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On b-ary binomial coefficients with negative entries

Lin Jiu [a] and Duanduan Wang [b]

Abstract. We extend the b-ary binomial coefficients by allowing negative entries, which is based on the generating function obtained in previous work. Besides an explicit expression for these coefficients involving restricted partition, several properties such as symmetry, congruence, Pascal-like recurrences and Chu-Vandermonde identities are presented. Finally, we also provide two other generalizations that partially satisfy Pascal-like recurrences.

1. Introduction

Besides the decimal number system, which is widely used in daily life, other number systems also have important applications, e.g., the binary number system for computers. Some mathematical formulas and constants have better applications and properties in special number systems. For instance, the famous BBP formula [**1**, Thm. 1] allows us to compute any hexadecimal and binary digit of π, without computing the preceding digits. Therefore, extensions of ordinary formulas into other bases are of importance and interest.

Callan [**2**, Thm. 2] originally extended the classical binomial identity

$$(X+Y)^n=\sum_{k=0}^{n}\binom{n}{k}X^kY^{n-k},$$

to the binary version

$$(X+Y)^{S_2(n)}=\sum_{0\leqslant k\lesssim_2 n}X^{S_2(n-k)}Y^{S_2(k)}. \tag{1.1}$$

Here, given any positive integer n, we denote its b-ary expansion as

$$n=n_{N-1}b^{N-1}+n_{N-2}b^{N-2}+\cdots+n_1b+n_0=\left(n_{N-1}\cdots n_0\right)_b.$$

Then, $S_b(n):=n_{N-1}+\cdots+n_0$ is the sum of all the digits of n, in base b. In addition, $0\leqslant k\lesssim_b n$ means the summation index k runs over all integers from 0 to n such that the b-ary addition $k+(n-k)=n$ is carry-free.

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An extension of (1.1), to any base b, is obtained as [**3**, Eq. 10]

$$(X+Y)^{S_b(n)}=\sum_{k=0}^{n}\binom{n}{k}_b X^{S_b(n-k)}Y^{S_b(k)}, \tag{1.2}$$

where, the b-ary binomial coefficient is defined by [**3**, Eq. 11]

$$\binom{n}{k}_b=\prod_{l=0}^{N-1}\binom{n_l}{k_l}, \tag{1.3}$$

for nonnegative integers n and k. Here, we assume $k=(k_{N-1}\cdots k_0)_b$ and $N=\min\{m\in\mathbb{N}: n_s=k_s=0 \text{ for all } s>m\}$. Namely, if n has N_1 digits and k has N_2 digits, in base b, then $N=\max\{N_1,N_2\}$. This setup for N shall be applied throughout this paper. It can be observed that the carry-free condition, specified in (1.1), is eliminated in (1.2), due to definition (1.3). Moreover, the generating function of the b-ary binomial coefficients is given by [**3**, Eq. 13]

$$\prod_{l=0}^{N-1}\left(1+x^{b^l}\right)^{n_l}=\sum_{k=0}^{n}\binom{n}{k}_b x^k. \tag{1.4}$$

A natural question that arises is to extend the b-ary binomial coefficients with negative entries. For the classical binomial coefficients, Loeb [**4**, Thm. 4.1] defined, for $n,k\in\mathbb{Z}$,

$$\binom{n}{k}:=\lim_{\varepsilon\to 0}\frac{\Gamma(n+1+\varepsilon)}{\Gamma(k+1+\varepsilon)\Gamma(n-k+1+\varepsilon)},$$

which also admit a combinatorial interpretation [**4**, Thm. 5.2] by counting the number of elements in a hybrid set. Alternatively, it can be defined as the coefficient of x^k in the power series of $(1+x)^n$ [**4**, Prop. 4.5], denoted by $\left[x^k\right]$:

$$\binom{n}{k}:=\left[x^k\right](1+x)^n, \tag{1.5}$$

where if k is negative, the inverse power series is applied. More precisely, for positive n, the following two series expansions will be considered as follows.

$$(1+x)^{-n}=\sum_{j=0}^{\infty}\binom{-n}{j}x^j=\sum_{j=n}^{\infty}\binom{-n}{-j}\frac{1}{x^j}. \tag{1.6}$$

The first three cases, $n = 1$, 2, and 3, are listed here. Calculations in later examples will consult these expressions:

$$(1+x)^{-1} = \sum_{j=0}^{\infty}(-1)^j x^j = \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{x^j},$$
$$(1+x)^{-2} = \sum_{j=0}^{\infty}(-1)^j (j+1) x^j = \sum_{j=2}^{\infty}\frac{(-1)^{j}(j-1)}{x^j},$$
$$(1+x)^{-3} = \sum_{j=0}^{\infty}(-1)^j \tfrac{(j+1)(j+2)}{2} x^j = \sum_{j=3}^{\infty}\frac{(-1)^{j+1}(j-1)(j-2)}{2x^j}.$$

The main purpose of this work is to extend the b-ary binomial coefficients with negative entries, similarly as in (1.5). A definition, examples and an explicit expression are introduced in Section 2. In Section 3, we study some properties, such as symmetry, congruence, Pascal-like recurrences, and Chu-Vandermonde identities. Finally in Section 4, we introduce two other natural but different generalizations, partially satisfying the Pascal-like recurrences.

2. Definition and an explicit expression

First of all, we adopt the convention that a negative integer has all its digits nonpositive, in any base b. Namely, if $n = (n_{N-1}n_{N-2}\cdots n_1 n_0)_b > 0$, then

$$-n = ((-n_{N-1})(-n_{N-2})\cdots(-n_1)(-n_0))_b\,,$$

which is compatible with the b-ary expansion that

$$-n = (-n_{N-1})b^{N-1} + (-n_{N-2})b^{N-2} + \cdots + (-n_1)b + (-n_0).$$

It also indicates $S_b(-n) = -n_{N-1} - n_{N-2} - \cdots - n_1 - n_0 = -S_b(n)$.

We now extend the b-ary binomial coefficients with negative entries as follows.

Definition 2.1. Let $n, k \in \mathbb{Z}$ with $n = (n_{N-1}\cdots n_0)_b$ and $k = (k_{N-1}\cdots k_0)_b$. Define

$$\binom{n}{k}_b := [x^k]\prod_{l=0}^{N-1}\left(1+x^{b^l}\right)^{n_l}, \tag{2.1}$$

where, when both n and k are negative, it is defined as the coefficient of x^k in the inverse power series of the function on the right-hand side.

Remark 2.1. For simplicity, we denote the generating function above by

$$f_{n,b}(x) := \prod_{l=0}^{N-1}\left(1+x^{b^l}\right)^{n_l}.$$

Also, for the negative case, we assume n is positive and consider the expansions

$$\sum_{k=0}^{\infty}\binom{-n}{k}_b x^k = f_{-n,b}(x) = \sum_{k=1}^{\infty}\binom{-n}{-k}_b \frac{1}{x^k}. \tag{2.2}$$

Example 2.1. Let $b = 4$ and $n = 6$, so that $-n = -6 = ((-1)(-2))_4$ and

$$f_{-6,4}(x) = \left(1 + x^4\right)^{-1} (1 + x)^{-2}.$$

(1) For $k = 7$, as $x \to 0$,

$$f_{-6,4}(x) = 1 - 2x + 3x^2 - 4x^3 + 4x^4 - 4x^5 + 4x^6 - 4x^7 + O\left(x^8\right),$$

and then

$$\binom{-6}{7}_4 = -4.$$

(2) For $k = -8$, as $x \to \infty$,

$$f_{-6,4}(x) = x^{-6} - 2x^{-7} + 3x^{-8} + O\left(x^{-9}\right),$$

and then

$$\binom{-6}{-8}_4 = 3.$$

The next proposition gives an explicit expression of the b-ary binomial coefficients with negative entries.

Proposition 2.1. *Let $n = \left(n_{N-1} \cdots n_0\right)_b$ be positive. Then,*

$$\binom{-n}{k}_b = \begin{cases} \displaystyle\sum_{(j_{N-1},\ldots,j_0)\in\mathcal{P}_n(k,\mathbf{b}_N)} \prod_{l=0}^{N-1} \binom{-n_l}{j_l}, & \text{if } k \geqslant 0; \\ \displaystyle\sum_{(j_{N-1},\ldots,j_0)\in\mathcal{P}_n^*(-k,\mathbf{b}_N)} \prod_{l=0}^{N-1} \binom{-n_l}{-j_l}, & \text{if } k < 0, \end{cases} \tag{2.3}$$

where

- $\mathbf{b}_N := \{1, b, \ldots, b^{N-1}\}$;
- $\mathcal{P}_n(k, \mathbf{b}_N)$ *is the set of* restricted partitions *of k into parts in $\mathbf{b}_N$, i.e., N-tuples of nonnegative integers $(j_{N-1}, \ldots, j_0)$ such that*

$$j_{N-1} b^{N-1} + \cdots + j_1 b^1 + j_0 = k; \tag{2.4}$$

- *and $\mathcal{P}_n^*(-k, \mathbf{b}_N)$ is the subset of $\mathcal{P}_n(-k, \mathbf{b}_N)$, consisting of those N-tuples with extra restrictions: $j_l \geqslant n_l$, for $l = 0, \ldots, N-1$.*

Remark 2.2. If $b > \max\{n, |k|\}$, then both n and k have one digit, i.e., $N = 1$. In this case, (2.3) reduces to (1.5).

Example 2.2. We revisit Example 2.1 where $b = 4$ and $-n = -6 = ((-1)(-2))_4$. Then, $N = 2$ implies $\mathbf{4}_2 = \{1, 4\}$.

(1) For $k = 7 = 1 \cdot 4 + 3 \cdot 1 = 0 \cdot 4 + 7 \cdot 1$, we see $\mathcal{P}_6(7, \mathbf{4}_2) = \{(1, 3), (0, 7)\}$. Thus,

$$\binom{-6}{7}_4 = \binom{-1}{1}\binom{-2}{3} + \binom{-1}{0}\binom{-2}{7} = (-1) \cdot (-4) + 1 \cdot (-8) = -4.$$

(2) For $k = -8$, it is not difficult to see that $\mathcal{P}_6^*(8, \mathbf{4}_2) = \{(1, 4)\}$. Therefore,

$$\binom{-6}{-8}_4 = \binom{-1}{-1}\binom{-2}{-4} = 1 \cdot 3 = 3.$$

PROOF OF PROPOSITION 2.1. (1) If $k > 0$, we apply (1.6) to each factor of $f_{-n,b}(x)$, namely

$$f_{-n,b}(x) = \prod_{l=0}^{N-1}\left(\sum_{j_l=0}^{\infty}\binom{-n_l}{j_l}x^{j_l b^l}\right) = \sum_{j_0,\dots,j_{N-1}=0}^{\infty}\left(\prod_{l=0}^{N-1}\binom{-n_l}{j_l}\right)x^{\sum_{l=0}^{N-1} j_l b^l}.$$

By comparing coefficients, the first case in (2.3) is confirmed.

(2) Similarly if $k < 0$, by the inverse power series in (1.6),

$$f_{-n,b}(x) = \prod_{l=0}^{N-1}\left(\sum_{j_l=n_l}^{\infty}\binom{-n_l}{-j_l}\frac{1}{x^{j_l b^l}}\right) = \sum_{j_l=n_l}^{\infty}\left(\prod_{l=0}^{N-1}\binom{-n_l}{-j_l}\right)\frac{1}{x^{\sum_{l=0}^{N-1} j_l b^l}}.$$

Note that all the $j_l's$, $l = 0,\dots,N-1$, start from n_l instead of 0. We need to additionally restrict the N-tuples $(j_{N-1},\dots,j_0) \in \mathcal{P}_n^*(-k,\mathbf{b}_N)$.

□

REMARK 2.3. When both n and k are positive, $f_{n,b}(x)$ is a polynomial

$$f_{n,b}(x) = \prod_{l=0}^{N-1}\left(1 + x^{b^l}\right)^{n_l} = \prod_{l=0}^{N-1}\left(\sum_{j_l=0}^{n_l}\binom{n_l}{j_l}x^{j_l b^l}\right).$$

Then, it requires $0 \leqslant j_l \leqslant n_l$, for $l = 0,\dots,N-1$, which restricts (2.4) to have only one solution: the unique expression of k in base b: $j_l = k_l$. This explains why (1.3) and (1.4) define the same coefficient, when $n \geqslant 0$. In the last section, we shall consider two other natural generalizations of (1.3) for $n < 0$ that are different from (2.1).

3. Properties

3.1. Symmetry.

PROPOSITION 3.1. *For any $n, k \in \mathbb{Z}$,* $\binom{n}{k}_b = \binom{n}{n-k}_b$.

PROOF. Since the nonnegative case is already proven in [**3**, Thm. 10], we let n, k be positive integers and it suffices to show that

$$\binom{-n}{k}_b = \binom{-n}{-n-k}_b \quad\text{and}\quad \binom{-n}{-k}_b = \binom{-n}{-n+k}_b.$$

(1) By the left expansion in (2.2),

$$\binom{-n}{k}_b = \frac{f_{-n,b}^{(k)}(0)}{k!}.$$

On the other hand, note that

$$f_{-n,b}\left(1/x\right) = \prod_{l=0}^{N-1}\left(1 + x^{-b^l}\right)^{-n_l} = \prod_{l=0}^{N-1} x^{n_l b^l}\left(x^{b^l}+1\right)^{-n_l} = x^n f_{-n,b}(x).$$

We apply the higher-order product rule to get that

$$f_{-n,b}^{(n+k)}(1/x)=\sum_{l=0}^{n+k}\binom{n+k}{l}\frac{\mathrm{d}^l(x^n)}{\mathrm{d}x^l}f_{-n,b}^{(n+k-l)}(x).$$

Letting $x\to\infty$, all terms on the right-hand side vanish except for one, corresponding to $l=n$. Thus,

$$\binom{-n}{-n-k}_b=\frac{\lim_{x\to 0}f_{-n,b}^{(n+k)}(1/x)}{(n+k)!}=\frac{\binom{n+k}{n}n!f_{-n,b}^{(k)}(0)}{(n+k)!}=\binom{-n}{k}_b.$$

(2) To obtain the second identity, since $k\geqslant n\Leftrightarrow -n+k\geqslant 0$, we merely replace k by $-n+k$ in the first identity.

Now, the remaining case is that when $k<n$, in which case both $-k$ and $-n+k$ are negative. In fact, we have

$$\binom{-n}{-k}_b=\binom{-n}{-n+k}_b=0,$$

which can be seen either from (2.3), where in this case $\mathcal{P}_n^*(-k,\mathbf{b}_N)=\emptyset$; or from the following direct calculation:

$$\sum_{k=1}^{\infty}\binom{-n}{-k}_b\frac{1}{x^k}=\frac{1}{x^n}\prod_{l=0}^{N-1}\left(1+\frac{1}{x^{b^l}}\right)^{-n_l}=\frac{1}{x^n}\prod_{l=0}^{N-1}\left(\sum_{j_l=1}^{\infty}\frac{(-1)^{j_l}}{x^{j_lb^l}}\right)^{n_l}. \qquad \square$$

Remark 3.1. The proof above suggests that (2.2) can be slightly modified to allow the summation index on the right-hand side to begin at $k=n$.

$$\sum_{k=0}^{\infty}\binom{-n}{k}_b x^k=f_{-n,b}(x)=\sum_{k=n}^{\infty}\binom{-n}{-k}_b x^{-k}. \tag{3.1}$$

3.2. Congruence. As the authors in [**3**, Thm. 12] have pointed out, Lucas' congruence theorem,

$$\binom{n}{k}\equiv\binom{n}{k}_p=\prod_{l=0}^{N-1}\binom{n_l}{k_l}\pmod p$$

where p is prime, becomes obvious by letting $b=p$ in the generating function (1.4) and by using the elementary congruence $(1+x)^n\equiv f_{n,b}(x) \pmod p$. The reciprocal of this congruence indicates that Lucas' congruence theorem also holds for negative entries, which we state without proof.

Proposition 3.2. *For $n,k\in\mathbb{Z}$ and a prime p,* $\binom{n}{k}\equiv\binom{n}{k}_p \pmod p$.

3.3. Pascal-like recurrences. As proven in [**3**, Thm. 10], for nonnegative integers n and k,

$$\binom{n}{k}_b + \binom{n}{k-1}_b = \binom{n+1}{k}_b$$

holds when $\binom{n+1}{k}_b \neq 0$. The next proposition shows that for negative entries, the recurrence holds similarly, under an indivisibility restriction.

PROPOSITION 3.3. *Let* $n = (n_{N-1} \cdots n_0) > 0$ *and* $k \in \mathbb{Z}$. *If* $b \nmid n$, *then*

$$\binom{-n}{k}_b + \binom{-n}{k-1}_b = \binom{-n+1}{k}_b. \tag{3.2}$$

PROOF. Since $b \nmid n \Leftrightarrow n_0 \neq 0$, we have $-n+1 = ((-n_{N-1}) \cdots (-(n_0-1))_b$. It follows that

$$f_{-n+1,b}(x) = (1+x)^{-n_0+1} \prod_{l=1}^{N-1} \left(1 + x^{b^l}\right)^{-n_l} = (1+x) f_{-n,b}(x).$$

This recurrence gives (3.2) by expanding both sides and comparing coefficients of the two expansions in (3.1). □

REMARK 3.2. If $n_s \neq 0$ for some $s \in \{0, \ldots, N-1\}$, then a similar calculation yields

$$\binom{-n}{k}_b + \binom{-n}{k-b^s}_b = \binom{-n+b^s}{k}_b. \tag{3.3}$$

Now, in the next proposition, we consider the case $b \mid n$.

PROPOSITION 3.4. *Let* n *be positive with* $n_0 = \cdots = n_{s-1} = 0$ *and* $n_s \neq 0$, *for some* $s \in \{0, \ldots, N-1\}$. *For any* $m \in \{0, \ldots, s-1\}$, *if* $k \geqslant b^s$ *or* $k \leqslant -n + b^m$, *then*

$$\binom{-n+b^s}{k}_b = \sum_{b^m \leqslant j \leqslant b^s,\ b^m \mid j} \binom{b^s - b^m}{j}_b \binom{-n+b^m}{k-j}_b. \tag{3.4}$$

PROOF. Since $n - b^m = (n_{N-1} \cdots (n_s - 1)(b-1) \cdots (b-1)0 \cdots 0)_b$, we see that

$$f_{-n+b^m,b}(x) = \frac{\prod_{l=m}^{s-1} \left(1 + x^{b^l}\right)^{1-b}}{(1 + x^{b^s})^{n_s - 1} \prod_{l=s+1}^{N-1} (1 + x^{b^l})^{n_l}} = \frac{1 + x^{b^s}}{\prod_{l=m}^{s-1} \left(1 + x^{b^l}\right)^{b-1}} f_{-n,b}(x),$$

or equivalently,

$$\begin{aligned}\left(1+x^{b^s}\right) f_{-n,b}(x) &= \left(\prod_{l=m}^{s-1}\left(1+x^{b^l}\right)^{b-1}\right) f_{-n+b^m,b}(x)\\ &= \prod_{l=m}^{s-1}\left(\sum_{j_l=0}^{b-1}\binom{b-1}{j_l}x^{j_l b^l}\right) f_{-n+b^m,b}(x)\\ &= \left(\sum_{b^m\leqslant j\leqslant b^s,\ b^m\mid j}\binom{b^s-b^m}{j}_b x^j\right) f_{-n+b^m,b}(x),\end{aligned}$$

where in the last step, $j=(j_{N-1}\cdots j_0)_b$ runs over all integers between b^m and b^s with $j_l=0$, $l=0,\dots,m-1$, which is equivalent to $b^m \mid j$. Comparing coefficients and applying (3.3) to the left-hand side complete the proof. □

3.4. Chu-Vandermonde identities. Observe that (3.4) is similar to the Chu-Vandermonde identity, but with a restriction on the summation index. To obtain identities that are more closely resemble the Chu-Vandermonde identity, suppose both $n=(n_{N-1}\cdots n_0)_b$ and $m=(m_{N-1}\cdots m_0)_b$ are positive, and $n+m$ in base b is carry-free. Then, by definition,

$$f_{-(n+m),b}(x) = f_{-n,b}(x) f_{-m,b}(x), \tag{3.5}$$

which, by way of series expansions, gives

$$\sum_{k=0}^{\infty}\binom{-n-m}{k}_b x^k = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{-n}{k-j}_b\binom{-m}{j}_b x^k.$$

This leads to the following generalized Chu-Vandermonde identities.

Proposition 3.5. *Let n and m be positive integers such that $n+m$ in base b is carry-free. Then, for $k \geqslant m$, we have*

$$\binom{-n-m}{k}_b = \sum_{j=0}^{k}\binom{-n}{k-j}_b\binom{-m}{j}_b,$$

and for $k \geqslant n+m$,

$$\binom{-n-m}{-k}_b = \sum_{j=1}^{k-1}\binom{-n}{-k+j}_b\binom{-m}{-j}_b.$$

Remark 3.3. The second identity is obtained by the inverse power series of (3.5), which requires both $-k+j$ and $-j$ to be negative.

Next, we consider the mixture of positive and negative cases. Let both n, m be positive such that $n>m$ and $m+(n-m)$ is carry-free in base b. Then, by definition,

$$f_{n-m,b}(x) = f_{n,b}(x) f_{-m,b}(x) \quad \text{and} \quad f_{-n+m,b}(x) = f_{-n,b}(x) f_{m,b}(x). \tag{3.6}$$

From the first identity in (3.6), we have

$$\sum_{k=0}^{n-m} \binom{n-m}{k}_b x^k = \left(\sum_{s=0}^{n} \binom{n}{s}_b x^s\right)\left(\sum_{j=0}^{\infty} \binom{-m}{j}_b x^j\right),$$

which leads to

$$\binom{n-m}{k}_b = \sum_{j=0}^{k} \binom{n}{k-j}_b \binom{-m}{j}_b.$$

Alternatively, if we apply the expansion on the right-hand side of (3.1) to $f_{-m,b}(x)$ in (3.6), we obtain

$$\sum_{k=0}^{n-m} \binom{n-m}{k}_b x^k = \left(\sum_{s=0}^{n} \binom{n}{s}_b x^s\right)\left(\sum_{j=m}^{\infty} \binom{-m}{-j}_b x^{-j}\right),$$

which yields

$$\binom{n-m}{k}_b = \sum_{s=k+1}^{n} \binom{n}{s}_b \binom{-m}{k-s}_b.$$

A similar argument can be applied to the second identity in (3.6) to obtain additional generalized Chu-Vandermonde identities, which we summarize in the next proposition'

Proposition 3.6. *Given positive integers n and m, such that $n > m$ and $m + (n-m)$ is carry-free in base b, we have*

(1) *for $0 \leqslant k \leqslant n-m$,*

$$\binom{n-m}{k}_b = \sum_{j=0}^{k} \binom{n}{k-j}_b \binom{-m}{j}_b = \sum_{s=k+1}^{n} \binom{n}{s}_b \binom{-m}{k-s}_b;$$

(2) *for $k \geqslant 0$,*

$$\binom{-n+m}{k}_b = \sum_{j=0}^{k} \binom{-n}{k-j}_b \binom{m}{j}_b;$$

(3) *and for $k \geqslant n-m$,*

$$\binom{-n+m}{-k}_b = \sum_{j=0}^{k} \binom{-n}{-k-j}_b \binom{m}{j}_b.$$

4. other possible generalizations

Naturally, we can generalize the b-ary binomial coefficients by defining, for positive integer n

$$\binom{-n}{k}_b^{*} := \prod_{l=0}^{N-1} \binom{-n_l}{k_l}. \tag{4.1}$$

However, this definition is not equivalent to (2.1) and yields different values. For instance, comparing the following values with those in Example 2.1:

$$\binom{-6}{7}_4^* = \binom{-1}{1}\binom{-2}{3} = 4 \quad \text{and} \quad \binom{-6}{-8}_4^* = \binom{-1}{-2}\binom{-2}{0} = -1.$$

We have also attempted to extend (1.2) by expanding its left-hand side as follows:

$$(X+Y)^{S_b(n)} = \prod_{l=0}^{N-1}(X+Y)^{n_l} = \prod_{l=0}^{N-1}\left(\sum_{j_l=0}^{n_l}\binom{n_l}{j_l}X^{-n_l-j_l}Y^{j_l}\right),$$

while in the negative case,

$$(X+Y)^{-S_b(n)} = \prod_{l=0}^{N-1}X^{-n_l}\left(1+\frac{Y}{X}\right)^{-n_l} = \prod_{l=0}^{N-1}\left(\sum_{j_l=0}^{\infty}\binom{-n_l}{j_l}X^{-n_l-j_l}Y^{j_l}\right).$$

When Y has power $j_{N-1}+\cdots+j_0 = S_b(k)$, X has the power $-S_b(n)-S_b(k)$. Using the inverse power series leads to similar results. Thus, by equating coefficients with the right-hand side of (1.2), we obtain the generalization

$$\begin{aligned}\binom{-n}{k}_b^{**} &:= \left[Y^{S_b(k)}X^{S_b(-n)-S_b(k)}\right](X+Y)^{-S_b(n)} \\ &= \sum_{\substack{j_l\geqslant 0 \text{ for } l=0,\ldots,N-1 \\ j_{N-1}+\cdots+j_0=S_b(k)}}\left(\prod_{l=0}^{N-1}\binom{-n_l}{j_l}\right),\end{aligned}$$

and

$$\begin{aligned}\binom{-n}{-k}_b^{**} &:= \left[X^{S_b(-n)-S_b(-k)}Y^{S_b(-k)}\right](X+Y)^{-S_b(n)} \\ &= \sum_{\substack{j_l\geqslant n_l \text{ for } l=0,\ldots,N-1 \\ j_{N-1}+\cdots+j_0=S_b(k)}}\left(\prod_{l=0}^{N-1}\binom{-n_l}{-j_l}\right).\end{aligned}$$

Again, this generalization is not equivalent to (1.2). For instance, the following values (obtained using the two series expansions of $(1+x)^{-3}$, since $S_4(-6)=-3$,) do not agree with those in Example 2.1:

$$\binom{-6}{7}_4^{**} = 15 \quad \text{and} \quad \binom{-6}{-8}_4^{**} = 0.$$

Moreover, this generalization does not seem to correctly extend (1.2), since when $S_b(k') = S_b(k)$, the definition above implies $\binom{-n}{k}_b^{**} = \binom{-n}{k'}_b^{**}$. However, in the positive case, one of them can be zero while the other is not, due to the carry-free condition. For example, when $b=2$, $S_2(3)=2=S_2(6)$, and

$$\binom{6}{6}_2 = 1 \neq 0 = \binom{6}{3}_2.$$

In the positive case, from the two expansions

$$\sum_{j=0}^{S_b(n)} \binom{S_b(n)}{j} X^j Y^{S_b(n)-j} = (X+Y)^{S_b(n)} = \sum_{k=0}^{n} \binom{n}{k}_b X^{S_b(k)} Y^{S_b(n-k)},$$

we should have

$$\binom{S_b(n)}{j} = \sum_{0\leqslant k\leqslant n,\ S_b(k)=j} \binom{n}{k}_b.$$

Note that the finite sum on the left-hand side becomes an infinite series in the negative case, which makes the extension of (1.2) unclear.

Although the two aforementioned extensions do not satisfy basic properties studied in Section 3, they satisfy similar Pascal-like recurrence, for positive k.

PROPOSITION 4.1. *For positive integers n and k, if $b \nmid k$ and $b \nmid n$, then*

$$\binom{-n}{k}_b^* + \binom{-n}{k-1}_b^* = \binom{-n+1}{k}_b^* \quad \text{and} \quad \binom{-n}{k}_b^{**} + \binom{-n}{k-1}_b^{**} = \binom{-n+1}{k}_b^{**}.$$

PROOF. We shall make use of the result [**4**, Prop. 4.4] that the generalized binomial coefficients also satisfy the Pascal-like recurrence:

$$\binom{-n}{k} + \binom{-n}{k-1} = \binom{-n+1}{k}.$$

Note that $b \nmid k$ and $b \nmid n$ are equivalent to both k_0 and n_0 are nonzero, so that

$$k-1 = (k_{N-1}\cdots k_1(k_0-1))_b \quad \text{and} \quad -n+1 = ((-n_{N-1})\cdots(-n_1)(-n_0+1))_b\,.$$

(1) Directly, we have

$$\begin{aligned}\binom{-n}{k}_b^* + \binom{-n}{k-1}_b^* &= \prod_{l=0}^{N-1}\binom{-n_l}{k_l} + \binom{-n_0}{k_0-1}\prod_{l=1}^{N-1}\binom{-n_l}{k_l}\\ &= \left(\binom{-n_0}{k_0} + \binom{-n_0}{k_0-1}\right)\prod_{l=1}^{N-1}\binom{-n_l}{k_l}\\ &= \binom{-n_0+1}{k_0}\prod_{l=1}^{N-1}\binom{-n_l}{k_l} = \binom{-n+1}{k}_b^*.\end{aligned}$$

(2) By definition,

$$\binom{-n}{k-1}_b^{**} = \sum_{\substack{j_l'\geqslant 0 \text{ for } l=0,\ldots,N-1\\ j_{N-1}'+\cdots+j_0'=S_b(k-1)}} \prod_{l=0}^{N-1}\binom{-n_l}{j_l}$$

and

$$\binom{-n}{k}_b^{**} = \sum_{\substack{j_l\geqslant 0 \text{ for } l=0,\ldots,N-1\\ j_{N-1}+\cdots+j_0=S_b(k)}} \prod_{l=0}^{N-1}\binom{-n_l}{j_l}$$

$$= \sum_{\substack{j_0>0, j_l \geqslant 0 \text{ for } l=1,\ldots,N-1 \\ j_{N-1}+\cdots+j_0=S_b(k)}} \prod_{l=0}^{N-1} \binom{-n_l}{j_l} + \sum_{\substack{j_0=0, j_l \geqslant 0 \text{ for } l=0,\ldots,N-1 \\ j_{N-1}+\cdots+j_0=S_b(k)}} \prod_{l=0}^{N-1} \binom{-n_l}{j_l}.$$

Note that $S_b(k-1) = S_b(k) - 1$ and if $j_0 > 0$, there is a one-to-one correspondence

$$\left\{ (j_{N-1}, \ldots, j_0) : \sum_{l=0}^{N-1} j_l = S_b(k) \right\} \longleftrightarrow \left\{ (j'_{N-1}, \ldots, j'_0) : \sum_{l=0}^{N-1} j'_l = S_b(k-1) \right\}$$
$$(j_{N-1}, \ldots, j_0) \quad \rightleftharpoons \quad (j_{N-1}, \ldots, j_0 - 1).$$

Thus, we can rewrite

$$\binom{-n}{k-1}_b^{**} = \sum_{\substack{j_0>0, j_l \geqslant 0 \text{ for } l=1,\ldots,N-1 \\ j_{N-1}+\cdots+j_0=S_b(k)}} \binom{-n_0}{j_0-1} \prod_{l=1}^{N-1} \binom{-n_l}{j_l}.$$

In addition, when $j_0 = 0$,

$$\binom{-n_0}{j_0} = 1 = \binom{-n_0+1}{j_0}.$$

Therefore,

$$\binom{-n}{k}_b^{**} + \binom{-n}{k-1}_b^{**}$$
$$= \sum_{\substack{j_0=0, j_l \geqslant 0 \text{ for } l=1,\ldots,N-1 \\ j_{N-1}+\cdots+j_0=S_b(k)}} \binom{-n_0+1}{j_0} \prod_{l=0}^{N-1} \binom{-n_l}{j_l}$$
$$+ \sum_{\substack{j_0>0, j_l \geqslant 0 \text{ for } l=1,\ldots,N-1 \\ j_{N-1}+\cdots+j_0=S_b(k)}} \left(\binom{-n_0}{j_0} + \binom{-n_0}{j_0-1} \right) \prod_{l=1}^{N-1} \binom{-n_l}{j_l}$$
$$= \sum_{\substack{j_l \geqslant 0 \text{ for } l=0,\ldots,N-1 \\ j_{N-1}+\cdots+j_0=S_b(k)}} \binom{-n_0+1}{j_0} \prod_{l=1}^{N-1} \binom{-n_l}{j_l} = \binom{-n+1}{k}_b^{**}. \qquad \square$$

Remark 4.1. (1) Note that by definition, if $k-1$ has more digits than n, recurrence for $\binom{-n}{k}_b^*$ holds vacuously, since all the three terms vanish.

(2) When $k < 0$, for "most" values of k, Prop. 4.1 holds for $\binom{-n}{k}_b^*$. For instance, if $b \nmid n$ and $k \equiv 1 \pmod{p}$, then there are cases where Prop. 4.1 fails to hold. However, these are not all the exceptions. Please see Table 1, which lists values of

$$\binom{-n}{-k}_4^* + \binom{-n}{-k-1}_4^* - \binom{-n+1}{-k-1}_4^*,$$

for $1 \leqslant n \leqslant 10$ and $1 \leqslant k \leqslant 20$. Non-zero values occur where the Pascal-like recurrence fails. We shall leave it as part of our future work, to understand

$$\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 2 & 0 & 0 & -1 & -3 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0
\end{pmatrix}$$

TABLE 1. $\binom{-n}{-k}_4^* + \binom{-n}{-k-1}_4^* - \binom{-n+1}{-k-1}_4^*$ for $\begin{matrix} 1 \leqslant n \leqslant 10 \\ 1 \leqslant k \leqslant 20 \end{matrix}$

why these exceptions occur and resolve differences between the three generalizations that have been presented.

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[a]ZU CHONGZHI CENTER FOR MATHEMATICS AND COMPUTATIONAL SCIENCES,
DUKE KUNSHAN UNIVERSITY,
KUNSHAN, JIANGSU PROVINCE,
215316 P.R.CHINA.
Email address: lin.jiu@dukekunshan.edu.cn

[b]MASTER OF SCIENCE CADIDATE,MS,
HEALTH DATE SCIENCES, DARTMOUTH COLLEGE,
U.S.A .
Email address: Duanduan.Wang.HSE@dartmouth.edu