**SCIENTIA**
Series A: *Mathematical Sciences*, Vol. 31 (2021), 1–14
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446


# Cutting the lemniscate of Fagnano into equal parts

Lori Mancuso[a], Victor H. Moll[b], Judith Nowalski [a] and Leonardo Solanilla[c]

Abstract. The work of Fagnano on the lemniscate dealing with equal division of chords is reexamined. The results are expressed in terms of analytic properties of the corresponding integrals.

## 1. Introduction

In his classical article on Elliptic Functions, written for the German Encyclopedia of the Mathematical Sciences, Fricke [**5**] tells that Jacobi set the date of birth of elliptic functions (*Geburtstag der elliptischen Funktionen*) on December 23, 1751. This day Euler received the *Produzioni matematiche* of an italian noble, Giulio Carlo, *Conte Fagnano e Marchese de' Toschi e di Sant'Onofrio*. This book contained the *Metodo per misurare la leminicata* [**4**], a cornerstone for the subsequent development of the addition formulas of elliptic functions. Certainly, this paper builds a bridge between the early works of the Bernoullis [**1**], d'Alembert [**2**], Landen [**7**], among others, and those of Euler [**3**].

The study of the lemniscate by J. Bernoulli began as a continuation of the work of Cassini on the curve defined by a geometric condition: it is the locus of all points the product of whose distances from two fixed points is constant. Recall that the ellipse corresponds to the analog definition, replacing product by sums. The Bernoulli lemniscate is the locus of points whose product of distances from two foci equals the square of the interfocal distance. The Bernoulli lemniscate has Cartesian equation

$$(x^2+y^2)^2 - 2a^2(x^2-y^2) = 0,$$

with polar form $r^2 = 2a^2\cos(2\theta)$.

The lemniscate is an interesting classical curve. C. L. Siegel has chosen the duplication of the lemniscate as his starting point in [**8**], for his development of the theory of elliptic functions. This approach seems indeed to be one of the best ways to start to understand general elliptic functions. The whole idea traces back to Gauss' lemniscate sine and cosine functions [**6**], which provide an appealing example leading to a Pythagorean-like identity and addition formulas (in particular, half-angle and double

2010 *Mathematics Subject Classification.* Primary 11A25, Secondary 11P99.
*Key words and phrases.* Lemniscate, integrals, division of chords.

angle formulas), analogous to those of the usual trigonometric functions. This results lead directly to the modern theory of elliptic functions.

In this paper we present the original ideas of Fagnano to divide the lemniscatic quadrant into 2, 3 and 5 equal parts. The only tools available to him were the consequences of the Fundamental Theorem of Calculus. So, he must device clever substitutions conducive to determine integer multiples of the lemniscate arclength, conveniently written in polar coordinates. Each substitution results in a polynomial of even degree and a zero of such a polynomial yields to a lemniscatic chord giving the solution to a division problem. The intent of this Section 2 is to prove some calculus lemmas concerning the transformation of elliptic integrals. As it was usual in Fagnano's times, these lemmas can be explain geometrically in terms of arclengths of well-known curves. In Section 3, we establish the theorems yielding the way to cut the lemniscate into 2, 3 and 5 equal parts. At the end, we draw some conclusions on Fagnano's method.

## 2. The computation of lemniscatic arcs and transformations of elliptic integrals

This section describes the computation of a lemniscatic arc. Consider the special case with $a = 1/\sqrt{2}$, so the equation reduces to $(x^2+y^2)^2 = x^2 - y^2$ or $r^2 = \cos 2\theta$.

Lemma 2.1. *Let $Z \in (0,1)$. Then the length of the lemniscatic arc form $x = 0$ to $x = Z$ is given by*

$$L(Z) = \int_0^Z \frac{du}{\sqrt{1-u^4}}. \tag{2.1}$$

Proof. Introduce a parameter $u$ so that $x^2+y^2 = u^2$ and $x^2 - y^2 = u^4$. That is,

$$x^2 = \frac{u^2+u^4}{2} = \frac{u^2(1+u^2)}{2} \quad \text{and} \quad y^2 = \frac{u^2(1-u^2)}{2}.$$

Then, implicit differentiation yields

$$dx = \frac{u(1+2u^2)}{x} du \quad \text{and} \quad dy = \frac{u(1-2u^2)}{y} du.$$

It follows that

$$(dx)^2 + (dy)^2 = 4\left(\frac{(1+2u^2)^2(1-u^2) + (1-2u^2)^2(1+u^2)}{1-u^4}\right)(du)^2 = \frac{8}{1-u^4}(du)^2.$$

□

The next two results present identities among elliptic integrals. The proof are direct, some details are provided.

Lemma 2.2. *Assume $Z \in [0,1)$. Then*

$$\int_0^Z \frac{dz}{\sqrt{1-z^4}} = \int_0^Z \sqrt{\frac{1+z^2}{1-z^2}}\, dz + \int_1^T \frac{t^2\, dt}{\sqrt{t^4-1}} - ZT \tag{2.2}$$

*where $T = \dfrac{\sqrt{1+Z^2}}{\sqrt{1-Z^2}}$.*

PROOF. The change of variables $t^2 = (1+z^2)/(1-z^2)$ produces

$$dt = \frac{1}{t}\frac{2z\,dz}{(1-z^2)^2}.$$

This leads to

$$\frac{t^2\cdot dt}{\sqrt{t^4-1}} = \sqrt{\frac{1+z^2}{1-z^2}}\cdot\frac{1-z^2}{\sqrt{4z^2}}\cdot\frac{2z\,dz}{(1-z^2)^2} = \sqrt{\frac{1+z^2}{(1-z^2)^3}}dz.$$

The result now follows from the identity

$$\begin{aligned}\frac{dz}{\sqrt{1-z^4}} - tdz + zdt + tdz &= \frac{dz}{\sqrt{(1+z^2)(1-z^2)}} + z\frac{1}{\sqrt{1+z^2}}\frac{2z\,dz}{\sqrt{(1-z^2)^3}}\\ &= \sqrt{\frac{1+z^2}{(1-z^2)^3}}\,dz.\end{aligned}$$

□

LEMMA 2.3. *Assume $x$ and $Z$ are variables connected by the relation*

$$X = \frac{\sqrt{1-\sqrt{1-Z^4}}}{Z}.$$

*Then*

$$\int_0^Z \frac{dz}{\sqrt{1-z^4}} = \int_0^X \frac{\sqrt{2}\cdot dx}{\sqrt{1+x^4}} \tag{2.3}$$

$$= \frac{3\sqrt{2}}{2}\int_0^X \sqrt{1+x^4}\;dx - \frac{\sqrt{2}}{2}\sqrt{1+X^4}. \tag{2.4}$$

PROOF. For $z \in (0,1]$, define the function

$$x = \frac{\sqrt{1-\sqrt{1-z^4}}}{z}.$$

Direct differentiation gives

$$\begin{aligned} dx &= \frac{\frac{1}{2\sqrt{1-\sqrt{1-z^4}}}\cdot\frac{-1}{2\sqrt{1-z^4}}\cdot(-4z^3)z - \sqrt{1-\sqrt{1-z^4}}}{z^2}\cdot dz\\ &= \frac{z^4-(1-\sqrt{1-z^4})(\sqrt{1-z^4})}{z^2\sqrt{1-\sqrt{1-z^4}}\sqrt{1-z^4}}dz\\ &= \frac{z^4-\sqrt{1-z^4}+1-z^4}{z^2\sqrt{1-\sqrt{1-z^4}}\sqrt{1-z^4}}dz = \frac{\sqrt{1-\sqrt{1-z^4}}}{z^2\sqrt{1-z^4}}dz.\end{aligned}$$

Also,

$$\begin{aligned}\frac{\sqrt{1+x^4}}{\sqrt{2}} &= \frac{\sqrt{1+\frac{(1-\sqrt{1-z^4})^2}{z^4}}}{\sqrt{2}} = \frac{\sqrt{z^4+1-2\sqrt{1-z^4}+1-z^4}}{\sqrt{2}\cdot z^2}\\ &= \frac{\sqrt{1-\sqrt{1-z^4}}}{z^2}.\end{aligned}$$

The quotient of the last two identities yields

$$\frac{\sqrt{2}\,dx}{\sqrt{1+x^4}} = \frac{dz}{\sqrt{1-z^4}}.$$

The final step in the proof comes from the expression

$$\begin{aligned}\frac{3}{2}\sqrt{1+x^4}dx - \frac{1}{2}d(x\sqrt{1+x^4}) &= \frac{3}{2}\sqrt{1+x^4}dx - \frac{1}{2}\sqrt{1+x^4} - \frac{x^4}{\sqrt{1+x^4}}dx \\ &= \frac{1}{\sqrt{1+x^4}}dx.\end{aligned}$$

□

Lemma 2.4. *Let variables $U, Z$ be related by*

$$U^2 = \frac{1-Z^2}{1+Z^2}$$

*Then*

$$\int_0^Z \frac{dz}{\sqrt{1-z^4}} = \int_U^1 \frac{du}{\sqrt{1-u^4}}. \tag{2.5}$$

Proof. We derive implicitly $u$ to obtain

$$\begin{aligned}du &= \frac{-z(1+z^2)-(1-z^2)z}{\frac{\sqrt{1-z^2}}{\sqrt{1+z^2}}(1+z^2)^2}\,dz \\ &= -\frac{2z}{1+z^2} \times \frac{dz}{\sqrt{1-z^4}}.\end{aligned}$$

Now the connection

$$\frac{1}{\sqrt{1-u^4}} = \frac{1}{\sqrt{1-\left(\frac{1-z^2}{1+z^2}\right)^2}} = \frac{1+z^2}{\sqrt{4z^2}} = \frac{1+z^2}{2z}.$$

produces

$$\frac{du}{\sqrt{1-u^4}} = -\frac{dz}{\sqrt{1-z^4}}.$$

Integration gives the result. □

Lemma 2.5. *Let $T$ and $Z$ be related by the expression*

$$\frac{\sqrt{1-T^4}}{\sqrt{2}T} = \frac{\sqrt{1-\sqrt{1-Z^4}}}{Z}.$$

*Then*

$$\int_0^Z \frac{dz}{\sqrt{1-z^4}} = \int_T^1 \frac{2\,dt}{\sqrt{1-t^4}}. \tag{2.6}$$

PROOF. We follow the method described by Fagnano. Consider the substitution in Lemma 2.3

$$X = \frac{\sqrt{1-\sqrt{1-Z^4}}}{Z}$$

with

$$X = \frac{\sqrt{1-T^4}}{\sqrt{2}T}.$$

Then, Lemma 2.3 gives

$$\frac{dz}{\sqrt{1-z^4}} = \frac{\sqrt{2}\,dx}{\sqrt{1+x^4}} = \frac{\sqrt{2}\,dx}{\sqrt{1+\frac{(1-t^4)^2}{4t^4}}} = \sqrt{2}\,dx \times \frac{1-t^4}{1+t^4}.$$

However, differentiation of the remaining relation yields

$$dx = \frac{1}{\sqrt{2}}\frac{-\frac{2t^4}{\sqrt{1-t^4}} - \sqrt{1-t^4}}{1-t^4}dt = -\sqrt{2}\frac{1+t^4}{(1-t^4)\sqrt{1-t^4}}.$$

This is just but an interesting application of the chain rule. □

An analogous statement is the last relation among elliptic integrals.

LEMMA 2.6. *Assume $U$ and $Z$ are related by*

$$\frac{\sqrt{2}U}{\sqrt{1-U^4}} = \frac{\sqrt{1-\sqrt{1-Z^4}}}{Z}. \tag{2.7}$$

*Then*

$$\int_0^Z \frac{dz}{\sqrt{1-z^4}} = \int_0^U \frac{2\cdot du}{\sqrt{1-u^4}}. \tag{2.8}$$

PROOF. Although the method in the previous proof is applicable, we use a different approach. Differentiation of

$$\frac{\sqrt{2}u}{\sqrt{1-u^4}} = \frac{\sqrt{1-\sqrt{1-z^4}}}{z},$$

provides

$$\sqrt{2}\cdot\frac{\sqrt{1-u^4}+\frac{2u^4}{\sqrt{1-u^4}}}{1-u^4}\,du = \frac{1}{z^2}\left(\frac{z^4}{\sqrt{1-z^4}\sqrt{1-\sqrt{1-z^4}}} - \sqrt{1-\sqrt{1-z^4}}\right)dz.$$

That is

$$\sqrt{2}\,\frac{1+u^4}{1-u^4}\,\frac{du}{\sqrt{1-u^4}} = \frac{\sqrt{1-\sqrt{1-z^4}}}{z^2}\cdot\frac{dz}{\sqrt{1-z^4}}, \tag{2.9}$$

and also

$$u^4 + \left(\frac{2z^2}{1-\sqrt{1-z^4}}\right)u^2 - 1 = 0.$$

It follows that

$$u^2 = \frac{-z^2}{1-\sqrt{1-z^4}} + \frac{\sqrt{z^4+1-2\sqrt{1-z^4}+1-z^4}}{1-\sqrt{1-z^4}} = \frac{-z^2}{1-\sqrt{1-z^4}} + \frac{\sqrt{2}}{\sqrt{1-\sqrt{1-z^4}}}$$

and therefore

$$u^2 + \frac{1-u^4}{2u^2} = \frac{1+u^4}{2u^2} = \frac{\sqrt{2}}{\sqrt{1-\sqrt{1-z^4}}}. \tag{2.10}$$

Finally, substituting $1+u^4$ in (2.9) gives

$$\frac{2\,du}{\sqrt{1-u^4}} = \frac{dz}{\sqrt{1-z^4}},$$

and the result follows from here. □

## 3. Division of the lemniscate into equals parts

This section presents a geometric form of the question of dividing the total length of a lemniscate into $N$ equal parts. Some restrictions on the number $N$ come from the methods employed in the reduction of elliptic integrals.

Theorem 3.1. *The arc $OP$ of the lemniscate*

$$L = \left\{(x,y) : x^2+y^2 = \sqrt{x^2-y^2}\right\} \tag{3.1}$$

*(see Figure 1) may be computed as*

$$OP = AB + CD - Z \times T, \quad 0 \leqslant z < 1, \tag{3.2}$$

*Where $AB$ is an elliptic arc on the ellipse of semiaxes $1, \sqrt{2}$ (see Figure 2), $CD$ is an hyperbolic arc on an equilateral hyperbola (see Figure 3) with unit semiaxis, $z$ is the horizontal coordinate of point $B$ and $t$ is the distance from point $D$ to the origin.*

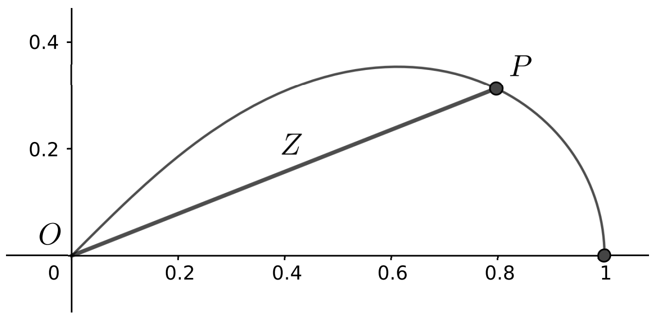


Figure 1. Lemniscatic quadrant $z = \sqrt{\cos 2\theta}$, $0 \leqslant \theta < \pi/2$.

Proof. Write the equation of the lemniscate in polar coordinates $x = z\cos\theta$, $y = \sin\theta$ to obtain

$$z^2 = \cos 2\theta. \tag{3.3}$$

Then, it follows that

$$\text{arc } OP = \int_0^Z \sqrt{1+\left(z\frac{d\theta}{dz}\right)^2}\,dz = \int_0^Z \sqrt{1+\frac{z^4}{1-\cos^2 2\theta}}\,dz = \int_0^Z \frac{dz}{\sqrt{1-z^4}}. \tag{3.4}$$

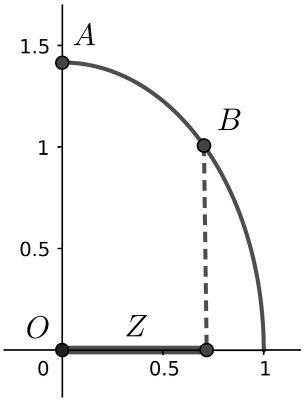


FIGURE 2. Elliptic quadrant $y = \sqrt{2}\sqrt{1-z^2}$, $0 \leqslant z \leqslant 1$.

Now, one sees that for the ellipse in Figure 2

$$z^2 + \frac{y^2}{2} = 1, \quad 0 \leqslant z \leqslant 1, \quad y \geqslant 0, \tag{3.5}$$

*i. e.* $y = \sqrt{2}\sqrt{1-z^2}$,

$$\text{arc } AB = \int_0^Z \sqrt{1 + \left(\frac{dy}{dz}\right)^2} dz = \int_0^Z \sqrt{1 + \frac{2z^2}{1-z^2}} dz = \int_0^Z \sqrt{\frac{1+z^2}{1-z^2}} dz. \tag{3.6}$$

Finally, with the parametrization $x = t\cos\theta$, $y = t\sin\theta$, in the equilateral hyperbola of equation $x^2 - y^2 = 1$, shown in Figure 3, we find that $t^2 = \sec 2\theta$. Therefore,

$$\text{arc } CD = \int_1^T \sqrt{1 + \left(t\frac{d\theta}{dt}\right)^2} dt = \int_1^T \sqrt{1 + \cot^2 2\theta} \cdot dt = \int_1^T \frac{dt}{\sqrt{1-t^4}}. \tag{3.7}$$

Now replace the integrals appearing in formula (2.2) of Lemma 2.2 by the corresponding arcs to produce the identity (3.2).

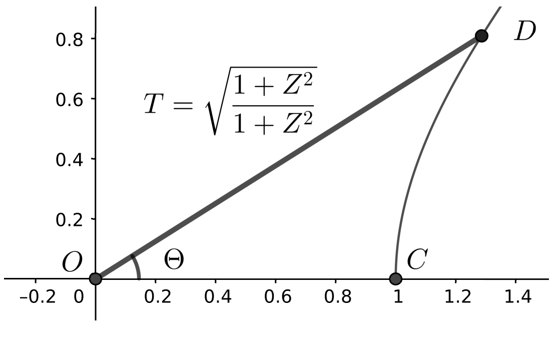


FIGURE 3. Hyperbolic quadrant $t = \sqrt{\sec 2\theta}$, $0 \leqslant \theta \leqslant \pi/2$.

∎

Theorem 3.2. *The lemniscatic arc $OP$, shown in Figure 1, can be calculated as*

$$OP = \frac{3\sqrt{2}}{2} \times EF - \frac{\sqrt{2}}{2} \times FG, \tag{3.8}$$

*where $EF$ is the arc of the cubic polynomial $y = \frac{1}{3}x^3$ shown in Figure 4 and $FG$ is the straight line segment of the tangent line at $F$ to the point $G$ of the intersection of this line and the $y$-axis.*

Proof. As before (Figure 1),

$$\text{arc } OP = \int_0^Z \frac{dz}{\sqrt{1-z^4}}. \tag{3.9}$$

Using now the relation $y = \frac{1}{3}x^3$, it follows that (see Figure 4)

$$\text{arc } EF = \int_0^X \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^X \sqrt{1+x^4} dx. \tag{3.10}$$

Now observe that the vertical distance $GH$ in Figure 4 equals to $X^3$ and therefore

$$\text{straight line segment } EF = X\sqrt{1+X^4}. \tag{3.11}$$

Expressing the lengths of the segments as elliptic integrals and using Lemma 2.3 give the result. ∎

The last statement provides geometric procedures to cut a lemniscatic arc into equal parts by only straightedge and compass constructions.

Theorem 3.3. *The part of the of the lemniscate on the first quadrant is given by*

$$\mathfrak{Q} = \left\{(x,y) : x^2 + y^2 = \sqrt{x^2 - y^2}, 0 \leqslant x \leqslant 1, y \geqslant 0\right\}. \tag{3.12}$$

*Then,*

(1) *If $a \in \mathfrak{Q}$ us the point for which the chord (straight segment to the origin) $OA = \sqrt{\sqrt{2}-1}$, then*

$$\textit{arc } OA = \textit{arc } AE = \frac{1}{2} \textit{ arc } OE. \tag{3.13}$$

(2) *If $B$ is the point of $\mathfrak{Q}$ for which the chord $OB = \sqrt[4]{2\sqrt{3}-3}$, then*

$$\textit{arc } OB = \frac{2}{3} \textit{ arc } OE \quad \textit{and} \quad \textit{arc } BE = \frac{1}{3} \textit{ arc } OE. \tag{3.14}$$

(3) *There is a $w \in [0,1)$ with the following property: if $C \in \mathfrak{Q}$ has chord $OC = w$, then*

$$\textit{arc } CE = \frac{1}{5} \textit{ arc } OE. \tag{3.15}$$

*This value is a solution of and equation of degree 8.*

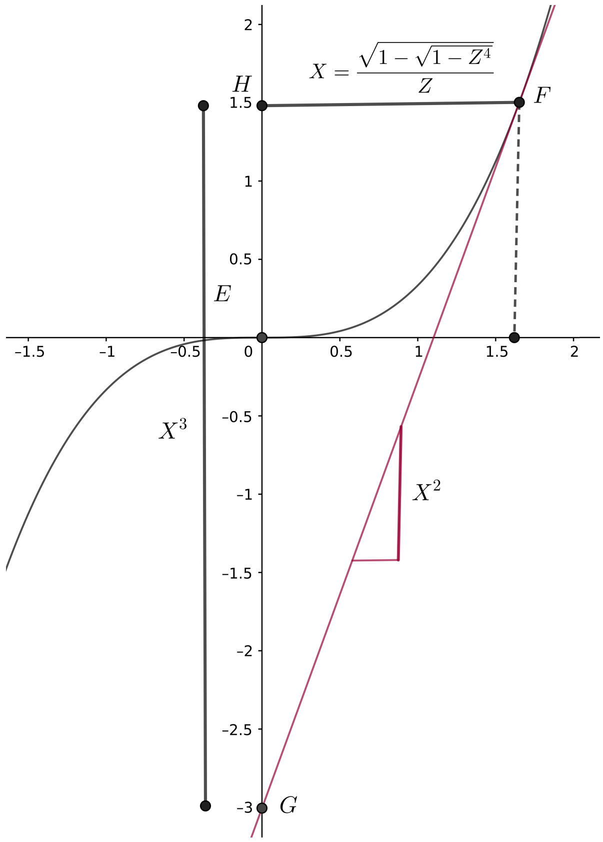


FIGURE 4. Graph of polynomial $y = \frac{1}{3}x^3$.

(4) *Given a division of the arc $= E$ into $n$ parts of equal length, there is a geometric procedure (involving only field operations and square roots) which divides each of these parts into two equal parts, producing a division into $2n$ equal parts. Therefore, for any positive integer $m$, the lemniscate can be divided algebraically into $2 \times 2^m$, $3 \times 2^m$, $5 \times 2^m$ equals parts with compass and straightedge or ruler.*

PROOF. The proof is divided into a number of steps.

`Step 1.` Figure 5 shows

$$\text{arc } OA = \int_0^Z \frac{dz}{\sqrt{1-z^4}}.$$

Let $Z = U$ be the fixed point of the the transformation

$$u = \sqrt{\frac{1-z^2}{1+z^2}},$$

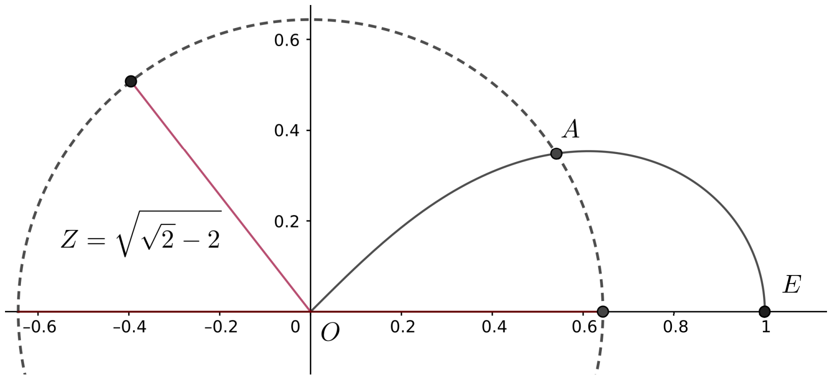


FIGURE 5. Division of the lemniscate into two equal parts with rule and compass.

appearing in Lemma 2.4. That is, the value $Z = \sqrt{\sqrt{2} - 1}$ produces the identity

$$\text{arc } OA = \text{arc } AE = \frac{1}{2}\text{arc } OE.$$

`Step 2`. This part comes from obtaining a fixed point of the transformation in Lemma 2.5, namely

$$\frac{\sqrt{1 - t^4}}{\sqrt{2}t} = \frac{\sqrt{1 - \sqrt{1 - z^4}}}{z}.$$

This produces

$$Z = T = \sqrt[4]{2\sqrt{3} - 3}.$$

In Figure 6 this means that

$$\text{arc } OB = 2 \times \text{arc } BE = \frac{2}{3} \text{ arc } OE.$$

This proves the second part of the theorem.

`Step 3`. Now start with Lemma 2.6. Consider the illustration provided in Figure 7. Assume the chord $OI = U$ is respectively related to the chord $OS = Z$ as in (2.7). Then

$$\text{arc } OS = 2 \times \text{arc } OI.$$

Now by Lemma 2.5, there is point $L$ in the lemniscate sucht that

$$\text{arc } OI = 2 \times \text{arc } LE.$$

Therefore

$$\text{arc } OS = 4 \times \text{arc } LE.$$

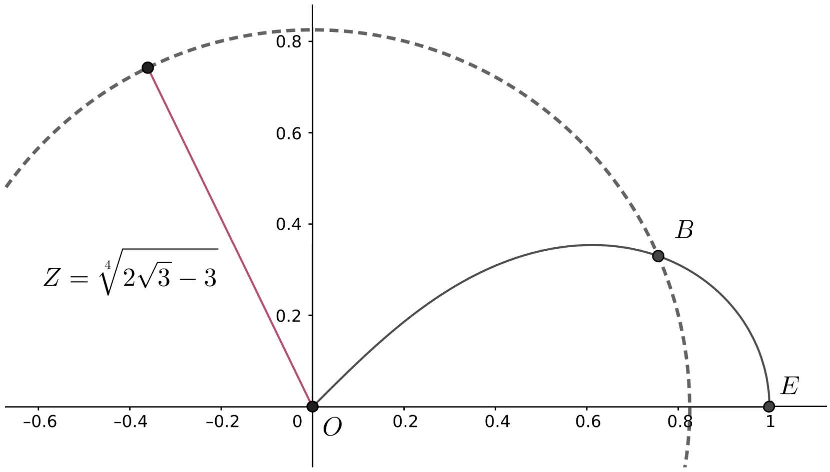


FIGURE 6. Division of the lemniscate into three equal parts with rule and compass.

Now let $S$ and $L$ collapse to a point, say $C$ (see Figure 7). Then,

$$\text{arc } CE = \frac{1}{5} \times \text{arc } OE,$$
$$\text{arc } OI = \frac{2}{5} \times \text{arc } OE,$$
$$\text{arc } IC = \text{arc } OI = \frac{2}{5} \times \text{arc } OE.$$

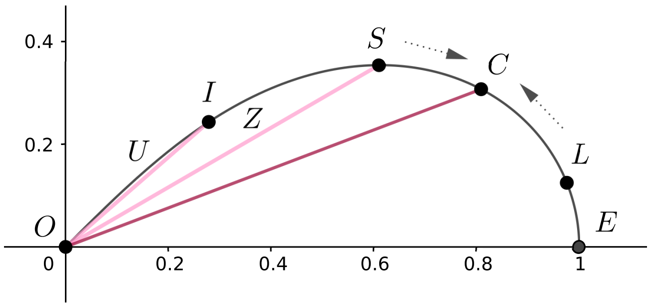


FIGURE 7. Division of the lemniscate into five equal parts with rule and compass.

Lemma 2.4 is now used to produce points $B$ and $F$ such that

$$\text{arc } OI = \text{arc } FE,$$
$$\text{arc } OB = \text{arc } CE.$$

This proves that the points $B, I, F$ and $C$ cut the quadrant $\mathfrak{Q}$ into five equal arcs. To complete the proof, it remains to find value $Z =$ chord $OC$. This value comes from the relation

$$Z = \frac{2U\sqrt{1-U^4}}{1+U^4}$$

arising from formula (2.10) in the proof of Lemma 2.6. Similarly, one also needs the relation

$$U = \frac{2V\sqrt{1-V^4}}{1+V^4}$$

coming from Lemma 2.5. Furthermore, as $S$ and $L$ collapse to $C$, it follows that $V = Z$. This produces fixed point equation

$$Z = (f \circ f)(Z), \quad \text{with} \quad f(Z) = \frac{2Z\sqrt{1-Z^4}}{1+Z^4}.$$

It is not hard to see this last equation has a non-trivial root in $[0, 1]$ and this root can be expressed with basic operations and square roots. The solution set of this equation comprises the solutions of $f(Z) = Z$, which split the lemniscate into three equal parts. Besides, it gives new solutions leading the cutting of the lemniscate into five parts. In particular, the sought solution to our problem is

$$Z = \sqrt[4]{-13 + 6\sqrt{5} + 2\sqrt{85 - 38\sqrt{5}}} \approx 0.93351782.$$

This value, together with the previous values of $Z$ accomplishing the split of the lemniscate into 2 and 3 parts, can be checked numerically without difficulty.

`Step 4`. The final part of the Theorem follows from the next statement.

Lemma 3.1. *Suppose the lemniscate is divided into $n$ arcs of equal length. Then each of these arcs can be divided, by an algebraic procedure, into two equal parts leading to a division into $2n$ equal parts.*

Proof. Let the arcs be $OX_1$, $OX_2, \ldots, OX_{n-1}$ and $OE$. Divide the odd arcs $OX_{2j-1}$ into halves using Lemma 2.6 and then use the chords of these halves and Lemma 2.4 to cut the remaining arcs. □

□

## 4. Conclusions

The last part of this work contains some historical comments.

- For Fagnano, definite integrals are not only simple real quantities but actual arc lengths of certain curves. Also, he does not use integral limits. He writes, for instance,

$$\int dx\sqrt{1+x^4}$$

and explains that *esprime l'arco $AQ$ della parabola cubica primaria corrispondente all'abscissa $AF = z$...*, it represents the arc length of the cubic parabola $y = x^3/3$ corresponding to the abscissa $AF = z$.

- First, Fagnano seeks clever transformations to express the arc length of the lemniscate in terms of arc lengths of other known curves. Then he finds transformations to cut an arc of the lemniscate into equal parts. In any case, these transformations only involve field operations and square roots; that is, straightedge and compass constructions.
- Fagnano's transformations are indeed solutions to first-order separable ordinary differential equations. In order to cut the lemniscate into three equal parts, for example, he finds the solution

$$\frac{\sqrt{1-t^4}}{\sqrt{2}t} = \frac{\sqrt{1-\sqrt{1-z^4}}}{z}$$

of the ordinary equation

$$\frac{dz}{\sqrt{1-z^4}} = -\frac{2dt}{\sqrt{1-t^4}},$$

furnished with the correct initial condition. In general, these differential equations express positive integer ratios between two arc length differentials of the lemniscate. Later on Euler noticed these facts and used them to foresee the addition properties and the periodicity of elliptic functions.
- Gauss formulated a sufficient condition for the constructibility (rule and compass) of regular polygons, which is equivalent to the division of circle into equal arcs. In his *Disquisitiones Arithmeticae* he announced that this holds for a wider class of transcendentals. After a while Abel proved that the lemniscate can be split into $n$ parts if and only if $n$ is of the form Gauss discovered.
- Neither Legendre's canonical forms for elliptic transcendentals nor Jacobi's theory of elliptic transformations were available to Fagnano. He was able to see the tip of the iceberg of all these developments.

Fagnano's previous results address two types of problems:

- To express the lemniscate arclength in terms of the arclength of other curves such as ellipses, hyperbolas, parabolas and straight lines.
- To construct substitutions $t = t(z)$ of the lemniscate arclength element

$$\frac{dz}{\sqrt{1-z^4}}$$

yielding a new arc element

$$n \times \frac{dt}{\sqrt{1-t^4}},$$

for certain (negative or positive) integer $n$.

The first of these problems is related to the fact that elliptic integrals do not possess an elementary antiderivative, that its, a function constructed from a finite number of algebraic, exponential, trigonometric and logarithmic functions, by using field operations. When this happens, it might be useful to express a given elliptic integral in terms of other (elliptic, however more familiar) integrals.

The second problem deals with the division of the lemniscate quadrant into an integer number of equal parts. Fagnano was able to prove that the lemniscate can be cut into 2, 3 and 5 parts by means of a ruler-and-compass construction. These means that each division can be expressed by finite field operations of algebraic expressions and square roots.

## References

[1] J. Bernoulli. Curvatura laminae elasticae. *Acta Eruditorum*, 13:262–276, 1694.

[2] J. D'Alembert. Lemniscate. *Encyclopedia, ou Dictionnaire des sciences, des arts et des matieres*, 9:382, 1765.

[3] L. Euler. De integratione aequationis differentialis $\frac{m\,dx}{\sqrt{1-x^4}} = \frac{n\,dy}{\sqrt{1-y^4}}$. In *Opera Omnia*, volume 20 of *I*, pages 58–79. Teubner, Berlin, 1738-1914.

[4] G. C. Fagnano. Metodo per misurare la lemniscata. In Societa Editrice Dante Alighieri di Albrighi Segati e C., editor, *Opere Matematiche*, volume 2, pages 293–313, 1911.

[5] R. Fricke. Elliptische Funktionen. In *Encyklopadie der mathematischen Wissenschaften*, volume 2, pages 177–348. B. G. Teubner, 1913.

[6] C. F. Gauss. De curva lemniscata. *Werke*, 3:413–432, 1876.

[7] J. Landen. A disquisition concerning certain fluents, which are assignable by the arcs of the conic sections; wherein are investigated some new and useful theorems for computing such fluents. *Philos. Trans. Royal Soc. London*, 61:298–309, 1771.

[8] C. L. Siegel. *Topics in Complex Function Theory I. Elliptic Functions and Uniformization.* Wiley-Interscience, 1969.

*Received 21 09 2020 revised 22 11 2020*

[a]Department of Mathematics,
University of New Orleans ,
New Orleans, LA 70148, U.S.A.
*E-mail address*: LMancuso@uno.edu
*E-mail address*: jnowalski@uno.edu

[b]Department of Mathematics,
Tulane University, New Orleans,
LA 70118, U.S.A
*E-mail address*: vhm@tulane.edu

[c]Departmento de Mathemáticas y Estadística,
Universidad del Tolima Ibagué,
Colombia
*E-mail address*: leonsolc@ut.edu.co