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# Closed-form expressions for Farhi's constant and related integrals and its generalization

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Abstract. In a recent work, Farhi developed a Fourier series expansion for the function $\ln\Gamma(x)$ on the interval $(0,1)$, which allowed him to derive a nice formula for the constant $\eta := 2\int_0^1 \ln\Gamma(x)\,\sin(2\pi x)\,dx$. At the end of that paper, he asks whether $\eta$ could be written in terms of other known mathematical constants. Here in this work, after deriving a simple closed-form expression for $\eta$, I show how it can be used for evaluating other related integrals, as well as certain logarithmic series, which allows for a generalization in the form of a continuous function $\eta(x)$, $x \in [0,1]$. Finally, from the Fourier series expansion of $\ln\Gamma(x)$, $x \in (0,1)$, I make use of Parseval's theorem to derive a closed-form expression for $\int_0^1 \ln^2\Gamma(x)\,dx$.

## 1. Introduction

The Fourier series expansion of the real function $\ln\Gamma(x)$ over the open interval $(0,1)$, where $\Gamma(x) := \int_0^\infty t^{\,x-1}\,e^{-t}\,dt$ is the classical Gamma function, was developed by Farhi in a recent note [**8**]. There, he shows that

$$\ln\Gamma(x) = \frac{1}{2}\ln\pi + \pi\,\eta\left(\frac{1}{2} - x\right) - \frac{1}{2}\ln\sin(\pi x) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\ln n}{n}\,\sin(2\pi n x) \tag{1.1}$$

holds for all $x \in (0,1)$, where

$$\eta := 2\int_0^1 \ln\Gamma(x)\,\sin(2\pi x)\,dx \tag{1.2}$$

is *Farhi's constant.* For this improper integral, numerical integration yields $\eta = 0.768747\ldots$ At the end of Ref. [**8**], Farhi asks if $\eta$ could be written in terms of other known mathematical constants.

Here in this paper, we make use of Farhi's formula itself to derive a closed-form expression for $\eta$ in terms of $\pi$, $\gamma$, and $\ln(2\pi)$ only, where $\gamma := \lim_{n\to\infty}(H_n - \ln n)$ is the Euler–Mascheroni constant, $H_n := \sum_{k=1}^{n} 1/k$ being the $n$-th harmonic number. We also show how to generalize Eq. (1.2) and how Eq. (1.1) can be taken into account for

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evaluating some logarithmic series and improper integrals, e.g. $\int_0^{1/2} \ln\Gamma(x)\,\sin(2\pi x)\,dx$, $\int_0^1 \ln\Gamma(x)\,\sin(\pi x)\,dx$, $\int_0^1 \psi(x)\,\sin^2(\pi x)\,dx$, $\int_0^1 \ln^2\Gamma(x)\,dx$, and $\int_0^{1/2} \psi(x)\,\sin^2(\pi x)\,dx$, where $\psi(x) := \frac{d}{dx}\ln\Gamma(x)$ is the digamma function.

## 2. Farhi's constant and some related integrals and series

LEMMA 2.1 (Farhi's constant). *The exact closed-form*

$$\eta = \frac{\gamma + \ln(2\pi)}{\pi}$$

*holds, where $\eta$ is the improper integral in Eq.* (1.2).

PROOF. On taking $x = 1/4$ in Farhi's formula, our Eq. (1.1), one finds

$$\ln\Gamma\left(\frac{1}{4}\right) = \frac{1}{2}\ln\pi + \frac{1}{4}\pi\,\eta + \frac{1}{4}\ln 2 + \frac{1}{\pi}\,S\,, \tag{2.1}$$

where

$$S := \sum_{n=1}^{\infty} \frac{\ln n}{n}\,\sin\left(\frac{\pi}{2}\,n\right) = \sum_{m=1}^{\infty} (-1)^{m+1}\,\frac{\ln(2m-1)}{2m-1}\,, \tag{2.2}$$

which agrees with Corollary 3 of Ref. [**8**]. Now, we take into account a functional relation established by Coffey in Eq. (3.13b) of Ref. [**4**], namely

$$\gamma_1\left(\frac{a+1}{2}\right) - \gamma_1\left(\frac{a}{2}\right) = \ln 2\,\left[\psi\left(\frac{a+1}{2}\right) - \psi\left(\frac{a}{2}\right)\right] + 2\sum_{k=0}^{\infty} (-1)^{k+1}\,\frac{\ln(k+a)}{k+a}\,, \tag{2.3}$$

where $\gamma_1(a)$ is the first generalized Stieltjes constant, a coefficient in the Laurent series expansion of the Hurwitz zeta function $\zeta(s,a)$ about its simple pole at $s = 1$, i.e. $\frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\,\gamma_n(a)\,(s-1)^n$. Here, $\zeta(s,a) := \sum_{n=0}^{\infty} 1/(n+a)^s$, $a \neq 0, -1, -2, \ldots$, a series that converges for all complex $s$ with $\Re(s) > 1$. In Eq. (2.3), we correct a mistake in the Stieltjes constants at the left-hand side of the equation corresponding to this one, as it appears in Ref. [**4**]. For $a = 1/2$, one finds

$$S = \frac{1}{4}\left[\gamma_1\left(\frac{1}{4}\right) - \gamma_1\left(\frac{3}{4}\right) + \pi\,\ln 4\right]. \tag{2.4}$$

On the other hand, in Eq. (3.21) of Ref. [**4**] Coffey showed that

$$\begin{aligned}\gamma_1\left(\frac{1}{4}\right) - \gamma_1\left(\frac{3}{4}\right) &= -\,\pi\left[\ln(8\pi) + \gamma - 2\,\ln\left(\frac{\Gamma(1/4)}{\Gamma(3/4)}\right)\right] \\ &= \pi\left[4\,\ln\Gamma\left(\frac{1}{4}\right) - 2\,\ln 4 - 3\,\ln\pi - \gamma\right],\end{aligned} \tag{2.5}$$

where the last step demanded the use of the reflection formula for $\Gamma(x)$, namely

$$\Gamma(1-x)\,\Gamma(x) = \frac{\pi}{\sin(\pi x)}\,. \tag{2.6}$$

On substituting the result of Eq. (2.5) in Eq. (2.4), one finds

$$S = \pi\,\ln\Gamma\left(\frac{1}{4}\right) - \frac{\pi}{2}\ln 2 - \frac{3}{4}\,\pi\ln\pi - \frac{\pi}{4}\,\gamma\,. \tag{2.7}$$

The proof completes by substituting this expression in Eq. (2.1). $\square$

The closed-form expression for $\eta$ established above could also have been deduced from Eq. (6.443.1) of Ref. [**10**] (the case $n = 1$), or it could have been determined by comparing Eq. (1.1) with the Kummer's Fourier series for $\ln\Gamma(x)$ mentioned by Connon in Eq. (2.9) of Ref. [**6**] (or Eq. (5.8) of Ref. [**7**]):

$$\ln\Gamma(x) = \frac{1}{2}\ln\left(\frac{\pi}{\sin(\pi x)}\right) + \left(\frac{1}{2} - x\right)\left[\gamma + \ln(2\pi)\right] + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\ln n}{n}\,\sin(2\pi n x)\,. \tag{2.8}$$

Note that in Corollary 9.6.53 of Ref. [**5**] Cohen uses Abel summation to derive a formula similar to that by Connon which holds for all real $x \notin \mathbb{Z}$. Another nice generalization of Farhi's formula was proposed by Blagouchine in Ex. 20(c) of Ref. [**2**], namely

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\ln(b\,n)}{n}\,\sin(n\,\phi) = \pi\,\ln\Gamma\left(\frac{\phi}{2\pi}\right) + \frac{\pi}{2}\,\ln\sin\left(\frac{\phi}{2}\right) - \frac{\pi}{2}\,\ln\pi \\ + \frac{\phi - \pi}{2}\left[\gamma + \ln\left(\frac{2\pi}{b}\right)\right],\end{aligned} \tag{2.9}$$

which is valid for all $b > 0$ and $\phi \in (0, 2\pi)$. On substituting $\phi = 2\pi x$, one shows that Connon's formula is the particular case $b = 1$ of Eq. (2.9).

Since the Farhi constant $\eta$ is defined as an improper integral, other similar integrals can be explored, as follows.

THEOREM 2.1 (An integral involving $\psi(x)$). *The following exact closed-form result holds:*

$$\int_0^1 \psi(x)\,\sin^2(\pi x)\,dx = -\,\frac{\gamma + \ln(2\pi)}{2}\,.$$

PROOF. From Lemma 2.1, one has

$$\pi\,\eta = 2\pi\int_0^1 \ln\Gamma(x)\left[2\sin(\pi x)\,\cos(\pi x)\right]\,dx = \gamma + \ln(2\pi)\,, \tag{2.10}$$

which implies that

$$I := \int_0^1 \ln\Gamma(x)\,\sin(\pi x)\,\cos(\pi x)\,dx = \frac{\gamma + \ln(2\pi)}{4\pi}\,. \tag{2.11}$$

Integration by parts then yields

$$\begin{aligned}I = \left[\ln\Gamma(x)\,\frac{\sin^2(\pi x)}{\pi}\right]_{0^+}^{1} - \int_0^1 \frac{\sin(\pi x)}{\pi}\,\left[\psi(x)\,\sin(\pi x) + \pi\,\cos(\pi x)\,\ln\Gamma(x)\right]dx \\ = 0 - \frac{1}{\pi}\int_0^1 \psi(x)\,\sin^2(\pi x)\,dx - \int_0^1 \sin(\pi x)\,\cos(\pi x)\,\ln\Gamma(x)\,dx \\ = -\frac{1}{\pi}\int_0^1 \psi(x)\,\sin^2(\pi x)\,dx - I\,.\end{aligned} \tag{2.12}$$

Therefore

$$2\,I = -\,\frac{1}{\pi}\int_0^1 \psi(x)\,\sin^2(\pi x)\,dx\,. \tag{2.13}$$

The substitution of this result in Eq. (2.11) completes the proof. □

A simple closed-form expression can also be determined for the integral obtained by halving the argument of the sine function in Eq. (1.2), namely $\int_0^1 \ln\Gamma(x)\,\sin(\pi x)\,dx = 0.462053\ldots$

THEOREM 2.2 (An integral involving $\ln\Gamma(x)$). *The following exact closed-form result holds:*

$$\int_0^1 \ln\Gamma(x)\,\sin(\pi x)\,dx = \frac{\ln\pi - \ln 2 + 1}{\pi}\,.$$

PROOF. On taking into account the logarithmic form of Eq. (2.6), namely

$$\ln\Gamma(1-x) + \ln\Gamma(x) = \ln\pi - \ln\sin(\pi x)\,, \tag{2.14}$$

one finds

$$\begin{aligned}
\int_0^1 [\ln\Gamma(1-x) + \ln\Gamma(x)]\sin(\pi x)\,dx \\
= \int_0^1 [\ln\pi - \ln\sin(\pi x)]\sin(\pi x)\,dx \\
= \ln\pi\int_0^1 \sin(\pi x)\,dx - \int_0^1 \ln\sin(\pi x)\,\sin(\pi x)\,dx \\
= \ln\pi\cdot\frac{2}{\pi} - \frac{1}{\pi}\int_0^\pi \sin\tilde\theta\,\ln\sin\tilde\theta\,d\tilde\theta\,. 
\end{aligned} \tag{2.15}$$

The symmetries of the first integrand with respect to $x = 1/2$ and of $\sin\tilde\theta$ with respect to $\tilde\theta = \pi/2$ lead to

$$\begin{aligned}
2\int_0^1 \ln\Gamma(x)\,\sin(\pi x)\,dx &= \frac{2}{\pi}\,\ln\pi - \frac{2}{\pi}\int_0^{\pi/2}\sin\tilde\theta\,\ln\sin\tilde\theta\,d\tilde\theta \\
&= 2\,\frac{\ln\pi}{\pi} - 2\,\frac{\ln 2 - 1}{\pi}\,.
\end{aligned} \tag{2.16}$$

□

On searching for other closed-form results, I have realized that the integration of both sides of Farhi's formula, our Eq. (1.1), could lead to an interesting result. As usual, $\mathrm{Cl}_2(\theta) := \Im\big[\mathrm{Li}_2\left(e^{i\,\theta}\right)\big] = \sum_{n=1}^{\infty}\sin(n\,\theta)/n^2$ is the Clausen function and $\zeta'(s,a)$ denotes a partial derivative of $\zeta(s,a)$ with respect to $s$.

THEOREM 2.3 (Integration of Farhi's formula). *For all* $x \in (0,1)$,

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\ln n}{n^2}\cos(2\pi n\,x) = \pi^2\left(x\,(1-x) - \frac{1}{6}\right)[\,\gamma + \ln(2\pi) - 1] + \frac{\pi}{2}\,\mathrm{Cl}_2(2\pi x) \\
- 2\pi^2\,\zeta'(-1,x)\,.
\end{aligned}$$

PROOF. We begin noting that, in the interval $(0,1)$, the logarithmic series above corresponds to $-\frac{1}{2}\frac{\partial}{\partial s}\left[\mathrm{Li}_s\left(e^{2\pi x\, i}\right)+\mathrm{Li}_s\left(e^{-2\pi x\, i}\right)\right]_{s=2}$, a result similar to that in Eq. (1.18) of Ref. [**12**]. The integration of both sides of Eq. (1.1) from 0 to $x$, for any $x \in (0,1)$, yields

$$\begin{aligned}
\int_0^x \ln\Gamma(t)\,dt &= \frac{\ln\pi}{2}\,x + \frac{\pi\,\eta}{2}\,x - \frac{\pi\,\eta}{2}\,x^2 - \frac{1}{2}\int_0^x \ln\sin(\pi t)\,dt \\
&\quad + \frac{1}{\pi}\int_0^x \left[\sum_{n=1}^{\infty}\frac{\ln n}{n}\sin(2\pi n\,t)\right]dt \\
&= \left[\frac{\ln\pi}{2}+\frac{\pi\eta}{2}\right]x - \frac{\pi\eta}{2}\,x^2 - \frac{1}{2}\int_0^x \ln\sin(\pi t)\,dt \\
&\quad + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\ln n}{n}\int_0^x \sin(2\pi n\,t)\,dt\,,
\end{aligned} \tag{2.17}$$

where $\pi\,\eta = \gamma + \ln(2\pi)$, as proved in Lemma 2.1. The absolute convergence of the last series above, for all $x \in (0,1)$, is sufficient for the application of Fubini's theorem — see, e.g., the corollary of Theorem 7.16 in Ref. [**14**] —, which validates the interchange of the integral and the series in the last step. Now, let us solve each integral separately. Firstly, by definition, $\int_0^x \ln\Gamma(t)\,dt = \psi^{(-2)}(x)$, known as the *negapolygamma* function, for which Adamchik showed in Ref. [**1**], as a particular case of his Eq. (14), that

$$\psi^{(-2)}(x) = \frac{x\,(1-x)}{2} + \frac{x}{2}\,\ln(2\pi) - \zeta'(-1) + \zeta'(-1,x)\,, \tag{2.18}$$

where $\zeta'(-1) = 1/12 - \ln A$, $A$ being the Glaisher-Kinkelin constant (in virtue of Eq. (5.2) of Ref. [**6**], namely $\zeta'(-1) - \zeta'(-1,t) = \ln G(1+t) - t\,\ln\Gamma(t)$, it is possible to write Eq. (2.18) in terms of the Barnes $G$-function, but we shall not explore this here). Secondly,

$$\int_0^x \ln\,\sin(\pi t)\,dt = -\,\frac{\mathrm{Cl}_2(2\pi x)}{2\pi} - x\,\ln 2\,. \tag{2.19}$$

Finally, on substituting $\int_0^x \sin(2\pi n\,t)\,dt = -\left[\cos(2\pi n\,x) - 1\right]/(2\pi n)$ in the last series in Eq. (2.17), one finds

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\ln n}{n}\int_0^x \sin(2\pi n\,t)\,dt &= -\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\ln n}{n^2}\left[\cos(2\pi n\,x) - 1\right] \\
&= -\frac{1}{2\pi}\left[\sum_{n=1}^{\infty}\frac{\ln n}{n^2}\cos(2\pi n\,x) - \sum_{n=1}^{\infty}\frac{\ln n}{n^2}\right],
\end{aligned} \tag{2.20}$$

which can be further simplified by noting that

$$\sum_{n=1}^{\infty}\frac{\ln n}{n^2} = -\,\zeta'(2) = \frac{\pi^2}{6}\left[12\,\ln A - \gamma - \ln(2\pi)\right], \tag{2.21}$$

as shown by Glaisher in 1894 [**9**]. The proof completes by substituting the four closed-form expressions above there in Eq. (2.17). $\square$

REMARK 2.1. On putting $x = 1/2$ (or $x = 1/4$) in Theorem 2.3, one finds

$$\sum_{n=1}^{\infty} (-1)^n \, \frac{\ln n}{n^2} = \frac{\pi^2}{12} \, [\, \gamma + \ln{(2\pi)} - 1] - 2\pi^2 \, \zeta'\left(-1, \frac{1}{2}\right)$$
$$= \frac{\pi^2}{12} \, [\, \gamma + \ln{(4\pi)} - 12 \, \ln A \,] \, .$$

Since the function $\cos{(2\pi n \, x)}$, $x \in (0,1)$, remains the same when we exchange $x$ by $1-x$, then both the first term in the right-hand side and the series in Theorem 2.3 also do, whereas $\mathrm{Cl}_2(2\pi x)$ changes the sign. This simple observation leads to

COROLLARY 2.1 (Reflection formula for $\zeta'(-1,x)$ ). *For all* $x \in [0,1]$*,*

$$\zeta'(-1,x) - \zeta'(-1,1-x) = \frac{\mathrm{Cl}_2(2\pi x)}{2\pi} \, .$$

This reflection formula corresponds to the case $n = 1$ of Eq. (21) in Ref. [**13**]. Note that it remains valid at both endpoints $x = 0$ and $x = 1$ because $\zeta'(-1,0) = \zeta'(-1,1) = \zeta'(-1)$ and $\mathrm{Cl}_2(0) = \mathrm{Cl}_2(2\pi) = 0$.

REMARK 2.2. For $x = 1/4$, the reflection formula above yields

$$\zeta'\left(-1, \frac{1}{4}\right) - \zeta'\left(-1, \frac{3}{4}\right) = \frac{\mathrm{Cl}_2(\pi/2)}{2\pi} = \frac{G}{2\pi} \, ,$$

where $G := \sum_{n=0}^{\infty} (-1)^n/(2n+1)^2$ is Catalan's constant.

The presence of the factor $\sin{(2\pi x)}$ accompanying the log-Gamma function in Eq. (1.2) suggests that further results can be found by multiplying both sides of Farhi's formula by that factor before integration. Then, let us define the real function

$$\eta(x) := 2 \int_0^x \ln \Gamma(t) \, \sin{(2\pi t)} \, dt \, , \quad x \in (0,1) \, , \tag{2.22}$$

as a generalization of Farhi's constant $\eta$.

THEOREM 2.4 (Another integral of Farhi's formula). *For all* $x \in (0,1)$*, one has*

$$\eta(x) = \frac{\gamma + \ln{(2\pi)}}{\pi} \left[ \sin^2\!\left(\frac{\theta}{2}\right) + \frac{\theta \, \cos\theta - \sin\theta}{2\pi} \right] + \frac{\sin^2(\theta/2)}{2\pi} \left[ 1 + 2 \, \ln\!\left(\frac{\pi}{\sin{(\theta/2)}}\right) \right]$$
$$+ \frac{\cos\theta}{\pi^2} \sum_{n=2}^{\infty} \frac{\ln n}{n\,(n^2-1)} \, \sin{(n\theta)} - \frac{\sin\theta}{\pi^2} \sum_{n=2}^{\infty} \frac{\ln n}{n^2-1} \, \cos{(n\theta)} \, ,$$

*where* $\theta = 2\pi x$.

PROOF. The multiplication of both sides of Eq. (1.1) by $\sin(2\pi t)$, followed by integration from 0 to $x$, for any $x \in (0,1)$, yields

$$\begin{aligned}
&\int_0^x \ln\Gamma(t)\,\sin(2\pi t)\,dt \\
&= \frac{\ln\pi}{4\pi}\int_0^\theta \sin\tilde\theta\,d\tilde\theta + \frac{\gamma+\ln(2\pi)}{4\pi}\int_0^\theta \left(1-\frac{\tilde\theta}{\pi}\right)\sin\tilde\theta\,d\tilde\theta \\
&\quad - \frac{1}{2\pi}\int_0^\theta \ln\left[\sin\left(\frac{\tilde\theta}{2}\right)\right]\sin\left(\frac{\tilde\theta}{2}\right)\cos\left(\frac{\tilde\theta}{2}\right)d\tilde\theta \\
&\quad + \frac{1}{2\pi^2}\sum_{n=2}^{\infty}\frac{\ln n}{n}\int_0^\theta \sin\left(n\tilde\theta\right)\,\sin\tilde\theta\,d\tilde\theta\,,
\end{aligned}\tag{2.23}$$

where $\theta = 2\pi x$ and $\tilde\theta$ is just a dummy variable. On applying the trigonometric identity $\sin\alpha\,\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$ to the last integral above, one finds

$$\begin{aligned}
&2\int_0^x \ln\Gamma(t)\,\sin(2\pi t)\,dt = \frac{\ln\pi}{2\pi}\,(1-\cos\theta) \\
&+\frac{\gamma+\ln(2\pi)}{2\pi}\left(1-\cos\theta-\frac{1}{\pi}\int_0^\theta \tilde\theta\,\sin\tilde\theta\,d\tilde\theta\right) - \frac{2}{\pi}\int_0^b u\,\ln u\,du \\
&+\frac{1}{2\pi^2}\sum_{n=2}^{\infty}\frac{\ln n}{n}\int_0^\theta\left\{\cos\left[(n-1)\,\tilde\theta\right]-\cos\left[(n+1)\,\tilde\theta\right]\right\}d\tilde\theta\,,
\end{aligned}\tag{2.24}$$

where $b = \sin(\theta/2)$. The remaining integrals are evaluated either applying integration by parts, i.e.

$$\int_0^\theta \tilde\theta\,\sin\tilde\theta\,d\tilde\theta = \sin\theta - \theta\,\cos\theta\,,\tag{2.25}$$

$$\int_0^b u\,\ln u\,du = \frac{b^2}{4}\,(2\ln b - 1)\,,\tag{2.26}$$

or using the simple substitution $u = \alpha\,\tilde\theta$, i.e.

$$\begin{aligned}
&\int_0^\theta\left\{\cos\left[(n-1)\,\tilde\theta\right]-\cos\left[(n+1)\,\tilde\theta\right]\right\}d\tilde\theta = \frac{\sin\left[(n-1)\,\theta\right]}{n-1}-\frac{\sin\left[(n+1)\,\theta\right]}{n+1} \\
&= \frac{(n+1)\left[\sin(n\theta)\,\cos\theta-\sin\theta\,\cos(n\theta)\right]-(n-1)\left[\sin(n\theta)\,\cos\theta+\sin\theta\,\cos(n\theta)\right]}{n^2-1} \\
&= 2\,\frac{\sin(n\theta)\,\cos\theta - n\,\sin\theta\,\cos(n\theta)}{n^2-1}\,.
\end{aligned}\tag{2.27}$$

The substitution of the last three expressions, above, there in Eq. (2.24) completes the proof. ☐

REMARK 2.3. In particular, for $x = 1/2$ our Theorem 2.4 promptly yields

$$\eta\left(\frac{1}{2}\right) = 2\int_0^{1/2}\ln\Gamma(t)\,\sin(2\pi t)\,dt = \frac{\gamma+\ln(2\pi)+2\ln\pi+1}{2\pi}\,.\tag{2.28}$$

The analytic expression of $\eta(x)$ established in Theorem 2.4 defines a continuous real function for all $x \in (0,1)$. However, it involves a logarithmic term which is undefined at the endpoints. This issue can be fixed *defining* $\eta(0) = \lim_{x\to 0^+} \eta(x)$ and $\eta(1) = \lim_{x\to 1^-} \eta(x)$, in a manner to make $\eta(x)$ continuous for all $x \in [0,1]$.

THEOREM 2.5 (Continuous extension of $\eta(x)$ ). *The domain over which the function* $\eta(x) = 2\int_0^x \ln\Gamma(t)\,\sin(2\pi t)\,dt$ *is continuous can be extended to the closed interval* $[0,1]$ *by defining* $\eta(0) := 0$ *and* $\eta(1) := \eta = [\gamma + \ln(2\pi)]/\pi$.

PROOF. In the analytic expression established for $\eta(x)$ in Theorem 2.4, all terms promptly nullify at $x = 0$, except the one involving $\ln[\pi/\sin(\theta/2)]$, where $\theta = 2\pi x$. The choice $\eta(0) = 0$ then comes from the limit

$$
\begin{aligned}
\lim_{\theta\to 0^+} \sin^2\!\left(\frac{\theta}{2}\right) \ln\!\left[\frac{\pi}{\sin(\theta/2)}\right] &= \ln\pi \lim_{\theta\to 0^+} \sin^2\!\left(\frac{\theta}{2}\right) - \lim_{\theta\to 0^+} \sin^2\!\left(\frac{\theta}{2}\right) \ln\!\left[\sin\!\left(\frac{\theta}{2}\right)\right] \\
&= -\lim_{\alpha\to 0^+} \sin^2\!\alpha\, \ln(\sin\alpha) = \lim_{\alpha\to 0^+} \frac{\ln(\csc\alpha)}{\csc^2\alpha} \\
&= \lim_{y\to+\infty} \frac{\ln y}{y^2} = \lim_{y\to\infty} \frac{1}{2\,y^2} = 0\,, \qquad (2.29)
\end{aligned}
$$

where the l'Hôpital rule was applied in the last step.

The appropriate choice for $\eta(1)$ is found by taking $x \to 1^-$ (or, equivalently, $\theta \to 2\pi^-$) in Theorem 2.4, which yields

$$
\begin{aligned}
\lim_{x\to 1^-} \eta(x) &= \frac{\gamma + \ln(2\pi)}{\pi} + \frac{1}{2\pi} \lim_{\theta\to 2\pi^-} \sin^2\!\left(\frac{\theta}{2}\right) \left[1 + 2\ln\!\left(\frac{\pi}{\sin(\theta/2)}\right)\right] \\
&= \eta + \frac{1}{2\pi} \lim_{\theta/2\to\pi^-} \sin^2\!\left(\frac{\theta}{2}\right) + \frac{1}{\pi} \lim_{\theta/2\to\pi^-} \sin^2\!\left(\frac{\theta}{2}\right) \ln\!\left(\frac{\pi}{\sin(\theta/2)}\right) \\
&= \eta + \frac{\ln\pi}{\pi} \lim_{\theta/2\to\pi^-} \sin^2\!\left(\frac{\theta}{2}\right) - \frac{1}{\pi} \lim_{\theta/2\to\pi^-} \sin^2\!\left(\frac{\theta}{2}\right) \ln\!\left[\sin\!\left(\frac{\theta}{2}\right)\right] \\
&= \eta - \frac{1}{\pi} \lim_{\alpha\to\pi^-} \sin^2\!\alpha\, \ln(\sin\alpha)\,. \qquad (2.30)
\end{aligned}
$$

As shown in Eq. (2.29), the last limit above is null. □

Note that our choices for $\eta(0)$ and $\eta(1)$ agree with the values obtained directly from the integral definition of $\eta(x)$ in Eq. (2.22), namely $\eta(0) = 2\int_0^0 \ln\Gamma(t)\,\sin(2\pi t)\,dt = 0$ and $\eta(1) = 2\int_0^1 \ln\Gamma(t)\,\sin(2\pi t)\,dt = \eta$, as defined in Eq. (1.2). Interestingly, we can use the function $\eta(x)$, as defined in Eq. (2.22), to generalize our Theorem 2.1.

THEOREM 2.6 (Generalization of the digamma integral). *For all* $x \in (0,1]$, *the closed-form result*

$$
\int_0^x \psi(t)\,\sin^2(\pi t)\,dt = \ln\Gamma(x)\,\sin^2(\pi x) - \frac{\pi}{2}\,\eta(x)
$$

*holds, where* $\eta(x)$ *is the function defined in Eq.* (2.22) *for* $x \in (0,1)$, *a domain extended to* $[0,1]$ *in Theorem 2.5.*

Proof. From the integral definition of $\eta(x)$ in Eq. (2.22), one has

$$\pi\,\eta(x) = 4\pi \int_0^x \ln\Gamma(t)\,\sin(\pi t)\,\cos(\pi t)\,dt = 4\int_0^{\pi x} \ln\Gamma\Big(\frac{y}{\pi}\Big)\sin y\,\cos y\,dy, \tag{2.31}$$

for all $x \in (0,1)$. Now, define the last integral, above, as $I$ and integrate it by parts, following the same steps as those in the proof of Theorem 2.1. After some algebra, one finds

$$2\,I = \frac{\pi}{2}\,\eta(x) = \ln\Gamma(x)\,\sin^2(\pi x) - \frac{1}{\pi}\int_0^{\pi x} \psi\Big(\frac{y}{\pi}\Big)\,\sin^2 y\,dy\,, \tag{2.32}$$

from which the closed-form expression for $\int_0^x \psi(t)\,\sin^2(\pi t)\,dt$ promptly follows. Finally, for $x = 1$ this closed-form expression reduces to

$$\int_0^1 \psi(t)\,\sin^2(\pi t)\,dt = 0 - \frac{\pi}{2}\,\eta(1) = -\frac{\pi\,\eta}{2}\,, \tag{2.33}$$

which agrees with Theorem 2.1. □

For $x = 1/2$, on taking into account the special value we have found in Eq. (2.28), one finds

Corollary 2.2.

$$\int_0^{1/2} \psi(t)\,\sin^2(\pi t)\,dt = -\frac{\gamma + \ln(2\pi) + 1}{4}\,.$$

Another useful generalization, from the point of view of Fourier series expansions, is obtained by inserting a positive integer parameter $k$ in the argument of the sine function in Eq. (1.2), as follows:

$$\eta_k := 2\int_0^1 \ln\Gamma(t)\,\sin(2\pi k\,t)\,dt\,. \tag{2.34}$$

The problem of finding closed-form expressions for the Fourier coefficients of $\ln\Gamma(x)$ on the interval $(0,1)$ was solved in details by Farhi in Ref. [**8**], the result being

$$a_0 = \int_0^1 \ln\Gamma(t)\,dt = \frac{1}{2}\,\ln(2\pi)\,, \tag{2.35}$$

$$a_k = 2\int_0^1 \ln\Gamma(t)\,\cos(2\pi k\,t)\,dt = \frac{1}{2k}\,, \quad k \geqslant 1\,, \tag{2.36}$$

$$b_k = 2\int_0^1 \ln\Gamma(t)\,\sin(2\pi k\,t)\,dt = \eta_k = \frac{\ln k}{\pi k} + \frac{\eta}{k}\,, \quad k \geqslant 1\,. \tag{2.37}$$

Here, we are considering the Fourier series in the form

$$a_0 + \sum_{k=1}^{\infty} a_k\,\cos(2\pi k x) + \sum_{k=1}^{\infty} b_k\,\sin(2\pi k x)\,, \tag{2.38}$$

as adopted in Ref. [**8**]. There in that paper, it was shown that this series, with the coefficients given in Eq. (2.37), converges to $\ln\Gamma(x)$ for all $x \in (0,1)$. In particular, for $x = 1/2$ one readily finds $\ln\Gamma(1/2) = \frac{1}{2}\,\ln\pi = \ln\sqrt{\pi}$, a well-known result. Other less-obvious results arise by considering distinct rational values of $x$. For instance, for $x = 1/3$ one finds

$$\ln\Gamma\left(\frac{1}{3}\right) = \frac{\sqrt{3}}{6\pi}\left[\gamma_1\left(\frac{1}{3}\right) - \gamma_1\left(\frac{2}{3}\right)\right] + \frac{\gamma}{6} + \frac{2}{3}\,\ln(2\pi) - \frac{\ln 3}{12}\,, \tag{2.39}$$

which agrees with the closed-form expressions for $\gamma_1\left(\frac{1}{3}\right)$ and $\gamma_1\left(\frac{2}{3}\right)$ found by Blagouchine in Eq. (61) of Ref. [**2**]. For $x = 1/4$, one finds

$$\begin{aligned}\ln\Gamma\left(\frac{1}{4}\right) &= \frac{1}{2}\sum_{k=1}^{\infty}\frac{\cos(k\pi/2)}{k} + \frac{\ln(2\pi)}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin(k\pi/2)}{k}\,(\pi\,\eta + \ln k)\\ &= \frac{\ln 2}{4} + \frac{\ln(2\pi)}{2} + \frac{1}{4}\left[\pi\,\eta + \frac{\gamma_1(1/4) - \gamma_1(3/4)}{\pi}\right], \end{aligned} \tag{2.40}$$

which simplifies just to our Eq. (2.5), so this evaluation can be viewed as an alternative derivation of Coffey's formula [**4**].

With the Fourier series expansion of $\ln\Gamma(x)$, $x \in (0,1)$, in hands, we can use Parseval's theorem to derive another closed-form result.

THEOREM 2.7 (Parseval's theorem for $\ln\Gamma(x)$ ).

$$\int_0^1 \ln^2\Gamma(t)\,dt = 2\ln A\,[\gamma + \ln(2\pi)] - \frac{\gamma^2}{12} + \frac{\pi^2}{48} + \frac{\ln(2\pi)}{6}\,[\,\ln(2\pi) - \gamma\,] + \frac{\zeta''(2)}{2\pi^2}\,.$$

PROOF. On applying Parseval's theorem to the Fourier series in Eq. (2.38), one finds

$$\int_0^1 \ln^2\Gamma(t)\;dt = a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty} a_k^2 + \frac{1}{2}\sum_{k=1}^{\infty} b_k^2\,, \tag{2.41}$$

where the Fourier coefficients are given in Eqs. (2.35)–(2.37). This expands to

$$\begin{aligned}\int_0^1 \ln^2\Gamma(t)\;dt &= \frac{\ln^2(2\pi)}{4} + \frac{1}{8}\,\zeta(2) + \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{\ln^2 k}{\pi\,k^2} + \frac{\eta^2}{k^2} + \frac{2\,\eta}{\pi}\,\frac{\ln k}{k^2}\right)\\ &= \frac{\ln^2(2\pi)}{4} + \frac{\pi^2}{48} + \frac{1}{2\pi}\sum_{k=2}^{\infty}\frac{\ln^2 k}{k^2} + \frac{\eta^2}{2}\cdot\frac{\pi^2}{6} + \frac{\eta}{\pi}\sum_{k=2}^{\infty}\frac{\ln k}{k^2}\,.\end{aligned} \tag{2.42}$$

The first series, above, reduces to $\zeta''(2)$. According to our Eq. (2.21), the last series simplifies to $(\pi^2/6)\;[12\,\ln A - \gamma - \ln(2\pi)]$. The substitution of these two summations in Eq. (2.42) completes the proof. □

The result of the above integral evaluation agrees with Eq. (31) of Ref. [**3**], as well as Eq. (6.441.6) of Ref. [**10**].

An interesting feature of the Fourier coefficients stated in Eqs. (2.35)–(2.37) arises from the analysis of its asymptotic behavior for large values of $k$. For $k = 1$, it is clear from Eq. (2.37) that $\eta_1 = \eta$. From Eq. (2.37) it also follows that

$$\eta_{2k} = \frac{\eta_k}{2} + \frac{\ln 2}{2\pi k}\,, \quad k \geqslant 1\,, \tag{2.43}$$

and

$$\eta_{2^k} = \frac{(\ln 2/\pi)\, k + \eta}{2^k}\,, \quad k \geqslant 0\,. \tag{2.44}$$

These two formulae were derived by Shamov using the Legendre duplication formula for $\Gamma(x)$, in his answer to a question by Silagadze in Ref. [**11**]. He indeed takes Eq. (2.44) into account for establishing the asymptotic behavior of $\eta_k$ for $k \to \infty$ and uses it to develop a heuristic proof of our Lemma 2.1. His reasoning is so aesthetic that it deserves to be mentioned here. He argues that the integrand in the definition of $\eta_k$ has only one singularity at $t = 0$, where $\Gamma(t) = -\ln t - \gamma\, t + \ldots$, so the asymptotic behavior of the Fourier coefficient $\eta_k$ should be the same as that of $\ln t$, i.e. $2\int_0^1 \sin(2\pi k t)\, \ln\!\left(t^{-1}\right) dt = \ln k/(\pi k) + \eta/k + \ldots$ He then notes that this result can be written as

$$-\,\frac{1}{\pi k} \int_0^{2\pi k} \frac{\cos z - 1}{z}\, dz = \frac{\gamma + \ln(2\pi\, k) - \mathrm{Ci}(2\pi k)}{\pi\, k}\,,$$

where $\mathrm{Ci}(x)$ is the cosine integral, which behaves as $(\sin x)/x$ for $x \to \infty$. A comparison of the terms of order $1/k$ in these expansions promptly yields $\eta = [\gamma + \ln(2\pi)]/\pi$, which agrees to our Lemma 2.1.

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## Conflict of interest

The author declares that he has no conflict of interest.

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