ as required to show.

4. Some generalisations

In this section we provide two examples that generalise two of the integrals given as examples in the previous section.

We start with a generalisation of the integral given in Example 3.3. The generalisation is for the form of the log-tangent integral given in (3.7). Before we give this generalisation we give as a lemma the values of two integrals we need that will serve as initial conditions to the recurrence relation we shall find for our generalised integral.

LEMMA 4.1. *For* $m \in \mathbb{N}$ *we have*

$$\mathfrak{T}_{m,0} = \int_0^\infty \frac{dx}{(1+x^2)^{m+1}} = \frac{\pi}{2^{2m+1}}\binom{2m}{m},$$

and

$$\mathfrak{T}_{m,1} = \int_0^\infty \frac{\log(x)}{(1+x^2)^{m+1}}\,dx = -\frac{\pi}{2^{2m+2}}\binom{2m}{m}\left(2H_{2m} - H_m\right).$$

PROOF. For the first of the integrals enforcing a substitution of $x \mapsto \sqrt{x}$ produces

$$\mathfrak{T}_{m,0} = \frac{1}{2}\int_0^\infty \frac{dx}{\sqrt{x}(1+x)^{m+1}} = \frac{1}{2}\,\mathrm{B}\left(\frac{1}{2}, m+\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)},$$

where the integral representation for the beta function given in (2.6) has been used. Since m is a positive integer we have

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}(2m)!}{2^{2m}m!},\ \Gamma(m+1) = m!,\ \text{and}\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{4.1}$$

Thus

$$\mathfrak{T}_{m,0} = \frac{\pi(2m)!}{2^{2m+1}(m!)^2} = \frac{\pi}{2^{2m+1}}\binom{2m}{m},$$

as required to prove.

For the second of the integrals enforcing a substitution of $x \mapsto \sqrt{x}$ produces

$$\begin{aligned}\mathfrak{T}_{m,1} &= \frac{1}{4}\int_0^\infty \frac{\log(x)}{\sqrt{x}(1+x)^{m+1}}\,dx = \frac{1}{4}\frac{\partial}{\partial s}\int_0^\infty \frac{x^{s-\frac{1}{2}}}{(1+x)^{m+1}}\,dx\Bigg|_{s=0} \\ &= \frac{1}{4}\frac{\partial}{\partial s}\,\mathrm{B}\left(s+\frac{1}{2}, m-s+\frac{1}{2}\right)\Bigg|_{s=0} = \frac{1}{4}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)}\left[\psi\left(\frac{1}{2}\right) - \psi\left(m+\frac{1}{2}\right)\right] \\ &= -\frac{\pi}{2^{2m+2}}\binom{2m}{m}(2H_{2m} - H_m),\end{aligned}$$

as required to prove. Note in the last line the results given in (2.24) and (4.1) together with the value of $\psi\left(\frac{1}{2}\right) = -\gamma - 2\log(2)$, which is entry **8.366.2** in [**19**], have been used. □

An alternative proof of $\mathfrak{T}_{m,1}$ that does not involve using the derivative of the beta function can be found in [**35**].

EXAMPLE 4.1. A generalisation of the integral given in Example 3.3 can be given. Consider a generalisation of the form of the log-tangent integral given by

$$\mathfrak{T}_{m,n} = \int_0^\infty \frac{\log^n(x)}{(1+x^2)^{m+1}}\,dx, \tag{4.2}$$

where $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Enforcing a substitution of $x \mapsto \tan x$, the integral in (4.2) becomes

$$\mathfrak{T}_{m,n} = \int_0^{\frac{\pi}{2}} \cos^{2m} x \log^n(\tan x)\,dx.$$

Thus

$$\begin{aligned} G(t) = \sum_{n=0}^\infty \frac{\mathfrak{T}_{m,n} t^n}{n!} &= \int_0^{\frac{\pi}{2}} \cos^{2m} x \sum_{n=0}^\infty \frac{(t\log(\tan x))^n}{n!}\,dx \\ &= \int_0^{\frac{\pi}{2}} \sin^t x \cos^{2m-t} x\,dx = \frac{1}{2}\,\mathrm{B}\left(\frac{2m+1-t}{2}, \frac{t+1}{2}\right). \end{aligned} \tag{4.3}$$

Here the integral representation for the beta function given in (2.5) has been used. Now as

$$\mathrm{B}\left(\frac{2m+1-t}{2}, \frac{t+1}{2}\right) = \frac{(-1)^m \pi}{2^m m!} \sec\left(\frac{\pi t}{2}\right) \prod_{k=1}^m (t-(2k-1)),$$

a result that can be proved using induction on m, (4.3) becomes

$$G(t) = \frac{(-1)^m \pi}{2^{m+1} m!} \sec\left(\frac{\pi t}{2}\right) \prod_{k=1}^m (t-(2k-1)),$$

or

$$\frac{G'(t)}{G(t)} = \frac{\pi}{2}\tan\left(\frac{\pi t}{2}\right) + \sum_{k=1}^m \frac{1}{t-(2k-1)}, \tag{4.4}$$

after the logarithmic derivative with respect to t has been taken. For the finite sum, if its summand is expanded as an infinite geometric series, for $|t| < 1$ we have

$$\begin{aligned} \sum_{k=1}^m \frac{1}{t-(2k-1)} &= -\sum_{k=1}^m \frac{1}{2k-1} \frac{1}{1-\left(\frac{t}{2k-1}\right)} = -\sum_{k=1}^m \frac{1}{2k-1} \sum_{i=0}^\infty \left(\frac{t}{2k-1}\right)^i \\ &= -\sum_{i=0}^\infty t^i \sum_{k=1}^m \frac{1}{(2k-1)^{i+1}} = -\sum_{n=1}^\infty \left(H_{2m}^{(n+1)} - \frac{1}{2^{n+1}} H_m^{(n+1)}\right) t^n. \end{aligned} \tag{4.5}$$

Here the change made in the order of the finite and infinite summations is permissible since for $|t| < 1$ the sequence $\{a_{ik}\} = \left\{t^i/(2k-1)^{i+1}\right\}$ converges as $i \to \infty$. Note the result given in (2.21) has been used while the dummy summation index appearing in the series in the last equality has been reverted from i back to n. If the tangent term appearing in (4.4) is now expanded using the Maclaurin series expansion in (2.14), when combined with the result given in (4.5) the expression in (4.4) can be rewritten as

$$\frac{G'(t)}{G(t)} = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{2^{n+1}(2^{n+1}-1)|B_{n+1}|}{(n+1)!}\left(\frac{\pi}{2}\right)^n t^n - \sum_{n=0}\left(H_{2m}^{(n+1)} - \frac{1}{2^{n+1}}H_m^{(n+1)}\right)t^n,$$

or as

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\mathfrak{T}_{m,n+1}t^n}{n!} &= \frac{1}{t}\sum_{n=1}^{\infty}\frac{\mathfrak{T}_{m,n}t^n}{n!}\cdot\sum_{n=1}^{\infty}\frac{2^{n+1}(2^{n+1}-1)|B_{n+1}|}{(n+1)!}\left(\frac{\pi}{2}\right)^{n+1}t^n\\ &\quad -\sum_{n=0}^{\infty}\frac{\mathfrak{T}_{m,n}t^n}{n!}\cdot\sum_{n=0}^{\infty}\left(H_{2m}^{(n+1)}-\frac{1}{2^{n+1}}H_m^{(n+1)}\right)t^n. \qquad (4.6)\end{aligned}$$

Employing the Cauchy product for power series in the form of (2.26) for the first of the products and in the form of (2.25) for the second of the products appearing in (4.6), if all the sums in the summation index n are adjusted to start at one we have

$$\begin{aligned}\mathfrak{T}_{m,1} + \sum_{n=1}^{\infty}\frac{\mathfrak{T}_{m,n+1}t^n}{n!} &= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{2^{k+1}(2^{k+1}-1)n!|B_{k+1}|}{(k+1)!(n-k)!}\left(\frac{\pi}{2}\right)^{k+1}\mathfrak{T}_{m,n-k}\right)\frac{t^n}{n!}\\ &\quad -\frac{1}{2}(2H_{2m}-H_m)\mathfrak{T}_{m,0} - \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}\frac{n!\left(2^{k+1}H_{2m}^{(k+1)}-H_m^{(k+1)}\right)\mathfrak{T}_{m,n-k}}{2^{k+1}(n-k)!}\right)\frac{t^n}{n!}. \qquad (4.7)\end{aligned}$$

From the two initial conditions given in Lemma 4.1 the first term on the left of the equality in (4.7) and the second term on the right cancel. Comparing equal coefficients for $t^n/n!$ we then see that

$$\begin{aligned}\mathfrak{T}_{m,n+1} &= \sum_{k=1}^{n}\frac{2^{k+1}(2^{k+1}-1)n!|B_{k+1}|}{(k+1)!(n-k)!}\left(\frac{\pi}{2}\right)^{k+1}\mathfrak{T}_{m,n-k}\\ &\quad -\sum_{k=0}^{n}\frac{n!\left(2^{k+1}H_{2m}^{(k+1)}-H_m^{(k+1)}\right)}{2^{k+1}(n-k)!}\mathfrak{T}_{m,n-k}, \qquad (4.8)\end{aligned}$$

subject to the two initial conditions given in Lemma 4.1.

Remark 4.1. The method used to find the second of the integrals given in Lemma 4.1 involving the derivative of the beta function provides an alternative method for finding integral (4.2) for all positive integers n. Following this approach we find

$$\mathfrak{T}_{m,n} = \frac{1}{2^{n+1}}\frac{\partial^n}{\partial s^n}\,\mathrm{B}\left(s+\frac{1}{2}, m-s+\frac{1}{2}\right)\bigg|_{s=0}. \qquad (4.9)$$

Compared however to the recurrence relation given for $\mathfrak{T}_{m,n}$ which only involves elementary arithmetical operations of addition, multiplication, subtraction, and division, expression (4.9) in terms of an nth order derivative of the beta function is far from trivial.

REMARK 4.2. For the specialisations of $m = 1$ (left column) and $m = 2$ (right column) the first five integrals in each case are:

$$\int_0^\infty \frac{\log(x)}{(1+x^2)^2}\,dx = -\frac{\pi}{4}, \qquad \int_0^\infty \frac{\log(x)}{(1+x^2)^3}\,dx = -\frac{\pi}{4},$$
$$\int_0^\infty \frac{\log^2(x)}{(1+x^2)^2}\,dx = \frac{\pi^3}{16}, \qquad \int_0^\infty \frac{\log^2(x)}{(1+x^2)^3}\,dx = \frac{\pi}{8} + \frac{3\pi^3}{64},$$
$$\int_0^\infty \frac{\log^3(x)}{(1+x^2)^2}\,dx = -\frac{3\pi^3}{16}, \qquad \int_0^\infty \frac{\log^3(x)}{(1+x^2)^3}\,dx = -\frac{3\pi^3}{16},$$
$$\int_0^\infty \frac{\log^4(x)}{(1+x^2)^2}\,dx = \frac{5\pi^5}{64}, \qquad \int_0^\infty \frac{\log^4(x)}{(1+x^2)^3}\,dx = \frac{3\pi^3}{16} + \frac{15\pi^5}{256},$$
$$\int_0^\infty \frac{\log^5(x)}{(1+x^2)^2}\,dx = -\frac{25\pi^5}{64}, \qquad \int_0^\infty \frac{\log^5(x)}{(1+x^2)^3}\,dx = -\frac{25\pi^5}{64}.$$

From the values of the integrals for the specialisations $m = 1$ and $m = 2$, when the logarithmic term is raised to an odd power it appears as though

$$\int_0^\infty \frac{\log^{2n+1}(x)}{(1+x^2)^2}\,dx = \int_0^\infty \frac{\log^{2n+1}(x)}{(1+x^2)^3}\,dx,$$

for all $n \in \mathbb{N}_0$. That this is indeed the case can readily be shown. Consider

$$\mathfrak{T}_{1,2n+1} - \mathfrak{T}_{2,2n+1} = \int_0^\infty \frac{x^2 \log^{2n+1}(x)}{(1+x^2)^3}\,dx.$$

Enforcing a substitution of $x \mapsto \frac{1}{x}$ one readily sees that $\mathfrak{T}_{1,2n+1} - \mathfrak{T}_{2,2n+1} = -(\mathfrak{T}_{1,2n+1} - \mathfrak{T}_{2,2n+1})$ or $\mathfrak{T}_{1,2n+1} = \mathfrak{T}_{2,2n+1}$, as required to show.

REMARK 4.3. As an alternative for the evaluation of the integral (4.2) for the specialisations of $m = 1$ and $m = 2$, when solved using the exponential generating function technique in an identical manner to how the integral in Example 3.3 was evaluated, it can be shown that

$$\int_0^\infty \frac{\log^n(x)}{(1+x^2)^2}\,dx = \frac{1}{4}\left(\pi|E_n| - 2n|E_{n-1}|\right)\left(\frac{\pi}{2}\right)^n, \tag{4.10}$$

valid for $n = 1, 2, 3, \ldots$ and

$$\int_0^\infty \frac{\log^n(x)}{(1+x^2)^3}\,dx = \frac{1}{16}\left(3\pi|E_n| - 8n|E_{n-1}| + \frac{4}{\pi}n(n-1)|E_{n-2}|\right)\left(\frac{\pi}{2}\right)^n, \tag{4.11}$$

valid for $n = 2, 3, 4, \ldots$. Having available explicit expressions such as (4.10) and (4.11) for integral (4.2) is obviously preferred over the value for the integral given in terms of the recurrence relation (4.8). The problem however with these explicit expressions is they are only valid for $n \geqslant m$. At least for the first few lowest orders of m such as (4.10) and (4.11) this is not a problem as one does not have to find separately too many integrals for those cases where $0 \leqslant n < m$. For higher orders of m many more integrals would first need to be found before the explicit formula could be used, making the recurrence relation more useful in such situations. A generalisation of this approach leading to explicit formulae for integrals of all orders of m in terms of Euler numbers has not been forthcoming.

For our final example we generalise the log-sine integral given in Example 3.4. Before we give this generalisation we again give as a lemma the values of two integrals we need that will serve as initial conditions to the recurrence relation we shall find for our generalised integral.

LEMMA 4.2. *For $m \in \mathbb{N}_0$ we have*

$$\Theta_{m,0} = \int_0^{\frac{\pi}{2}} \sin^{2m} x \, dx = \frac{\pi}{2^{2m+1}} \binom{2m}{m},$$

and

$$\Theta_{m,1} = \int_0^{\frac{\pi}{2}} \sin^{2m} x \log(\sin x) \, dx = \frac{\pi}{2^{2m+1}} \binom{2m}{m} \left(H_{2m} - H_m - \log(2) \right).$$

PROOF. For the first of the integrals it is directly of the form of the beta function given in (2.5). Thus

$$\Theta_{m,0} = \frac{1}{2} \, \mathrm{B}\left(m + \frac{1}{2}, \frac{1}{2} \right) = \frac{\Gamma\left(m + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2\Gamma(m+1)},$$

with the desired result following on applying the results for the gamma functions given in (4.1).

For the second of the integrals

$$\Theta_{m,1} = \frac{\partial}{\partial s} \int_0^{\frac{\pi}{2}} \sin^{2m+s} x \, dx \bigg|_{s=0} = \frac{1}{2} \frac{\partial}{\partial s} \, \mathrm{B}\left(m + \frac{s}{2} + \frac{1}{2}, \frac{1}{2} \right) \bigg|_{s=0}$$
$$= \frac{\pi}{2^{2m+2}} \binom{2m}{m} \left[\psi\left(m + \frac{1}{2} \right) - \psi(m+1) \right],$$

with the desired result following on substituting the results (2.23) and (2.24) for the digamma function terms and completes the proof. □

EXAMPLE 4.2. We now give a generalisation of the log-sine integral appearing in Example 3.4. Consider

$$\Theta_{m,n} = \int_0^{\frac{\pi}{2}} \sin^{2m} x \log^n(\sin x) \, dx, \tag{4.12}$$

where $m, n \in \mathbb{N}_0$. So

$$G(t) = \sum_{n=0}^{\infty} \frac{\Theta_{m,n} t^n}{n!} = \int_0^{\frac{\pi}{2}} \sin^{2m} x \sum_{n=0}^{\infty} \frac{(t \log \sin x)^n}{n!} \, dx$$
$$= \int_0^{\frac{\pi}{2}} \sin^{2m+t} x \, dx = \frac{1}{2} \, \mathrm{B}\left(m + \frac{t+1}{2}, \frac{1}{2} \right) = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(m + \frac{t+1}{2}\right)}{\Gamma\left(m + 1 + \frac{t}{2}\right)},$$

where the integral representation given in (2.5) for the beta function has been used, or

$$G(t) = \frac{\pi}{2^{2m+t+1}} \frac{\Gamma(2m+t+1)}{\Gamma^2\left(m+1+\frac{t}{2}\right)}, \tag{4.13}$$

after the duplication formula for the gamma function of (2.8) has been applied to the gamma function term appearing in the numerator of (4.13). Taking the logarithmic derivative of (4.13) with respect to t we find

$$\frac{G'(t)}{G(t)} = -\log(2) + \psi(2m+t+1) - \psi\left(m+1+\frac{t}{2}\right). \tag{4.14}$$

Applying (2.10) to each of the digamma function terms appearing in (4.14) produces

$$\frac{G'(t)}{G(t)} = -\log(2) + \psi(t+1) - \psi\left(\frac{t}{2}+1\right) + \sum_{k=1}^{2m} \frac{1}{t+k} - 2\sum_{k=1}^{m} \frac{1}{t+2k}. \tag{4.15}$$

Noting that

$$\begin{aligned}\sum_{k=1}^{2m} \frac{1}{t+k} - 2\sum_{k=1}^{m} \frac{1}{t+2k} &= \sum_{k=1}^{2m} \frac{(-1)^{k+1}}{t+k} = \sum_{k=1}^{2m} \frac{(-1)^{k+1}}{k} \frac{1}{1+\frac{t}{k}} \\ &= \sum_{k=1}^{2m} \frac{(-1)^{k+1}}{k} \sum_{i=0}^{\infty} (-1)^i \left(\frac{t}{k}\right)^i = \sum_{i=0}^{\infty} (-1)^i t^i \sum_{k=1}^{2m} \frac{(-1)^{k+1}}{k^{i+1}} \\ &= \sum_{n=0}^{\infty} (-1)^n \overline{H}_{2m}^{(n+1)} t^n,\end{aligned}$$

on reverting the dummy summation index from i back to n. Here the change made in the order of the finite and infinite summations is permissible since for $|t| < 1$ the sequence $\{a_{ik}\} = \{(-1)^{k+i+1} t^i / k^{i+1}\}$ converges as $i \to \infty$. Note the definition for the generalised nth skew-harmonic number of order p given in (2.20) has been used. Substituting this result into (4.15), after expanding the difference between two digamma functions terms appearing in (4.15) using (2.27) we find

$$\frac{G'(t)}{G(t)} = -\log(2) - \sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{1}{2^n}\right) \zeta(n+1) t^n + \sum_{n=0}^{\infty} (-1)^n \overline{H}_{2m}^{(n+1)} t^n,$$

or

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\Theta_{m,n+1} t^n}{n!} &= -\log(2) \sum_{n=0}^{\infty} \frac{\Theta_{m,n} t^n}{n!} + \left(\sum_{n=0}^{\infty} \frac{\Theta_{m,n} t^n}{n!}\right) \cdot \left(\sum_{n=0}^{\infty} (-1)^n \overline{H}_{2m}^{(n+1)} t^n\right) \\ &\quad - \frac{1}{t} \left(\sum_{n=1}^{\infty} \frac{\Theta_{m,n-1} t^n}{(n-1)!}\right) \cdot \left(\sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{1}{2^n}\right) \zeta(n+1) t^n\right)\end{aligned} \tag{4.16}$$

Employing the Cauchy product for power series in the form of (2.25) for the first of the products and in the form of (2.26) for the second of the products appearing in

(4.16) before adjusting all sums in the summation index n to start at one, we have

$$\begin{aligned}\Theta_{m,1}+\sum_{n=1}^{\infty}\frac{\Theta_{m,n+1}t^n}{n!}&=-\log(2)\Theta_{m,0}-\log(2)\sum_{n=1}^{\infty}\frac{\Theta_{m,n}t^n}{n!}\\&\quad-\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1)^k\left(1-\frac{1}{2^k}\right)\frac{\zeta(k+1)n!\,\Theta_{m,n-k}}{(n-k)!}\right)\frac{t^n}{n!}\\&\quad+\overline{H}_{2m}\Theta_{m,0}+\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}(-1)^k\frac{\overline{H}_{2m}^{(k+1)}n!\,\Theta_{m,n-k}}{(n-k)!}\right)\frac{t^n}{n!}.\end{aligned}\tag{4.17}$$

From the two initial conditions given in Lemma 4.2 together with the Botez–Catalan identity which is (2.22) when $p=1$, the first term on the left of the equality in (4.17) cancels with the first and fourth terms on the right. Comparing equal coefficients for $t^n/n!$ we finally see that

$$\begin{aligned}\Theta_{m,n+1}&=-\log(2)\Theta_{m,n}-\sum_{k=1}^{n}(-1)^k\left(1-\frac{1}{2^k}\right)\frac{\zeta(k+1)n!}{(n-k)!}\Theta_{m,n-k}\\&\quad+\sum_{k=0}^{n}\frac{(-1)^k\overline{H}_{2m}^{(k+1)}n!}{(n-k)!}\Theta_{m,n-k},\end{aligned}\tag{4.18}$$

subject to the two initial conditions given in Lemma 4.2.

Remark 4.4. An evaluation for the special case of

$$\Theta_{1,2}=\int_0^{\frac{\pi}{2}}\sin^2 x\log^2(\sin x)\,dx=\frac{\pi^3}{48}-\frac{\pi}{8}-\frac{\pi}{4}\log(2)+\frac{\pi}{4}\log^2(2),$$

was given as a problem in [**14, 46**].

5. A class of infinite sums

In this section we give a class of series involving the central binomial coefficients that are related to the log-sine integral (1.2) and its generalisation (4.12). We begin with a lemma.

Lemma 5.1. *If $n\in\mathbb{N}$ and $k,m\in\mathbb{N}_0$ then*

$$\frac{1}{(2m+2k+1)^n}=\frac{(-1)^{n-1}}{(n-1)!}\int_0^1 t^{2m+2k}\log^{n-1}(t)\,dt.\tag{5.1}$$

Proof. Induction on n gives the result at once. ☐

The class of series we are interested in involve the central binomial coefficients. We give this in the following theorem.

Theorem 5.1. *If $n\in\mathbb{N}$ and $k\in\mathbb{N}_0$ then*

$$\sum_{m=0}^{\infty}\binom{2m}{m}\frac{1}{4^m(2m+2k+1)^n}=\frac{(-1)^{n-1}}{(n-1)!}\Theta_{k,n-1}.\tag{5.2}$$

Here $\Theta_{k,n}$ is the term found from the recurrence relation given in (4.18).

PROOF. Using the result given in (5.1) the series can be expressed as

$$\sum_{m=0}^{\infty}\binom{2m}{m}\frac{1}{4^m(2m+2k+1)^n}=\frac{(-1)^{n-1}}{(n-1)!}\int_0^1 t^{2k}\log^{n-1}(t)\sum_{m=0}^{\infty}\binom{2m}{m}\frac{t^{2m}}{4^m}\,dt. \tag{5.3}$$

Here the interchange made between the integration and summation is permissible due to Fubini's theorem [**17**, p. 55, Theorem 2.25]. Recalling the well-known result of

$$\sum_{m=0}^{\infty}\binom{2m}{m}\frac{t^{2m}}{4^m}=\frac{1}{\sqrt{1-t^2}},\quad |t|<1,$$

allows one to rewrite (5.3) as

$$\sum_{m=0}^{\infty}\binom{2m}{m}\frac{1}{4^m(2m+2k+1)^n}=\int_0^1\frac{t^{2k}\log^{n-1}(t)}{\sqrt{1-t^2}}\,dt.$$

Enforcing a substitution of $t=\sin x$ yields

$$\begin{aligned}\sum_{m=0}^{\infty}\binom{2m}{m}\frac{1}{4^m(2m+2k+1)^n}&=\frac{(-1)^{n-1}}{(n-1)!}\int_0^1\sin^{2k}x\log^{n-1}(\sin x)\,dx\\&=\frac{(-1)^{n-1}}{(n-1)!}\Theta_{k,n-1},\end{aligned}$$

in view of (4.12), as required to prove. □

REMARK 5.1. When the value of n is small, expressions for the sums given by (5.2) can be written out explicitly. The first four of these are:

$$\sum_{m=0}^{\infty}\binom{2m}{m}\frac{1}{4^m(2m+2k+1)}=\frac{\pi}{2^{2k+1}}\binom{2k}{k}, \tag{5.4}$$

$$\sum_{m=0}^{\infty}\binom{2m}{m}\frac{1}{4^m(2m+2k+1)^2}=\frac{\pi}{2^{2k+1}}\binom{2k}{k}\left(\log(2)+H_k-H_{2k}\right), \tag{5.5}$$

$$\begin{aligned}\sum_{m=0}^{\infty}\binom{2m}{m}\frac{1}{4^m(2m+2k+1)^3}&=\frac{\pi}{2^{2k+3}}\binom{2k}{k}\left(\frac{\pi^2}{6}+H_k^{(2)}-2H_{2k}^{(2)}\right)\\&\quad+\frac{\pi}{2^{2k+2}}\binom{2k}{k}\left(\log(2)+H_k-H_{2k}\right)^2,\end{aligned} \tag{5.6}$$

$$\begin{aligned}\sum_{m=0}^{\infty}\binom{2m}{m}\frac{1}{4^m(2m+2k+1)^4}&=\frac{\pi}{3\cdot 2^{2k+3}}\binom{2k}{k}\left(3\zeta(3)+H_k^{(3)}-4H_{2k}^{(3)}\right)\\&\quad+\frac{\pi}{3\cdot 2^{2k+2}}\binom{2k}{k}\left(\log(2)+H_k-H_{2k}\right)^3\\&\quad+\frac{\pi}{2^{2k+3}}\binom{2k}{k}\left(\log(2)+H_k-H_{2k}\right)\left(\frac{\pi^2}{6}+H_k^{(2)}-2H_{2k}^{(2)}\right).\end{aligned} \tag{5.7}$$

In obtaining these expressions the result for the generalised skew-harmonic numbers with even indices have been written in terms of a difference between an even generalised harmonic number and a generalised harmonic number in accordance with the result given in (2.22).

REMARK 5.2. For the special case when $k = 0$ in (5.2) we have

$$\sum_{m=0}^{\infty} \binom{2m}{m} \frac{1}{4^m (2m+1)^n} = \frac{(-1)^{n-1}}{(n-1)!} \Theta_{0,n-1} = \frac{(-1)^{n-1}}{(n-1)!} S_{n-1}, \tag{5.8}$$

where S_n is the term given by the recurrence relation found in (3.16). Setting $n = 1$ to 5, using those values for S_0 to S_4 given by (3.17) to (3.20) the first five values for the series containing the central binomial coefficients when $k = 0$ are:

$$\sum_{m=0}^{\infty} \binom{2m}{m} \frac{1}{4^m (2m+1)} = \frac{\pi}{2}, \tag{5.9}$$

$$\sum_{m=0}^{\infty} \binom{2m}{m} \frac{1}{4^m (2m+1)^2} = \frac{\pi}{2} \log(2), \tag{5.10}$$

$$\sum_{m=0}^{\infty} \binom{2m}{m} \frac{1}{4^m (2m+1)^3} = \frac{\pi^3}{48} + \frac{\pi}{4} \log^2(2), \tag{5.11}$$

$$\sum_{m=0}^{\infty} \binom{2m}{m} \frac{1}{4^m (2m+1)^4} = \frac{\pi}{12} \log^3(2) + \frac{\pi^3}{48} \log(2) + \frac{\pi}{8} \zeta(3), \tag{5.12}$$

$$\sum_{m=0}^{\infty} \binom{2m}{m} \frac{1}{4^m (2m+1)^5} = \frac{19\pi^5}{11520} + \frac{\pi}{48} \log^4(2) + \frac{\pi^3}{96} \log^2(2) + \frac{\pi}{8} \log(2) \zeta(3). \tag{5.13}$$

Both [**6**, Eq. (12.5.5), p. 246] and [**36**, Proposition 5] gave the expression we found in (5.8) while a recent evaluation for series (5.11) can be found in [**45, 12**].

For the case when $k = 1$ in (5.2) the first five values for the next order series containing the central binomial coefficients are:

$$\sum_{m=0}^{\infty} \binom{2m}{m} \frac{1}{4^m (2m+3)} = \frac{\pi}{4}, \tag{5.14}$$

$$\sum_{m=0}^{\infty} \binom{2m}{m} \frac{1}{4^m (2m+3)^2} = \frac{\pi}{8} - \frac{\pi}{4} \log(2), \tag{5.15}$$

$$\sum_{m=0}^{\infty} \binom{2m}{m} \frac{1}{4^m (2m+3)^3} = \frac{\pi^3}{96} + \frac{\pi}{8} \log^2(2) - \frac{\pi}{16} - \frac{\pi}{8} \log(2), \tag{5.16}$$

$$\sum_{m=0}^{\infty} \binom{2m}{m} \frac{1}{4^m (2m+3)^4} = -\frac{\pi^3}{192} - \frac{\pi}{32} - \frac{\pi}{16} \log(2) - \frac{\pi}{16} \log^2(2) + \frac{\pi}{24} \log^3(2) + \frac{\pi^3}{96} \log(2) + \frac{\pi}{16} \zeta(3), \tag{5.17}$$

$$\sum_{m=0}^{\infty}\binom{2m}{m}\frac{1}{4^m(2m+3)^5}=\frac{19\pi^5}{23040}-\frac{\pi^3}{384}-\frac{\pi}{64}-\frac{\pi}{32}\log(2)-\frac{\pi}{32}\zeta(3)$$
$$-\frac{\pi}{32}\log^2(2)-\frac{\pi}{48}\log^3(2)-\frac{\pi^3}{192}\log(2)$$
$$+\frac{\pi}{96}\log^4(2)+\frac{\pi^3}{192}\log^2(2)+\frac{\pi}{16}\log(2)\zeta(3). \tag{5.18}$$

Observed how the expressions for the value of the series rapidly grow, quickly becoming long and more complicated as the value of n increases.

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