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On some definite integrals containing the inverse tangent function

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Abstract. Evaluations are presented for definite integrals of the form

$$\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right) f(ax)\,dx,$$

for numerous functions f subject to appropriate restrictions on the real numbers a, r, and b. Many new results pertaining to the particular inverse tangent function appearing in the integrand not found in the tables of Gradshteyn and Ryzhik are given.

1. Introduction

In entry **4.574.3** of [**2**] the following Fourier sine transform for the function $\arctan\left(\frac{2rx}{b^2+x^2}\right)$ appears

$$\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right)\sin(ax)\,dx = \frac{\pi}{a}\exp\left(-a\sqrt{r^2+b^2}\right)\sinh(ar). \tag{1.1}$$

Here $a > 0$ and $b, r \in \mathbb{R}$. It cites [**1**, p. 87, Entry 2.8.9]. Turning there no reference to any original source can be found. Intrigued, the current work is the result of a quest to prove (1.1). Here a large number of definite integrals containing the product between the inverse tangent function appearing in the integrand of (1.1) and various other functions will be evaluated. The definite integrals to be found are of the form

$$\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right) f(ax)\,dx. \tag{1.2}$$

Here f is a continuous function on $x > 0$ with suitable restrictions applied on the real numbers a, r, and b. Most of the definite integrals we present are not to be found in the classic table of integrals by Gradshteyn and Ryzhik [**2**]. Four of the definite integrals found are for the Laplace, Mellin, and Fourier sine and cosine transforms of

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$\arctan\left(\frac{2rx}{b^2+x^2}\right)$. The latter two integral transforms are known, the former two appear to be new.

Many of the results for the definite integrals we give in terms of a number of parameters contain what Moll terms 'fake' parameters [**5**, p. 34]. By a suitable change in variable and simple scaling a reduction in their number can be achieved allowing the final expression to be written in minimal form independent of one of these parameters. We shall indicate when the removal of a fake parameter leading to minimal expressions for our results can be made.

All integrals to be found will make use of the following lemma. It allows all definite integrals we are interested in finding to be written in terms of a double integral that can then be evaluated after the order of integration has been changed.

Lemma 1.1. *If* $r = \sinh\alpha$ *where* $\alpha \in \mathbb{R}$, *then*

$$\arctan\left(\frac{2rx}{1+x^2}\right) = \int_{e^{-\alpha}}^{e^{\alpha}} \frac{x}{t^2+x^2}\, dt. \tag{1.3}$$

Proof. Setting $r = \sinh\alpha = \frac{1}{2}\left(e^{\alpha} - e^{-\alpha}\right)$ where $\alpha \in \mathbb{R}$ we can write

$$\arctan\left(\frac{2rx}{1+x^2}\right) = \arctan\left(\frac{xe^{\alpha} - xe^{-\alpha}}{1+x^2}\right) = \arctan(xe^{\alpha}) - \arctan(xe^{-\alpha}), \tag{1.4}$$

where we have made use of the following result for the inverse tangent function

$$\arctan(u) - \arctan(v) = \arctan\left(\frac{u-v}{1+uv}\right), \quad uv > -1.$$

As $x > 0$ and $e^{\pm\alpha} > 0$ for all α, from the well-known identity for the inverse tangent function of

$$\arctan(u) = \frac{\pi}{2} - \arctan\left(\frac{1}{u}\right), \quad u > 0, \tag{1.5}$$

we can rewrite (1.4) as

$$\arctan\left(\frac{2rx}{1+x^2}\right) = \arctan\left(\frac{e^{\alpha}}{x}\right) - \arctan\left(\frac{e^{-\alpha}}{x}\right) = \int_{e^{-\alpha}}^{e^{\alpha}} \frac{x}{t^2+x^2}\, dt,$$

and completes the proof. □

Remark 1.1. From the limits of integration that appear in the integral given in Lemma 1.1 we will have a need to express these limits in terms of the parameter r. More generally if $\sinh\alpha = r/b$ where $b \neq 0$, expressing the hyperbolic sine function in terms of exponentials, on solving the quadratic equation that results one finds

$$e^{\pm\alpha} = \frac{\sqrt{r^2+b^2} \pm r}{b}. \tag{1.6}$$

Note the positive case for the square root has been chosen since $e^{\pm\alpha} > 0$ for all $\alpha \in \mathbb{R}$.

2. Functions leading to elementary functions

In this section we give a number of definite integrals involving the product between the inverse tangent function appearing in (1.2) and various functions f that lead to results which can be expressed in terms of elementary functions. Before presenting these, while the final form of all definite integrals given in this section will be in terms of elementary functions there will be a need for the gamma function and some other functions closely associated with it. For convenience, these we introduce here.

The *digamma function* is defined by

$$\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{2.1}$$

where $\Gamma(x)$ is the classical *gamma function* defined by the Eulerian integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt. \tag{2.2}$$

Closely connected to the gamma function is the *beta function* defined by

$$\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \quad x, y > 0, \tag{2.3}$$

which is related to the gamma function by the identity

$$\mathrm{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.4}$$

The *upper incomplete gamma function* is defined by

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t}\, dt, \tag{2.5}$$

and reduces to the gamma function when $x = 0$. The *error function* is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt. \tag{2.6}$$

When $a = \frac{1}{2}$ in (2.5) the upper incomplete gamma function and the error function are known to be related to each other in the following manner [**6**, p. 177, Eq. (8.4.6)].

$$\Gamma\left(\frac{1}{2}, x\right) = \sqrt{\pi}\left(1 - \operatorname{erf}\left(\sqrt{x}\right)\right). \tag{2.7}$$

The reader will find information about these functions in [**3**] and [**6**].

For our first definite integral we take f to be a simple rational function.

Proposition 2.1. *For $a \geqslant 0$ and $b, r \in \mathbb{R}$ the identity*

$$\int_0^\infty \arctan\left(\frac{2rx}{b^2 + x^2}\right) \frac{x}{x^2 + a^2}\, dx = \frac{\pi}{2} \log\left(\frac{a + \sqrt{r^2 + b^2} + r}{a + \sqrt{r^2 + b^2} - r}\right), \tag{2.8}$$

holds.

Proof. Let $I(a,r,b)$ denote the left-hand side of (2.8). Enforcing a substitution of $x \mapsto bx$ reduces the number of parameters by one. Doing so yields

$$I(a,r,b) = \int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right)\frac{x}{x^2+c^2}\,dx, \tag{2.9}$$

where $\sinh\alpha = r/b$ and $c = a/b$. Applying (1.3) to (2.9), a change in the order of integration which admits an elementary justification produces

$$I(a,r,b) = \int_{e^{-\alpha}}^{e^\alpha}\int_0^\infty \frac{x^2}{(t^2+x^2)(x^2+c^2)}\,dx\,dt.$$

Performing the inner integral which is elementary; it following after performing a partial decomposition of

$$\frac{x^2}{(t^2+x^2)(x^2+c^2)} = \frac{c^2}{(c^2-t^2)(c^2+x^2)} - \frac{t^2}{(c^2-t^2)(t^2+x^2)}; \tag{2.10}$$

we have

$$I(a,r,b) = \frac{\pi}{2}\int_{e^{-\alpha}}^{e^\alpha}\frac{dt}{t+c} = \frac{\pi}{2}\log\left(\frac{c+e^\alpha}{c+e^{-\alpha}}\right).$$

The desired result then follows on substituting for the values of c and α. □

Remark 2.1. The parameter b appearing in (2.8) is an example of a fake parameter. Enforcing a substitution of $x \mapsto bx$ and scaling a by ab yields the minimal form

$$\int_0^\infty \arctan\left(\frac{2rx}{1+x^2}\right)\frac{x}{x^2+a^2}\,dx = \frac{\pi}{2}\log\left(\frac{a+\sqrt{r^2+1}+r}{a+\sqrt{r^2+1}-r}\right). \tag{2.11}$$

Remark 2.2. Two particular cases immediately follow from Proposition 2.1. Setting $b=0$ in (2.8) gives

$$\int_0^\infty \arctan\left(\frac{2r}{x}\right)\frac{x}{x^2+a^2}\,dx = \frac{\pi}{2}\log\left(\frac{a+|r|+r}{a+|r|-r}\right), \tag{2.12}$$

and is valid for $a>0$ and $r\in\mathbb{R}$ while setting $a=0$ in (2.8) gives

$$\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right)\frac{dx}{x} = \pi\,\mathrm{arcsinh}\left(\frac{r}{b}\right), \tag{2.13}$$

and is valid for $b>0$ and $r\in\mathbb{R}$. In this last expression, b is a fake parameter and can be removed. Enforcing $x\mapsto bx$ and replacing r with rb one obtains the minimal form of

$$\int_0^\infty \arctan\left(\frac{2rx}{1+x^2}\right)\frac{dx}{x} = \pi\,\mathrm{arcsinh}(r), \tag{2.14}$$

a result which recently appeared as a problem [**7**].

Remark 2.3. Set $p = \frac{2}{a} > 0$ in (2.12). If a substitution of $x \mapsto \frac{2}{x}$ is enforced, for the case where $r \geqslant 0$ one finds

$$\int_0^\infty \frac{\arctan(rx)}{x(p^2+x^2)}\,dx = \frac{\pi}{2p^2}\log(1+rp). \tag{2.15}$$

This is entry **4.535.9** in [**2**].

EXAMPLE 2.1. An attractive special case is

$$\int_0^\infty \arctan\left(\frac{2x}{x^2+4}\right)\frac{x}{x^2+4}\,dx = \frac{\pi}{2}\log(\varphi), \tag{2.16}$$

where φ is the *golden ratio* $(1+\sqrt{5})/2$ and is found on setting $r = 1$ and $a = b = 2$ in (2.8).

EXAMPLE 2.2. A second attractive special case is

$$\int_0^\infty \arctan\left(\frac{6x}{16+x^2}\right)\frac{dx}{x} = \pi\log(2), \tag{2.17}$$

and is found on setting $r = 3$ and $b = 4$ in (2.13).

LEMMA 2.1. *For* $r \in \mathbb{R}$

$$\int_0^\infty \arctan\left(\frac{2rx}{1+x^2}\right)\frac{\log(x)}{x}\,dx = 0. \tag{2.18}$$

PROOF. The result is immediate on enforcing a substitution of $x \mapsto \frac{1}{x}$. □

PROPOSITION 2.2. *For* $a > 0$, $b > 0$, *and* $r \in \mathbb{R}$ *the identity*

$$\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right)\frac{\log(ax)}{x}\,dx = \pi\log(ab)\,\mathrm{arcsinh}\left(\frac{r}{b}\right), \tag{2.19}$$

holds.

PROOF. Let $I(a,r,b)$ denote the left-hand side of (2.19). Enforcing a substitution of $x \mapsto bx$ gives

$$\begin{aligned} I(a,r,b) = \log(ab)&\int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right)\frac{dx}{x} \\ &+ \int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right)\frac{\log(x)}{x}\,dx, \end{aligned} \tag{2.20}$$

where $\sinh\alpha = r/b$. The first of the integrals appearing in (2.20) is given by (2.13). From Lemma 2.1 the second of the integrals is equal to zero. The desired result then follows and completes the proof. □

REMARK 2.4. The parameter b appearing in (2.19) is fake. Enforcing a substitution of $x \mapsto bx$ and scaling r by rb and a by $\frac{a}{b}$ produces the minimal form

$$\int_0^\infty \arctan\left(\frac{2rx}{1+x^2}\right)\frac{\log(ax)}{x}\,dx = \pi\log(a)\,\mathrm{arcsinh}(r). \tag{2.21}$$

EXAMPLE 2.3. An attractive special case is

$$\int_0^\infty \arctan\left(\frac{8x}{9+x^2}\right)\frac{\log(x)}{x}\,dx = \pi\log^2(3), \tag{2.22}$$

and is found on setting $a = 1$, $r = 4$, and $b = 3$ in (2.19).

Remark 2.5. Differentiating (2.19) with respect to the parameter b gives

$$\int_0^\infty \frac{\log(ax)}{x^4+2x^2(b^2+2r^2)+b^4}\,dx = \frac{\pi}{4rb^2}\left\{\frac{r\log(ab)}{\sqrt{r^2+b^2}} - \operatorname{arcsinh}\left(\frac{r}{b}\right)\right\}, \tag{2.23}$$

valid for $r > 0$, $b > 0$. The parameter b is fake. Since

$$\lim_{r\to 0^+} \frac{1}{r}\operatorname{arcsinh}\left(\frac{r}{b}\right) = \frac{1}{b}, \tag{2.24}$$

when $r \to 0^+$, (2.23) reduces to

$$\int_0^\infty \frac{\log(ax)}{(x^2+b^2)^2}\,dx = \frac{\pi}{4b^3}\left(\log(ab)-1\right), \tag{2.25}$$

and is valid for $a > 0$, $b > 0$. Once more the parameter b is fake. Differentiating (2.19) with respect to the parameter r, combining the result with (2.23) produces

$$\int_0^\infty \frac{x^2\log(ax)}{x^4+2x^2(b^2+2r^2)+b^4}\,dx = \frac{\pi}{4r}\left\{\frac{r\log(ab)}{\sqrt{b^2+r^2}} + \operatorname{arcsinh}\left(\frac{r}{b}\right)\right\}, \tag{2.26}$$

valid for $a > 0$, $b > 0$. Here the parameter b is fake. Finally, as $r \to 0^+$, (2.26) reduces to

$$\int_0^\infty \frac{x^2\log(ax)}{(x^2+b^2)^2}\,dx = \frac{\pi}{4b}\left(1+\log(ab)\right), \tag{2.27}$$

and is valid for $a > 0$, $b > 0$. Once more the parameter b here is fake and we have made use of the limit given in (2.24).

Example 2.4. Setting $a = b = 1$ in (2.25) and (2.27) before subtracting the former from the latter we find

$$\int_0^\infty \frac{(1-x^2)\log(x)}{(x^2+1)^2}\,dx = -\frac{\pi}{2}. \tag{2.28}$$

This is entry **4.234.4** in [**2**]. It is surprising neither (2.25) nor (2.27) appear in [**2**].

Proposition 2.3. *For $a > 0$, $b > 0$, and $r \in \mathbb{R}$ the identity*

$$\begin{aligned}\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right)\frac{\arctan(ax)}{x^2}\,dx &= \frac{\pi}{2b^2}\Big\{2ab^2\operatorname{arcsinh}\left(\frac{r}{b}\right)\\ &\quad + \left(ab^2+\sqrt{r^2+b^2}+r\right)\log\left(a\sqrt{r^2+b^2}-ar+1\right)\\ &\quad - \left(ab^2+\sqrt{r^2+b^2}-r\right)\log\left(a\sqrt{r^2+b^2}+ar+1\right)\Big\},\end{aligned} \tag{2.29}$$

holds.

Proof. Let $I(a,r,b)$ denote the left-hand side of (2.29). Enforcing a substitution of $x \mapsto bx$ gives

$$I(a,r,b) = \frac{1}{b}\int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right)\frac{\arctan(cx)}{x^2}\,dx, \tag{2.30}$$

where $c = ab$ and $\sinh\alpha = r/b$. Applying (1.3) to (2.30), a change in the order of integration produces

$$I(a,r,b) = \frac{1}{b}\int_{e^{-\alpha}}^{e^{\alpha}}\int_0^\infty \frac{\arctan(cx)}{x(t^2+x^2)}\,dx\,dt. \tag{2.31}$$

The inner integral appears as entry **4.535.9** of [**2**]. Here

$$\int_0^\infty \frac{\arctan(cx)}{x(t^2+x^2)}\,dx = \frac{\pi}{2t^2}\log(1+ct), \quad c>0, t>0. \tag{2.32}$$

Thus (2.31) becomes

$$I(a,r,b) = \frac{\pi}{2b}\int_{e^{-\alpha}}^{e^{\alpha}} \frac{\log(1+ct)}{t^2}\,dt.$$

This integral is elementary. It can be found by integrating by parts first before applying a partial fraction decomposition. The result is

$$\begin{aligned} I(a,r,b) &= \frac{\pi}{2b}\left[c\log(t) - \frac{(1+ct)\log(1+ct)}{t}\right]_{e^{-\alpha}}^{e^{\alpha}} \\ &= \frac{\pi}{2b}\left\{2c\alpha + (c+e^{\alpha})\log(1+ce^{-\alpha}) - (c+e^{-\alpha})\log(1+ce^{\alpha})\right\}. \end{aligned}$$

The desired result then follows on substituting the values for c and α and completes the proof. □

REMARK 2.6. The parameter b appearing in (2.29) is a fake parameter. Enforcing a substitution of $x \mapsto bx$ and scaling r by rb and a by $\frac{a}{b}$ one can see the minimal form is equivalent to its formulation when $b=1$ is set in (2.29).

From the next result a number of interesting definite integrals follow as special cases.

PROPOSITION 2.4. *For $b>0$, $r\in\mathbb{R}$, and $s\in\mathbb{R}$ the identity*

$$\begin{aligned} \int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right)\arctan\left(\frac{2sx}{b^2+x^2}\right)dx &= \pi\left\{r\,\mathrm{arcsinh}\left(\frac{s}{b}\right) + s\,\mathrm{arcsinh}\left(\frac{r}{b}\right)\right. \\ &\left.+\left(\sqrt{r^2+b^2}+\sqrt{s^2+b^2}\right)\log\left[\frac{\cosh\left(\frac{1}{2}\mathrm{arcsinh}\left(\frac{r}{b}\right)-\frac{1}{2}\mathrm{arcsinh}\left(\frac{s}{b}\right)\right)}{\cosh\left(\frac{1}{2}\mathrm{arcsinh}\left(\frac{r}{b}\right)+\frac{1}{2}\mathrm{arcsinh}\left(\frac{s}{b}\right)\right)}\right]\right\} \end{aligned} \tag{2.33}$$

holds.

PROOF. Let $I(r,s,b)$ denote the left-hand side of (2.33). Enforcing a substitution of $x\mapsto bx$ gives

$$I(r,s,b) = b\int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right)\arctan\left(\frac{2x\sinh\lambda}{1+x^2}\right)dx, \tag{2.34}$$

where $\sinh\alpha = r/b$ and $\sinh\lambda = s/b$. Here $\alpha,\lambda\in\mathbb{R}$. The single integral appearing in (2.34) can be converted to a triple integral by applying the result of Lemma (1.1) to each of the inverse tangent functions appearing in the integrand. Doing so, after the order of integration is changed produces

$$I(r,s,b) = b\int_{e^{-\alpha}}^{e^{\alpha}}\int_{e^{-\lambda}}^{e^{\lambda}}\int_0^\infty \frac{x^2}{(t^2+x^2)(u^2+x^2)}\,dx\,du\,dt. \tag{2.35}$$

Making use of the partial fraction decomposition given in (2.10) we find

$$\begin{aligned}
I(r,s,b) &= b\int_{e^{-\alpha}}^{e^{\alpha}}\int_{e^{-\lambda}}^{e^{\lambda}} \left[\frac{t}{t^2-u^2}\arctan\left(\frac{x}{t}\right) - \frac{u}{t^2-u^2}\arctan\left(\frac{x}{u}\right)\right]_0^\infty du\,dt \\
&= \frac{\pi b}{2}\int_{e^{-\alpha}}^{e^{\alpha}}\int_{e^{-\lambda}}^{e^{\lambda}} \frac{1}{t+u}\,du\,dt = \frac{\pi b}{2}\int_{e^{-\alpha}}^{e^{\alpha}} \left[\log(t+u)\right]_{e^{-\lambda}}^{e^{\lambda}}\,dt \\
&= \frac{\pi b}{2}\int_{e^{-\alpha}}^{e^{\alpha}} \left(\log(t+e^{\lambda}) - \log(t+e^{-\lambda})\right)\,dt \\
&= \frac{\pi b}{2}\left[(e^{\lambda}+t)\log(e^{\lambda}+t) - (e^{-\lambda}+t)\log(e^{-\lambda}+t)\right]_{e^{-\alpha}}^{e^{\alpha}} \\
&= \pi b\left\{\lambda\sinh\alpha + \alpha\sinh\lambda + (\cosh\alpha+\cosh\lambda)\log\left(\frac{e^{\alpha}+e^{\lambda}}{1+e^{\alpha+\lambda}}\right)\right\}.
\end{aligned}$$

The desired result then follows on substituting the values for α and λ. □

Remark 2.7. The parameter b appearing in (2.33) is a fake parameter. Enforcing a substitution of $x \mapsto bx$ and scaling $r \mapsto rb$, $s \mapsto sb$, and $a \mapsto \frac{a}{b}$ one can see the minimal form is equivalent to its formulation when $b = 1$ is set in (2.33).

Remark 2.8. The special case $r = s$ gives

$$\int_0^\infty \arctan^2\left(\frac{2rx}{b^2+x^2}\right)dx = \pi\left\{2r\operatorname{arcsinh}\left(\frac{r}{b}\right) + \sqrt{r^2+b^2}\log\left(\frac{b^2}{r^2+b^2}\right)\right\}. \tag{2.36}$$

Here $r \in \mathbb{R}$ and $b > 0$. Again the parameter b is fake.

Example 2.5. A rather simple special case can be produced. It is to be found on setting $r = 3$ and $b = 4$ in (2.36)

$$\int_0^\infty \arctan^2\left(\frac{6x}{x^2+16}\right)dx = 2\pi\log\left(\frac{8192}{3125}\right). \tag{2.37}$$

Remark 2.9. For $b > 0$, on enforcing a substitution of $x \mapsto \frac{b^2}{x}$ in (2.34) one obtains the integral identity

$$\begin{aligned}
\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right)\arctan\left(\frac{2sx}{b^2+x^2}\right)dx = b^2\int_0^\infty &\arctan\left(\frac{2rx}{b^2+x^2}\right) \\
&\times\arctan\left(\frac{2sx}{b^2+x^2}\right)\frac{dx}{x^2}.
\end{aligned} \tag{2.38}$$

Here $r, s \in \mathbb{R}$. Once more the parameter b is fake.

Remark 2.10. Since

$$\lim_{b\to 0^+}\left[2r\operatorname{arcsinh}\left(\frac{r}{b}\right) + \sqrt{r^2+b^2}\log\left(\frac{b^2}{r^2+b^2}\right)\right] = 2r\log(2), \tag{2.39}$$

when $b \to 0^+$, (2.36) reduces to

$$\int_0^\infty \arctan^2\left(\frac{2r}{x}\right) dx = 2\pi r \log(2), \tag{2.40}$$

and is valid for $r \geqslant 0$. Enforcing a substitution of $x \mapsto \frac{2r}{x}$ one obtains

$$\int_0^\infty \left(\frac{\arctan x}{x}\right)^2 dx = \pi \log(2), \tag{2.41}$$

an integral that surprisingly does not appear in [**2**]. Differentiating (2.36) with respect to b gives

$$\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right) \frac{x}{x^4+2x^2(b^2+2r^2)+b^4}\, dx = \frac{\pi}{8r\sqrt{r^2+b^2}} \log\left(\frac{r^2+b^2}{b^2}\right), \tag{2.42}$$

valid for $r > 0$ and $b > 0$. Differentiating (2.36) with respect to r, combining the result with (2.42) produces

$$\begin{aligned}\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right) \frac{x^3}{x^4+2x^2(b^2+2r^2)+b^4}\, dx = {} & \frac{\pi(b^2+2r^2)}{8r\sqrt{r^2+b^2}} \log\left(\frac{b^2}{r^2+b^2}\right) \\ & + \frac{\pi}{2} \operatorname{arcsinh}\left(\frac{r}{b}\right),\end{aligned} \tag{2.43}$$

and is valid for $r > 0$ and $b > 0$. Finally, enforcing a substitution of $x \mapsto \frac{b^2}{x}$ in (2.43) produces

$$\begin{aligned}\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right) \frac{dx}{x\left(x^4+2x^2(b^2+2r^2)+b^4\right)} = {} & \frac{\pi}{2b^4}\left\{\operatorname{arcsinh}\left(\frac{r}{b}\right)\right. \\ & \left. + \frac{b^2+2r^2}{4r\sqrt{r^2+b^2}} \log\left(\frac{b^2}{b^2+r^2}\right)\right\},\end{aligned} \tag{2.44}$$

valid for $r > 0$ and $b > 0$. Note in all cases, the parameter b appearing in (2.42), (2.43), and (2.44) is fake.

In the next proposition the Mellin transform of $\arctan\left(\frac{2rx}{b^2+x^2}\right)$ is given.

PROPOSITION 2.5. *For $0 < |a| < 1$, $r \in \mathbb{R}$, and $b \in \mathbb{R}$ the identity*

$$\begin{aligned}\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right) x^{a-1}\, dx = {} & \frac{\pi}{2a} \sec\left(\frac{\pi a}{2}\right) \left\{\left(\sqrt{r^2+b^2}+r\right)^a\right. \\ & \left. - \left(\sqrt{r^2+b^2}-r\right)^a\right\},\end{aligned} \tag{2.45}$$

holds.

PROOF. Let $I(a,r,b)$ denote the left-hand side of (2.45). Enforcing a substitution of $x \mapsto bx$ gives

$$I(a,r,b) = b^a \int_0^\infty \arctan\left(\frac{2x \sinh \alpha}{1+x^2}\right) x^{a-1}\, dx, \tag{2.46}$$

where $\sinh\alpha = r/b$. Applying (1.3) to (2.46), a change in the order of integration produces

$$I(a,r,b) = b^a \int_{e^{-\alpha}}^{e^{\alpha}} \int_0^\infty \frac{x^a}{t^2+x^2}\,dx\,dt. \tag{2.47}$$

Enforcing a substitution of $x \mapsto t\sqrt{x}$ in (2.47) yields

$$I(a,r,b) = b^a \int_{e^{-\alpha}}^{e^{\alpha}} \frac{t^a}{2t} \int_0^\infty \frac{x^{\frac{a-1}{2}}}{1+x}\,dx\,dt. \tag{2.48}$$

The inner integral of (2.48) can be evaluated in terms of the beta function given in (2.3). Substituting $t = u/(1+u)$ into (2.3) before reverting the dummy variable u back to x we see that

$$\int_0^\infty \frac{x^{\frac{a-1}{2}}}{1+x}\,dx = \mathrm{B}\left(\frac{a+1}{2}, \frac{1-a}{2}\right) = \Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{1-a}{2}\right), \tag{2.49}$$

where identity (2.4) has been used. Euler's well-known reflexion formula for the gamma function

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)},$$

allows (2.49) to be rewritten as

$$\int_0^\infty \frac{x^{\frac{a-1}{2}}}{1+x}\,dx = \pi\sec\left(\frac{\pi a}{2}\right).$$

Thus (2.48) becomes

$$I(a,r,b) = \frac{\pi b^a}{2}\sec\left(\frac{\pi a}{2}\right)\int_{e^{-\alpha}}^{e^{\alpha}} t^{a-1}\,dt = \frac{\pi b^a}{a}\sec\left(\frac{\pi a}{2}\right)\sinh(a\alpha).$$

The desired result then follows on substituting for the value of α. This completes the proof. □

REMARK 2.11. The parameter b in (2.45) is a fake parameter. If $b = 0$, $r > 0$, and $0 < a < 1$ then (2.45) reduces to

$$\int_0^\infty \arctan\left(\frac{2r}{x}\right) x^{a-1}\,dx = \frac{\pi}{2a}\sec\left(\frac{\pi a}{2}\right)(2r)^a, \tag{2.50}$$

and represents the Mellin transform of $\arctan\left(\frac{2r}{x}\right)$. Setting $a = -(p+1)$ where $-2 < p < -1$, after enforcing a substitution of $x \mapsto 2r/x$ one obtains

$$\int_0^\infty x^p \arctan(x)\,dx = \frac{\pi}{2(p+1)}\operatorname{cosec}\left(\frac{p\pi}{2}\right). \tag{2.51}$$

This is entry **4.532.2** in [**2**].

NOTE 2.1. For the case when $a = 0$ in (2.45) this is given separately by (2.13).

We now give a proof for the Fourier sine transform given in (1.1).

PROOF. Denote the left-hand side of (1.1) by $I(a,r,b)$. Enforcing a substitution of $x \mapsto bx$ in (1.1) gives

$$I(a,r,b) = b \int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right)\sin(cx)\,dx, \tag{2.52}$$

where $c = ab$ and $\sinh\alpha = r/b$. Applying (1.3) to (2.52), a change in the order of integration produces

$$I(a,r,b) = b\int_{e^{-\alpha}}^{e^{\alpha}} \frac{x\sin(ax)}{t^2+x^2}\,dx\,dt. \tag{2.53}$$

The inner integral appears as entry **3.723.3** in [**2**]. Here

$$\int_0^\infty \frac{x\sin(cx)}{t^2+x^2}\,dx = \frac{\pi}{2}e^{-ct}, \quad c>0, t>0. \tag{2.54}$$

Thus (2.53) becomes

$$I(a,r,b) = \frac{\pi b}{2}\int_{e^{-\alpha}}^{e^{\alpha}} e^{-ct}\,dt = \frac{\pi}{2a}\left(e^{-ce^{-\alpha}} - e^{-ce^{\alpha}}\right). \tag{2.55}$$

The desired result then follows on substituting for c and α and rearranging algebraically the exponential terms that appear into a form corresponding to the definition for the hyperbolic sine function. This completes the proof. □

REMARK 2.12. If $b=0$, $a>0$, and $r>0$ then (1.1) reduces to

$$\int_0^\infty \arctan\left(\frac{2r}{x}\right)\sin(ax)\,dx = \frac{\pi}{a}e^{-ar}\sinh(ar). \tag{2.56}$$

This is entry **4.574.1** in [**2**] and represents the Fourier sine transform of $\arctan\left(\frac{2r}{x}\right)$.

REMARK 2.13. Differentiating (1.1) with respect to b gives

$$\int_0^\infty \frac{x\sin(ax)}{x^4+2x^2(b^2+2r^2)+b^4}\,dx = \frac{\pi}{4r\sqrt{r^2+b^2}}\exp\left(-a\sqrt{r^2+b^2}\right)\sinh(ar), \tag{2.57}$$

valid for $a>0$, $r>0$, and $b\in\mathbb{R}$ with b being a fake parameter. Setting $b=0$ in (2.57) the integral reduces to

$$\int_0^\infty \frac{\sin(ax)}{x(x^2+c^2)}\,dx = \frac{\pi}{2c^2}(1-e^{-ac}), \tag{2.58}$$

where we have set $c=2r$ and is valid for $a>0$ and $r>0$. This is entry **3.725.1** in [**2**]. Since

$$\lim_{r\to 0^+} \frac{\exp\left(-a\sqrt{r^2+b^2}\right)}{r\sqrt{r^2+b^2}} = \frac{ae^{-ab}}{b}, \tag{2.59}$$

and is valid for $a>0$ and $b>0$, as $r\to 0^+$ (2.57) reduces to

$$\int_0^\infty \frac{x\sin(ax)}{(x^2+b^2)^2}\,dx = \frac{\pi a}{4b}e^{-ab}, \tag{2.60}$$

and is valid for $a > 0$ and $b > 0$. This is entry **3.729.2** in [**2**]. Differentiating (1.1) with respect to r, combining the result with (2.57) produces

$$\int_0^\infty \frac{x^3 \sin(ax)}{x^4 + 2x^2(b^2 + 2r^2) + b^4} \, dx = \frac{\pi}{2} \exp\left(-a\sqrt{r^2 + b^2}\right) \cosh(ar) - \frac{\pi(b^2 + 2r^2)}{4r\sqrt{r^2 + b^2}} \exp\left(-a\sqrt{r^2 + b^2}\right) \sinh(ar), \tag{2.61}$$

valid for $a > 0$, $r > 0$, and $b \in \mathbb{R}$. Setting $b = 0$ in (2.61) the integral reduces to (2.54) with $t = 2r$ and $c = a$ which is entry **3.723.2** in [**2**]. Making use of the limit (2.59) we see that as $r \to 0^+$, (2.61) reduces to

$$\int_0^\infty \frac{x^3 \sin(ax)}{(x^2 + b^2)^2} \, dx = \frac{\pi}{4}(2 - ab)e^{-ab}, \tag{2.62}$$

and is valid for $a > 0$ and $b > 0$. This is entry **3.729.4** in [**2**].

3. Functions leading to special functions

In this section we give a number of definite integrals between the product of the inverse tangent function in (1.2) with a function f which on evaluation leads to special functions. We now give definitions of those special functions we are going to have a need for.

The *dilogarithm* is defined by

$$\operatorname{Li}_2(x) = -\int_0^x \frac{\log(1 - t)}{t} \, dt. \tag{3.1}$$

More generally, the *polylogarithm* is defined by

$$\operatorname{Li}_{s+1}(x) = \int_0^x \frac{\operatorname{Li}_s(t)}{t} \, dt, \tag{3.2}$$

where s is referred to as the order of the polylogarithm. Thus the dilogarithm ($s = 2$) is the integral of a function involving the logarithm, the trilogarithm ($s = 3$) the integral of a function involving the dilogarithm, and so forth. Further information about these functions can be found in [**4**].

The *exponential integral* function is defined by

$$\operatorname{Ei}(x) = \int_{-x}^\infty \frac{e^{-t}}{t} \, dt, \tag{3.3}$$

for $x < 0$. This appears as entry **8.211.1** in [**2**]. In the case $x > 0$ one uses the Cauchy principal value

$$\operatorname{Ei}(x) = -\lim_{\epsilon \to 0^+} \left[\int_{-x}^{\epsilon} \frac{e^{-t}}{t} \, dt + \int_{\epsilon}^\infty \frac{e^{-t}}{t} \, dt\right]. \tag{3.4}$$

This appears as entry **8.211.2** in [**2**]. The *sine integral* is defined by

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t} \, dt, \tag{3.5}$$

and the closely related *negative complementary sine integral* is defined by

$$\mathrm{si}(x) = -\int_x^\infty \frac{\sin t}{t}\,dt = -\frac{\pi}{2} + \mathrm{Si}(x). \tag{3.6}$$

These two special functions appear as entry **8.230.1** in [**2**]. The *cosine integral* is defined by

$$\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}\,dt. \tag{3.7}$$

This appears as entry **8.230.2** in [**2**].

PROPOSITION 3.1. *For $a > 0$, $b > 0$, and $r \in \mathbb{R}$ the identity*

$$\begin{aligned}\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right)\frac{\log(x^2+a^2)}{x}\,dx &= \pi\log(a)\,\mathrm{arcsinh}\left(\frac{r}{b}\right)\\ &+ \pi\,\mathrm{Li}_2\left(\frac{-\sqrt{r^2+b^2}+r}{a}\right) - \pi\,\mathrm{Li}_2\left(\frac{-\sqrt{r^2+b^2}-r}{a}\right),\end{aligned} \tag{3.8}$$

holds.

PROOF. Let $I(a,r,b)$ denote the left-hand side of (3.8). Enforcing a substitution of $x \mapsto bx$ gives

$$\begin{aligned}I(a,r,b) &= 2\log(b)\int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right)\frac{dx}{x}\\ &+ \int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right)\frac{\log(x^2+c^2)}{x}\,dx.\end{aligned} \tag{3.9}$$

Here $\sinh\alpha = r/\alpha$ and $c = a/b$. The first of the integrals in (3.9) is given by (2.13). For the second of the integrals, applying (1.3) to it, a change in the order of integration produces

$$I(a,r,b) = 2\pi\log(b)\,\mathrm{arcsinh}\left(\frac{r}{b}\right) + \int_{e^{-\alpha}}^{e^\alpha}\int_0^\infty \frac{\log(x^2+c^2)}{t^2+x^2}\,dx\,dt. \tag{3.10}$$

The inner integral appears as entry **4.295.22** in [**2**]. Here

$$\int_0^\infty \frac{\log(x^2+c^2)}{t^2+x^2}\,dx = \frac{\pi\log(t+c)}{t}, \quad c>0, t>0.$$

Thus (3.10) becomes

$$\begin{aligned}I(a,r,b) &= 2\log(b)\pi\,\mathrm{arcsinh}\left(\frac{r}{b}\right) + \pi\int_{e^{-\alpha}}^{e^\alpha}\frac{\log(c+t)}{t}\,dt\\ &= 2\log(b)\pi\,\mathrm{arcsinh}\left(\frac{r}{b}\right) + \pi\log(c)\int_{e^{-\alpha}}^{e^\alpha}\frac{dt}{t} + \pi\int_{e^{-\alpha}}^{e^\alpha}\log\left(1+\frac{t}{c}\right)\frac{dt}{t}.\end{aligned}$$

The latter integral that has appeared can be found in terms of the dilogarithm given in (3.1). The former integral is elementary. Thus

$$I(a,r,b) = 2\pi\log(b)\,\mathrm{arcsinh}\left(\frac{r}{b}\right) + \pi\left[2\alpha\log(c) + \mathrm{Li}_2\left(-\frac{e^{-\alpha}}{c}\right) - \mathrm{Li}_2\left(-\frac{e^{\alpha}}{c}\right)\right],$$

from which the desired result follows on substituting for the values of c and α. This completes the proof. □

REMARK 3.1. Differentiating (3.8) with respect to b gives

$$\int_0^\infty \frac{\log(x^2+a^2)}{x^4+2x^2(b^2+2r^2)+b^4}\,dx = \frac{\pi}{4rb^2}\left\{\log\left(\frac{a+\sqrt{r^2+b^2}-r}{a+\sqrt{r^2+b^2}+r}\right)\right.$$

$$\left. + \frac{r}{\sqrt{r^2+b^2}}\log\left(a^2+2a\sqrt{r^2+b^2}+b^2\right)\right\}, \tag{3.11}$$

and is valid for $a > 0$, $r > 0$, and $b > 0$. Noting that

$$\lim_{r\to 0^+}\frac{1}{r}\log\left(\frac{a+\sqrt{r^2+b^2}-r}{a+\sqrt{r^2+b^2}+r}\right) = -\frac{2}{a+b}, \quad a, b > 0, \tag{3.12}$$

as $r \to 0^+$, (3.11) becomes

$$\int_0^\infty \frac{\log(x^2+a^2)}{(x^2+b^2)^2} = \frac{\pi}{2b^3}\left(\log(a+b) - \frac{b}{a+b}\right), \tag{3.13}$$

and is valid for $a > 0$ and $b > 0$. This is entry **4.295.25** in [**2**] (with $b = g = 1$). Differentiating (3.8) with respect to r, combining the result with (3.11) produces

$$\int_0^\infty \frac{x^2\log(x^2+a^2)}{x^4+2x^2(b^2+2r^2)+b^4}\,dx = \frac{\pi}{4r}\left\{\frac{r}{\sqrt{r^2+b^2}}\log\left(a^2+2a\sqrt{r^2+b^2}+b^4\right)\right.$$

$$\left. - \log\left(\frac{a+\sqrt{r^2+b^2}-r}{a+\sqrt{r^2+b^2}+r}\right)\right\}, \tag{3.14}$$

valid for $a > 0$, $r > 0$, and $b > 0$. Finally as $r \to 0^+$, (3.14) reduces to

$$\int_0^\infty \frac{x^2\log(x^2+a^2)}{(x^2+b^2)^2}\,dx = \frac{\pi}{2b}\left(\log(a+b) + \frac{b}{a+b}\right), \tag{3.15}$$

where we have made use of the limit given in (3.12). The result is valid for $a > 0$ and $b > 0$. This is entry **4.295.26** in [**2**] (with $b = g = 1$).

PROPOSITION 3.2. *For $a > 0$, $b \neq 0$, and $r \in \mathbb{R}$ the identity*

$$\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right)\frac{\cos(ax)}{x}\,dx = \frac{\pi}{2}\left\{\mathrm{Ei}\left(-a\sqrt{r^2+b^2}-ar\right)\right.$$

$$\left. - \mathrm{Ei}\left(-a\sqrt{r^2+b^2}+ar\right)\right\} \tag{3.16}$$

holds.

PROOF. Let $I(a, r, b)$ denote the left-hand side of (3.16). Enforcing a substitution of $x \mapsto bx$ gives

$$I(a, r, b) = \int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right)\frac{\cos(cx)}{x}\,dx. \tag{3.17}$$

Here $c = ab$ and $\sinh\alpha = r/b$. Applying (1.3) to (3.17), a change in the order of integration produces

$$I(a,r,b) = \int_{e^{-\alpha}}^{e^{\alpha}} \int_0^\infty \frac{\cos(cx)}{t^2+x^2}\,dx\,dt. \tag{3.18}$$

The inner integral appears as entry **3.723.2** in [**2**]. Here

$$\int_0^\infty \frac{\cos(cx)}{t^2+x^2}\,dx = \frac{\pi}{2t}e^{-ct}, \quad c \geqslant 0, t > 0.$$

After making use of this result we are left with

$$I(a,r,b) = \frac{\pi}{2}\int_{e^{-\alpha}}^{e^{\alpha}} \frac{e^{-ct}}{t}\,dt.$$

The integral that has appeared can be found in terms of the exponential integral function given in (3.3). Thus

$$I(a,r,b) = \frac{\pi}{2}\left[\operatorname{Ei}(-ct)\right]_{e^{-\alpha}}^{e^{\alpha}} = \frac{\pi}{2}\left\{\operatorname{Ei}(-ce^{\alpha}) - \operatorname{Ei}(-ce^{-\alpha})\right\},$$

from which the desired result follows on substituting for the values of c and α. This competes the proof. ☐

Remark 3.2. The parameter b appearing in (3.16) is fake. Enforcing $x \mapsto bx$ and replacing r with rb and a with $\frac{a}{b}$ yields the minimal form

$$\begin{aligned}\int_0^\infty \arctan\left(\frac{2rx}{1+x^2}\right)\frac{\cos(ax)}{x}\,dx = \frac{\pi}{2}&\left\{\operatorname{Ei}\left(-a\sqrt{r^2+1}-ar\right)\right.\\ &\left.-\operatorname{Ei}\left(-a\sqrt{r^2+1}+ar\right)\right\}.\end{aligned} \tag{3.19}$$

In the next proposition we give the Laplace transform of $\arctan\left(\frac{2rx}{b^2+x^2}\right)$.

Proposition 3.3. *If $a > 0$, $r \in \mathbb{R}$, and $b > 0$ the identity*

$$\begin{aligned}\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right)e^{-ax}\,dx = \frac{1}{a}&\left\{\operatorname{si}\left(a\sqrt{r^2+b^2}+ar\right)\cos\left(a\sqrt{r^2+b^2}+ar\right)\right.\\ &-\operatorname{Ci}\left(a\sqrt{r^2+b^2}+ar\right)\sin\left(a\sqrt{r^2+b^2}+ar\right)\\ &-\operatorname{si}\left(a\sqrt{r^2+b^2}-ar\right)\cos\left(a\sqrt{r^2+b^2}-ar\right)\\ &\left.+\operatorname{Ci}\left(a\sqrt{r^2+b^2}-ar\right)\sin\left(a\sqrt{r^2+b^2}-ar\right)\right\}\end{aligned} \tag{3.20}$$

holds.

Proof. Let $I(a,r,b)$ denote the left-hand side of (3.20). Enforcing a substitution of $x \mapsto bx$ gives

$$I(a,r,b) = b\int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right)e^{-cx}\,dx, \tag{3.21}$$

where $c = ab$ and $\sinh\alpha = r/b$. On applying (1.3) to the integral, a change in the order of integration produces

$$I(a,r,b) = b\int_{e^{-\alpha}}^{e^{\alpha}}\int_0^\infty \frac{xe^{-cx}}{t^2+x^2}\,dx\,dt. \tag{3.22}$$

The inner integral appears as entry **3.354.2** in [**2**]. Here

$$\int_0^\infty \frac{xe^{-cx}}{t^2+x^2}\,dx = -\operatorname{Ci}(ct)\cos(ct) - \operatorname{si}(ct)\sin(ct), \quad t>0, c>0. \tag{3.23}$$

After making use of this result in (3.22), integrating by parts we find

$$\begin{aligned} I(a,r,b) &= -b\int_{e^{-\alpha}}^{e^{\alpha}} \left(\operatorname{Ci}(ct)\cos(ct) + \operatorname{si}(ct)\sin(ct)\right)\,dt \\ &= \frac{\pi}{c}\left[\operatorname{si}(ct)\cos(ct) - \operatorname{Ci}(ct)\sin(ct)\right]_{e^{-\alpha}}^{e^{\alpha}} \\ &= \frac{b}{c}\left(\operatorname{si}(ce^{\alpha})\cos(ce^{\alpha}) - \operatorname{Ci}(ce^{\alpha})\sin(ce^{\alpha})\right. \\ &\quad \left. - \operatorname{si}(ce^{-\alpha})\cos(ce^{-\alpha}) + \operatorname{Ci}(ce^{-\alpha})\sin(ce^{-\alpha})\right). \end{aligned}$$

The desired result then follows on substituting for the values of c and α. This completes the proof. ☐

Remark 3.3. The Laplace transform of $\arctan\left(\frac{2r}{x}\right)$ can also be found. Since

$$\lim_{b\to 0^+} \operatorname{si}\left(a\sqrt{r^2+b^2} - ar\right) = \lim_{b\to 0^+}\left[-\frac{\pi}{2} + \operatorname{Si}\left(a\sqrt{r^2+b^2} - ar\right)\right] = -\frac{\pi}{2}, \tag{3.24}$$

and

$$\lim_{b\to 0^+} \operatorname{Ci}\left(a\sqrt{r^2+b^2} - ar\right)\sin\left(a\sqrt{r^2+b^2} - ar\right) = 0, \tag{3.25}$$

when $b \to 0^+$, (3.20) reduces to

$$\int_0^\infty \arctan\left(\frac{2r}{x}\right)e^{-ax}\,dx = \frac{1}{a}\left\{\operatorname{si}(2ar)\cos(2ar) - \operatorname{Ci}(2ar)\sin(2ar) + \frac{\pi}{2}\right\}, \tag{3.26}$$

valid for $a > 0$ and $r > 0$. Differentiating (3.26) with respect to r yields (3.23) with $t = 2r > 0$ and is entry **3.354.2** in [**2**].

In the next proposition we give the Fourier cosine transform of $\arctan\left(\frac{2rx}{b^2+x^2}\right)$.

Proposition 3.4. *If $a > 0$, $r \in \mathbb{R}$, and $b > 0$ the identity*

$$\begin{aligned} \int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right)\cos(ax)\,dx = -\frac{1}{2a}&\left\{\exp\left(a\sqrt{r^2+b^2}+ar\right)\operatorname{Ei}\left(-a\sqrt{r^2+b^2}-ar\right)\right. \\ &- \exp\left(-a\sqrt{r^2+b^2}-ar\right)\operatorname{Ei}\left(a\sqrt{r^2+b^2}+ar\right) \\ &- \exp\left(a\sqrt{r^2+b^2}-ar\right)\operatorname{Ei}\left(-a\sqrt{r^2+b^2}+ar\right) \\ &\left.+ \exp\left(-a\sqrt{r^2+b^2}+ar\right)\operatorname{Ei}\left(a\sqrt{r^2+b^2}-ar\right)\right\}, \end{aligned} \tag{3.27}$$

holds.

PROOF. Let $I(a,r,b)$ denote the left-hand side of (3.27). Enforcing a substitution of $x \mapsto bx$ gives

$$I(a,r,b) = b\int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right)\cos(cx)\,dx, \tag{3.28}$$

where $c = ab$ and $\sinh\alpha = r/b$. On applying (1.3) to the integral, a change in the order of integration produces

$$I(a,r,b) = b\int_{e^{-\alpha}}^{e^\alpha}\int_0^\infty \frac{x\cos(cx)}{t^2+x^2}\,dx\,dt. \tag{3.29}$$

The inner integral appears as entry **3.723.5** in [**2**]. Here

$$\int_0^\infty \frac{x\cos(cx)}{t^2+x^2}\,dx = -\frac{1}{2}e^{-ct}\operatorname{Ei}(ct) - \frac{1}{2}e^{ct}\operatorname{Ei}(-ct), \quad c>0, t>0. \tag{3.30}$$

Making use of this result, the integral in (3.29) becomes

$$\begin{aligned} I(a,r,b) &= -\frac{b}{2}\int_{e^{-\alpha}}^{e^\alpha}\left(e^{-ct}\operatorname{Ei}(ct) + e^{ct}\operatorname{Ei}(-ct)\right)\,dt = -\frac{b}{2c}\left[e^{ct}\operatorname{Ei}(-ct) - e^{-ct}\operatorname{Ei}(ct)\right]_{e^{-\alpha}}^{e^\alpha} \\ &= -\frac{b}{2c}\left\{e^{ce^\alpha}\operatorname{Ei}(-ce^\alpha) - e^{-ce^\alpha}\operatorname{Ei}(ce^\alpha) - e^{ce^{-\alpha}}\operatorname{Ei}(-ce^{-\alpha}) + e^{-ce^{-\alpha}}\operatorname{Ei}(ce^{-\alpha}\right\}. \end{aligned}$$

The desired result then follows on substituting for the values of c and α. This completes the proof. ☐

REMARK 3.4. The Fourier cosine transform of $\arctan\left(\frac{2r}{x}\right)$ can also be found. For $a>0$ and $r>0$, since

$$\begin{aligned} \lim_{b\to 0^+}\Big\{&\exp\left(-a\sqrt{r^2+b^2}+ar\right)\operatorname{Ei}\left(a\sqrt{r^2+b^2}-ar\right) \\ &-\exp\left(a\sqrt{r^2+b^2}-ar\right)\operatorname{Ei}\left(-a\sqrt{r^2+b^2}+ar\right)\Big\} = 0, \end{aligned} \tag{3.31}$$

as $b \to 0^+$, (3.27) reduces to

$$\int_0^\infty \arctan\left(\frac{2r}{x}\right)\cos(ax)\,dx = \frac{1}{2a}\left\{e^{-2ar}\operatorname{Ei}(2ar) - e^{2ar}\operatorname{Ei}(-2ar)\right\}. \tag{3.32}$$

This is entry **1.8.7** in [**1**]. Differentiating (3.32) with respect to r yields (3.30) with $t = 2r > 0$ and is entry **3.723.5** in [**2**].

PROPOSITION 3.5. *If $a>0$, $r\in\mathbb{R}$, and $b>0$ the identity*

$$\begin{aligned} \int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right)\frac{dx}{e^{ax}-1} &= \operatorname{arcsinh}\left(\frac{r}{b}\right)\left(\sqrt{r^2+b^2}-\frac{\pi}{a}\right) \\ &+ r\left(\log\left(\frac{ab}{2\pi}\right)-1\right) + \frac{\pi}{a}\log\left\{\frac{\Gamma\left(\frac{a\sqrt{r^2+b^2}-ar}{2\pi}\right)}{\Gamma\left(\frac{a\sqrt{r^2+b^2}+ar}{2\pi}\right)}\right\}, \end{aligned} \tag{3.33}$$

holds.

Proof. Let $I(a,r,b)$ denote the left-hand side of (3.33). Enforcing a substitution of $x \mapsto bx$ gives

$$I(a,r,b) = b\int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right)\frac{dx}{e^{cx}-1}, \tag{3.34}$$

where $c = ab$ and $\sinh\alpha = r/b$. On applying (1.3) to the integral, a change in the order of integration produces

$$I(a,r,b) = b\int_{e^{-\alpha}}^{e^{\alpha}}\int_0^\infty \frac{x}{(t^2+x^2)(e^{cx}-1)}\,dx\,dt. \tag{3.35}$$

The inner integral appears as entry **3.415.1** in [**2**]. Here

$$\int_0^\infty \frac{x}{(t^2+x^2)(e^{cx}-1)}\,dx = \frac{1}{2}\left[\log\left(\frac{ct}{2\pi}\right) - \frac{\pi}{ct} - \psi\left(\frac{ct}{2\pi}\right)\right], \quad c>0, t>0. \tag{3.36}$$

Making use of this result, the integral in (3.35) becomes

$$I(a,r,b) = \frac{b}{2}\int_{e^{-\alpha}}^{e^{\alpha}}\left[\log\left(\frac{ct}{2\pi}\right) - \frac{\pi}{ct} - \psi\left(\frac{ct}{2\pi}\right)\right]dt. \tag{3.37}$$

The integral of the first two terms in the integrand of (3.37) are elementary. The integral of the third term comes from (2.1). The result is

$$I(a,r,b) = \frac{\pi}{2}\left[t\log\left(\frac{ct}{2\pi}\right) - t - \frac{\pi}{c}\log(t) - \frac{2\pi}{c}\log\Gamma\left(\frac{ct}{2\pi}\right)\right]_{e^{-\alpha}}^{e^{\alpha}} \tag{3.38}$$

Substituting in the upper and lower limits of integration followed by substituting for the values of c and α, the desired result then follows and completes the proof. □

Remark 3.5. It is easy to see the parameter b appearing in (3.33) is a fake parameter.

Example 3.1. A very curious special case is

$$\int_0^\infty \arctan\left(\frac{6x}{16+x^2}\right)\frac{dx}{e^{\pi x}-1} = \log\left(\frac{64}{3}\right) - 3, \tag{3.39}$$

and is found on setting $a = \pi$, $r = 3$, and $b = 4$ in (3.33).

Definite integrals where the function f in (1.2) contains the square of the sine function and the square of the cosine function will now be given.

Proposition 3.6. *If $a > 0$, $r \in \mathbb{R}$, and $b > 0$ the identity*

$$\begin{aligned}\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right)\frac{\sin^2(ax)}{x}\,dx = \frac{\pi}{4}\Big\{&2\operatorname{arcsinh}\left(\frac{r}{b}\right) + \operatorname{Ei}\left(-2a\sqrt{r^2+b^2}+2ar\right)\\ &- \operatorname{Ei}\left(-2a\sqrt{r^2+b^2}-2ar\right)\Big\},\end{aligned} \tag{3.40}$$

holds.

PROOF. Let $I(a,r,b)$ denote the left-hand side of (3.40). Enforcing a substitution of $x \mapsto bx$ gives

$$I(a,r,b) = \int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right)\frac{\sin^2(cx)}{x}\,dx, \tag{3.41}$$

where $c = ab$ and $\sinh\alpha = r/b$. On applying (1.3) to the integral, a change in the order of integration produces

$$I(a,r,b) = \int_{e^{-\alpha}}^{e^\alpha}\int_0^\infty \frac{\sin^2(cx)}{t^2+x^2}\,dx\,dt. \tag{3.42}$$

The inner integral appears as entry **3.742.1** or entry **3.824.1** in [**2**]. Here

$$\int_0^\infty \frac{\sin^2(cx)}{t^2+x^2}\,dx = \frac{\pi}{4t}(1-e^{-2ct}), \quad c>0, t>0. \tag{3.43}$$

After making use of this result in the integral in (3.42) one has

$$\begin{aligned} I(a,r,b) &= \frac{\pi}{4}\int_{e^{-\alpha}}^{e^\alpha}\frac{1-e^{-2ct}}{t}\,dt = \frac{\pi}{4}\left[\log(t) - \operatorname{Ei}(-2ct)\right]_{e^{-\alpha}}^{e^\alpha} \\ &= \frac{\pi}{4}\left(2\alpha + \operatorname{Ei}(-2ce^{-\alpha}) - \operatorname{Ei}(-2e^{\alpha})\right). \end{aligned} \tag{3.44}$$

In the first line on the right the integral definition for the exponential integral function given in (3.3) has been used. The desired result then follows on substituting for the values of c and α. This completes the proof. ☐

REMARK 3.6. The parameter b appearing in (3.40) is fake. Enforcing a substitution of $x \mapsto bx$ and simple scalings of $r \mapsto rb$ and $a \mapsto \frac{a}{b}$ we arrive at the minimal form for (3.40) of

$$\begin{aligned} \int_0^\infty \arctan\left(\frac{2rx}{1+x^2}\right)\frac{\sin^2(ax)}{x}\,dx = \frac{\pi}{4}\Big\{&2\operatorname{arcsinh}(r) + \operatorname{Ei}\left(-2a\sqrt{r^2+1}+2ar\right) \\ &- \operatorname{Ei}\left(-2a\sqrt{r^2+1}-2ar\right)\Big\}. \end{aligned} \tag{3.45}$$

PROPOSITION 3.7. *If $a>0$, $r \in \mathbb{R}$, and $b>0$ the identity*

$$\begin{aligned} \int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right)\frac{\cos^2(ax)}{x}\,dx = \frac{\pi}{4}\Big\{&2\operatorname{arcsinh}\left(\frac{r}{b}\right) + \operatorname{Ei}\left(-2a\sqrt{r^2+b^2}-2ar\right) \\ &- \operatorname{Ei}\left(-2a\sqrt{r^2+b^2}+2ar\right)\Big\}, \end{aligned} \tag{3.46}$$

holds.

PROOF. Let $I(a,r,b)$ denote the left-hand side of (3.46). Enforcing a substitution of $x \mapsto bx$ gives

$$I(a,r,b) = \int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right)\frac{\cos^2(cx)}{x}\,dx, \tag{3.47}$$

where $c = ab$ and $\sinh\alpha = r/b$. On applying (1.3) to the integral, a change in the order of integration produces

$$I(a,r,b) = \int_{e^{-\alpha}}^{e^{\alpha}} \int_0^\infty \frac{\cos^2(cx)}{t^2+x^2}\, dx\, dt. \tag{3.48}$$

The inner integral appears as entry **3.742.3** or entry **3.824.2** in [**2**]. Here

$$\int_0^\infty \frac{\cos^2(cx)}{t^2+x^2}\, dx = \frac{\pi}{4t}(1+e^{-2ct}), \quad c>0, t>0. \tag{3.49}$$

After making use of this result in the integral in (3.48) one has

$$\begin{aligned} I(a,r,b) &= \frac{\pi}{4}\int_{e^{-\alpha}}^{e^{\alpha}} \frac{1+e^{-2ct}}{t}\, dt = \frac{\pi}{4}\left[\log(t) + \mathrm{Ei}(-2ct)\right]_{e^{-\alpha}}^{e^{\alpha}} \\ &= \frac{\pi}{4}\left(2\alpha + \mathrm{Ei}(-2ce^{\alpha}) - \mathrm{Ei}(-2e^{-\alpha})\right). \end{aligned} \tag{3.50}$$

In the first line on the right the integral definition for the exponential integral function given in (3.3) has been used. The desired result then follows on substituting for the values of c and α. This completes the proof. □

REMARK 3.7. If (3.46) is added to (3.40), from the Pythagorean identity $\cos^2(ax)+\sin^2(ax) = 1$ one immediately obtains (2.13). On the other hand if (3.40) is subtracted from (3.46), from the double angle formula $\cos^2(ax)-\sin^2(ax) = \cos(2ax)$ one obtains (3.16) after a shift of $a \mapsto \frac{a}{2}$ is made.

REMARK 3.8. As was the case for (3.40), the parameter b appearing in (3.46) is also fake. In a similar fashion in what was done to arrive at minimal form (3.45), we find the minimal form for (3.46) is

$$\begin{aligned} \int_0^\infty \arctan\left(\frac{2rx}{1+x^2}\right)\frac{\cos^2(ax)}{x}\, dx = \frac{\pi}{4}\Big\{ & 2\,\mathrm{arcsinh}(r) + \mathrm{Ei}\left(-2a\sqrt{r^2+1} - 2ar\right) \\ & - \mathrm{Ei}\left(-2a\sqrt{r^2+1} + 2ar\right)\Big\}. \end{aligned} \tag{3.51}$$

Definite integrals where the function f in (1.2) contains the product between the sine and cosine functions with unequal arguments will now be given.

PROPOSITION 3.8. *If* $0 < a < d$, $r \in \mathbb{R}$, *and* $b > 0$ *the identity*

$$\begin{aligned} \int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right)\frac{\sin(ax)\sin(dx)}{x}\, dx = \frac{\pi}{4}\Big\{ & \mathrm{Ei}\left[(a-d)\left(\sqrt{r^2+b^2}+r\right)\right] \\ & - \mathrm{Ei}\left[-(a+d)\left(\sqrt{r^2+b^2}+r\right)\right] \\ & - \mathrm{Ei}\left[(a-d)\left(\sqrt{r^2+b^2}-r\right)\right] \\ & + \mathrm{Ei}\left[-(a+d)\left(\sqrt{r^2+b^2}-r\right)\right]\Big\}, \end{aligned} \tag{3.52}$$

holds.

PROOF. Let $I(a,b,r,d)$ denote the left-hand side of (3.52). Enforcing a substitution of $x \mapsto bx$ gives

$$I(a,b,r,d) = \int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right)\frac{\sin(cx)\sin(\mu x)}{x}\,dx, \tag{3.53}$$

where $c = ab$, $\mu = db$, and $\sinh\alpha = r/b$. On applying (1.3) to the integral, a change in the order of integration produces

$$I(a,b,r,d) = \int_{e^{-\alpha}}^{e^\alpha}\int_0^\infty \frac{\sin(cx)\sin(\mu x)}{t^2+x^2}\,dx\,dt. \tag{3.54}$$

The inner integral appears as entry **3.742.1** in [**2**]. Here

$$\int_0^\infty \frac{\sin(cx)\sin(\mu x)}{t^2+x^2}\,dx = \frac{\pi}{2t}e^{-\mu t}\sinh(ct), \quad 0 < c < \mu, t > 0. \tag{3.55}$$

After making use of this result in the integral (3.54) one has

$$\begin{aligned} I(a,b,r,d) &= \frac{\pi}{2}\int_{e^{-\alpha}}^{e^\alpha}\frac{e^{-\mu t}\sinh(ct)}{t}\,dt = \frac{\pi}{4}\int_{e^{-\alpha}}^{e^\alpha}\frac{e^{(c-\mu)t}-e^{-(c+\mu)t}}{t}\,dt \\ &= \frac{\pi}{4}\left[\mathrm{Ei}((c-\mu)t) - \mathrm{Ei}(-(c+\mu)t)\right]_{e^{-\alpha}}^{e^\alpha} \\ &= \frac{\pi}{4}\left\{\mathrm{Ei}((c-\mu)e^{-\alpha}) - \mathrm{Ei}(-(c+\mu)e^{\alpha}) - \mathrm{Ei}((c-\mu)e^{-\alpha})\right. \\ &\qquad \left. + \mathrm{Ei}(-(c+\mu)e^{-\alpha})\right\}. \end{aligned}$$

The desired result then follows on substituting for the values of c, μ, and α. This completes the proof. □

REMARK 3.9. Since

$$\lim_{b\to 0^+}\left\{\mathrm{Ei}\left[-(a+d)(\sqrt{r^2+b^2}-r)\right] - \mathrm{Ei}\left[(a-d)(\sqrt{r^2+b^2}-r)\right]\right\} = \log\left(\frac{d+a}{d-a}\right) \tag{3.56}$$

which is valid for $r > 0$ and $0 < a < d$, as $b \to 0^+$ (3.52) reduces to

$$\begin{aligned} \int_0^\infty \arctan\left(\frac{2r}{x}\right)\frac{\sin(ax)\sin(dx)}{x}\,dx &= \frac{\pi}{4}\left\{\mathrm{Ei}[2r(a-d)] - \mathrm{Ei}[-2r(a+d)]\right. \\ &\qquad \left. + \log\left(\frac{d+a}{d-a}\right)\right\}, \end{aligned} \tag{3.57}$$

and is valid for $r > 0$ and $0 < a < d$. Differentiating (3.57) with respect to r yields (3.55) with $t = 2r > 0$ which in entry **3.742.1** in [**2**].

Proposition 3.9. *If $0 < a < d$, $r \in \mathbb{R}$, and $b > 0$ the identity*

$$\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right)\frac{\cos(ax)\cos(dx)}{x}\,dx = \frac{\pi}{4}\left\{\mathrm{Ei}\left[(a-d)\left(\sqrt{r^2+b^2}+r\right)\right]\right.$$
$$+\,\mathrm{Ei}\left[-(a+d)\left(\sqrt{r^2+b^2}+r\right)\right]$$
$$-\,\mathrm{Ei}\left[(a-d)\left(\sqrt{r^2+b^2}-r\right)\right]$$
$$\left.-\,\mathrm{Ei}\left[-(a+d)\left(\sqrt{r^2+b^2}-r\right)\right]\right\}, \tag{3.58}$$

holds.

Proof. Let $I(a,b,r,d)$ denote the left-hand side of (3.58). Enforcing a substitution of $x \mapsto bx$ gives

$$I(a,b,r,d) = \int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right)\frac{\cos(cx)\cos(\mu x)}{x}\,dx, \tag{3.59}$$

where $c = ab$, $\mu = db$, and $\sinh\alpha = r/b$. On applying (1.3) to the integral, a change in the order of integration produces

$$I(a,b,r,d) = \int_{e^{-\alpha}}^{e^\alpha}\int_0^\infty \frac{\cos(cx)\cos(\mu x)}{t^2+x^2}\,dx\,dt. \tag{3.60}$$

The inner integral appears as entry **3.742.3** in [**2**]. Here

$$\int_0^\infty \frac{\cos(cx)\cos(\mu x)}{t^2+x^2}\,dx = \frac{\pi}{2t}e^{-\mu t}\cosh(ct), \quad 0 < c < \mu, t > 0. \tag{3.61}$$

After making use of this result, the integral in (3.60) becomes

$$\begin{aligned} I(a,b,r,d) &= \frac{\pi}{2}\int_{e^{-\alpha}}^{e^\alpha}\frac{e^{-\mu t}\cosh(ct)}{t}\,dt = \frac{\pi}{4}\int_{e^{-\alpha}}^{e^\alpha}\frac{e^{(c-\mu)t}+e^{-(c+\mu)t}}{t}\,dt \\ &= \frac{\pi}{4}\left[\mathrm{Ei}((c-\mu)t)+\mathrm{Ei}(-(c+\mu)t)\right]_{e^{-\alpha}}^{e^\alpha} \\ &= \frac{\pi}{4}\left\{\mathrm{Ei}((c-\mu)e^{-\alpha})+\mathrm{Ei}(-(c+\mu)e^{\alpha})-\mathrm{Ei}((c-\mu)e^{-\alpha})\right. \\ &\quad \left.-\,\mathrm{Ei}(-(c+\mu)e^{-\alpha})\right\}. \end{aligned}$$

The desired result then follows on substituting for the values of c, μ, and α. This completes the proof. □

Proposition 3.10. *If $0 < a < d$, $r \in \mathbb{R}$ and $b > 0$ the identity*

$$\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right)\frac{\sin(ax)\cos(dx)}{x}\,dx = \frac{\pi}{2(d-a)(d+a)}\left\{\exp\left(-d\sqrt{r^2+b^2}-dr\right)\right.$$
$$\times\left[a\cosh\left(a\sqrt{r^2+b^2}+ar\right)+d\sinh\left(a\sqrt{r^2+b^2}+ar\right)\right]$$
$$-\exp\left(-d\sqrt{r^2+b^2}+dr\right)$$
$$\left.\times\left[a\cosh\left(a\sqrt{r^2+b^2}-ar\right)+d\sinh\left(a\sqrt{r^2+b^2}-ar\right)\right]\right\}, \tag{3.62}$$

holds and if $0 < d < a$, $r \in \mathbb{R}$ *and* $b > 0$ *the identity*

$$\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right)\frac{\sin(ax)\cos(dx)}{x}dx = \frac{\pi}{2(a-d)(d+a)}\Big\{\exp\Big(-a\sqrt{r^2+b^2}-ar\Big) \times \Big[a\cosh\Big(d\sqrt{r^2+b^2}+dr\Big)+d\sinh\Big(d\sqrt{r^2+b^2}+dr\Big)\Big] - \exp\Big(-a\sqrt{r^2+b^2}+ar\Big) \times \Big[a\cosh\Big(d\sqrt{r^2+b^2}-dr\Big)+d\sinh\Big(d\sqrt{r^2+b^2}-dr\Big)\Big]\Big\}, \tag{3.63}$$

holds.

PROOF. Let $I(a, b, r, d)$ denote the left-hand side of the integral appearing in proposition 3.10. Enforcing a substitution of $x \mapsto bx$ gives

$$I(a, b, r, d) = b\int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right)\sin(cx)\cos(\mu x)\,dx, \tag{3.64}$$

where $c = ab$, $\mu = db$, and $\sinh\alpha = r/b$. On applying (1.3) to the integral, a change in the order of integration produces

$$I(a, b, r, d) = b\int_{e^{-\alpha}}^{e^\alpha}\int_0^\infty \frac{x\sin(cx)\cos(\mu x)}{t^2+x^2}\,dx\,dt. \tag{3.65}$$

The inner integral appears as entry **3.742.5** in [**2**]. Here

$$\int_0^\infty \frac{x\sin(cx)\cos(\mu x)}{t^2+x^2}\,dx = \begin{cases} -\dfrac{\pi}{2}e^{-\mu t}\sinh(ct) & 0 < c < \mu, t > 0 \\ \dfrac{\pi}{2}e^{-ct}\cosh(\mu t) & 0 < \mu < c, t > 0. \end{cases} \tag{3.66}$$

After making use of this result in the integral in (3.65) the resulting t-integration is elementary with the desired result readily following. This completes the proof. □

REMARK 3.10. The parameter b found in the integrals (3.52), (3.58), (3.62), and (3.63) is fake. Enforcing a substitution of $x \mapsto bx$ and applying simple scalings of $r \mapsto rb$, $a \mapsto \frac{a}{b}$, and $d \mapsto \frac{d}{b}$ give minimal forms equivalent to formulations obtained on setting $b = 1$ in each of these four integrals.

PROPOSITION 3.11. *If* $a > 0$, $0 < d < 2$, $r \in \mathbb{R}$, *and* $b \geqslant 0$ *the identity*

$$\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right)x^{d-2}\sin\left(ax-\frac{d\pi}{2}\right)dx = \frac{\pi}{2a^{d-1}}\Big\{\Gamma\left(d-1, a\sqrt{r^2+b^2}+ar\right) - \Gamma\left(d-1, a\sqrt{r^2+b^2}-ar\right)\Big\}, \tag{3.67}$$

holds.

PROOF. Let $I(a, d, r, b)$ denote the left-hand side of (3.67). Enforcing a substitution of $x \mapsto bx$ gives

$$I(a, d, r, b) = b^{d-1}\int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right)x^{d-2}\sin\left(cx-\frac{d\pi}{2}\right)dx, \tag{3.68}$$

where $c = ab$ and $\sinh \alpha = r/b$. On applying (1.3) to the integral, a change in the order of integration produces

$$I(a,d,r,b) = b^{d-1} \int_{e^{-\alpha}}^{e^{\alpha}} \int_0^\infty \frac{x^{d-1} \sin\left(cx - \frac{d\pi}{2}\right)}{t^2 + x^2} \, dx \, dt. \tag{3.69}$$

The inner integral appears as entry **3.767.1** in [**2**]. Here

$$\int_0^\infty \frac{x^{d-1} \sin\left(cx - \frac{d\pi}{2}\right)}{t^2 + x^2} \, dx = -\frac{\pi}{2} t^{d-2} e^{-ct}, \quad 0 < d < 2, c > 0, t > 0. \tag{3.70}$$

Making use of this result, the integral given in (3.69) becomes

$$\begin{aligned} I(a,d,r,b) &= -\frac{\pi}{2} b^{d-1} \int_{e^{-\alpha}}^{e^{\alpha}} t^{d-2} e^{-ct} \, dt = -\frac{\pi}{2} b^{d-1} \left[-\frac{1}{c^{d-1}} \Gamma(d-1, ct) \right]_{e^{-\alpha}}^{e^{\alpha}} \\ &= \frac{\pi}{2} \left(\frac{b}{c} \right)^{d-1} \left(\Gamma(d-1, ce^{\alpha}) - \Gamma(d-1, ce^{-\alpha}) \right). \end{aligned}$$

Here the integral definition for the upper incomplete gamma function given in (2.5) has been used. The desired result then follows on substituting for the values of c and α. This completes the proof. ☐

REMARK 3.11. The parameter b appearing in (3.67) is a fake parameter.

EXAMPLE 3.2. Setting $a = 1$, $d = \frac{3}{2}$, $b = 3$, and $r = 4$ in (3.67) we obtain the remarkable result of

$$\begin{aligned} \int_0^\infty \arctan\left(\frac{8x}{9 + x^2} \right) \sin\left(x + \frac{\pi}{4} \right) \frac{dx}{\sqrt{x}} &= \frac{\pi}{2} \left\{ \Gamma\left(\frac{1}{2}, 9 \right) - \Gamma\left(\frac{1}{2}, 1 \right) \right\} \\ &= \frac{\pi\sqrt{\pi}}{2} \left(\operatorname{erf}(1) - \operatorname{erf}(3) \right). \end{aligned} \tag{3.71}$$

Note the relation between the upper incomplete gamma function and the error function given in (2.7) has been used here.

REMARK 3.12. Setting $b = 0$ in (3.67) produces

$$\int_0^\infty \arctan\left(\frac{2r}{x} \right) x^{d-2} \sin\left(ax - \frac{d\pi}{2} \right) dx = \frac{\pi}{2a^{d-1}} \left(\Gamma(d-1, 2ar) - \Gamma(d-1) \right), \tag{3.72}$$

and is valid for $a > 0$, $r > 0$, and $0 < d < 2$. Setting $a = 1$, $r = 2$, and $d = \frac{3}{2}$ gives the more remarkable special case of

$$\int_0^\infty \arctan\left(\frac{4}{x} \right) \sin\left(x + \frac{\pi}{4} \right) \frac{dx}{\sqrt{x}} = \frac{\pi\sqrt{\pi}}{2} \operatorname{erf}(2). \tag{3.73}$$

Differentiating (3.72) with respect to r yields (3.70) with $t = 2r > 0$ and is entry **3.767.1** in [**2**].

Proposition 3.12. *If $a > 0$, $-1 < d < 1$, $r \in \mathbb{R}$, and $b \geqslant 0$ the identity*

$$\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right) x^{d-1} \cos\left(ax - \frac{d\pi}{2}\right) dx = \frac{\pi}{2a^d}\left\{\Gamma\left(d, a\sqrt{r^2+b^2} - ar\right)\right.$$
$$\left. -\Gamma\left(d, a\sqrt{r^2+b^2} + ar\right)\right\}, \tag{3.74}$$

holds.

Proof. Let $I(a,d,r,b)$ denote the left-hand side of (3.74). Enforcing a substitution of $x \mapsto bx$ gives

$$I(a,d,r,b) = b^d \int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right) x^{d-1} \cos\left(cx - \frac{d\pi}{2}\right) dx, \tag{3.75}$$

where $c = ab$ and $\sinh\alpha = r/b$. On applying (1.3) to the integral, a change in the order of integration produces

$$I(a,d,r,b) = b^d \int_{e^{-\alpha}}^{e^\alpha} \int_0^\infty \frac{x^d \cos\left(cx - \frac{d\pi}{2}\right)}{t^2+x^2}\, dx\, dt. \tag{3.76}$$

The inner integral appears as entry **3.767.2** in [**2**]. Here

$$\int_0^\infty \frac{x^d \cos\left(cx - \frac{d\pi}{2}\right)}{t^2+x^2}\, dx = \frac{\pi}{2} t^{d-1} e^{-ct}, \quad -1 < d < 1, c > 0, t > 0. \tag{3.77}$$

Making use of this result, the integral given in (3.76) becomes

$$I(a,d,r,b) = \frac{\pi}{2} b^d \int_{e^{-\alpha}}^{e^\alpha} t^{d-1} e^{-ct}\, dt = \frac{\pi}{2} b^d \left[-\frac{1}{c^d}\Gamma(d,ct)\right]_{e^{-\alpha}}^{e^\alpha}$$
$$= \frac{\pi}{2}\left(\frac{b}{c}\right)^d \left(\Gamma(d, ce^{-\alpha}) - \Gamma(d, ce^{\alpha})\right).$$

Here the integral definition for the upper incomplete gamma function given in (2.5) has been used. The desired result then follows on substituting for the values of c and α. This completes the proof. □

Remark 3.13. The parameter b appearing in (3.74) is a fake parameter.

Remark 3.14. Note that setting $a = 1$, $d = \frac{1}{2}$, $b = 3$, and $r = 4$ in (3.74) leads to the same remarkable result that was given in Example 3.2.

Remark 3.15. Setting $b = 0$ in (3.74) produces

$$\int_0^\infty \arctan\left(\frac{2r}{x}\right) x^{d-1} \cos\left(ax - \frac{d\pi}{2}\right) dx = \frac{\pi}{2a^d}\left(\Gamma(d) - \Gamma(d, 2ar)\right), \tag{3.78}$$

and is valid for $a > 0$, $r > 0$, and $-1 < d < 1$. Differentiating (3.78) with respect to r yields (3.77) with $t = 2r > 0$ and is entry **3.767.2** in [**2**].

Proposition 3.13. *If $a > 0$, $r \in \mathbb{R}$, and $b > 0$ the identity*

$$\int_0^\infty \arctan\left(\frac{2rx}{b^2+x^2}\right)\operatorname{si}(ax)\,dx = \frac{\pi}{2}\left\{\left(\sqrt{r^2+b^2}+r\right)\operatorname{Ei}\left(-a\sqrt{r^2+b^2}-ar\right)\right.$$
$$-\left(\sqrt{r^2+b^2}-r\right)\operatorname{Ei}\left(-a\sqrt{r^2+b^2}+ar\right)$$
$$\left.-\frac{2}{a}\exp\left(-a\sqrt{r^2+b^2}\right)\sinh(ar)\right\}, \tag{3.79}$$

holds.

Proof. Let $I(a,r,b)$ denote the left-hand side of (3.79). Enforcing a substitution of $x \mapsto bx$ gives

$$I(a,r,b) = b\int_0^\infty \arctan\left(\frac{2x\sinh\alpha}{1+x^2}\right)\operatorname{si}(cx)\,dx, \tag{3.80}$$

where $c = ab$ and $\sinh\alpha = r/b$. On applying (1.3) to the integral, a change in the order of integration produces

$$I(a,r,b) = b\int_{e^{-\alpha}}^{e^\alpha}\int_0^\infty \frac{x\operatorname{si}(cx)}{t^2+x^2}\,dx\,dt. \tag{3.81}$$

The inner integral appears as entry **6.244.1** in [**2**]. Here

$$\int_0^\infty \frac{x\operatorname{si}(cx)}{t^2+x^2}\,dx = \frac{\pi}{2}\operatorname{Ei}(-ct), \quad c>0, t>0. \tag{3.82}$$

After making use of this result in the integral in (3.81), integrating by parts gives

$$I(a,r,b) = \frac{\pi b}{2}\int_{e^{-\alpha}}^{e^\alpha}\operatorname{Ei}(-ct)\,dt = \frac{\pi b}{2}\left[t\operatorname{Ei}(-ct)+\frac{1}{c}e^{-ct}\right]_{e^{-\alpha}}^{e^\alpha}$$
$$= \frac{\pi b}{2}\left(e^\alpha\operatorname{Ei}(-ce^\alpha) - e^{-\alpha}\operatorname{Ei}(-ce^{-\alpha}) + \frac{e^{-ce^\alpha}-e^{-ce^{-\alpha}}}{c}\right).$$

The desired result then follows on substituting for the values of c and α. This completes the proof. ☐

Remark 3.16. If $a > 0$ and $r > 0$, since

$$\lim_{b\to 0^+}\left(\sqrt{r^2+b^2}-r\right)\operatorname{Ei}\left(-a\sqrt{r^2+b^2}+ar\right) = 0, \tag{3.83}$$

as $b \to 0^+$, (3.79) reduces to

$$\int_0^\infty \arctan\left(\frac{2r}{x}\right)\operatorname{si}(ax)\,dx = \pi\left[r\operatorname{Ei}(-2ar) - \frac{1}{a}e^{-ar}\sinh(ar)\right], \tag{3.84}$$

valid for $a > 0$ and $r > 0$. Differentiating (3.84) with respect to r yields (3.82) with $t = 2r > 0$ and is entry **6.244.1** in [**2**]. Differentiating (3.84) with respect to a yields (2.56) and is entry **4.574.1** in [**2**].

Remark 3.17. The parameter b appearing in (3.79) is fake. Enforcing $x \mapsto bx$ and scaling a by $\frac{a}{b}$ we immediately see the minimal form is equivalent to setting $b = 1$ in (3.79).

4. Some definite integrals for the case when $b = 1$ in the inverse tangent function appearing in (1.2)

A number of interesting definite integrals for the case when $b = 1$ appearing in the inverse tangent function found in (1.2) can be given for various functions f. These we now give.

PROPOSITION 4.1. *For $r \in \mathbb{R}$ the identity*

$$\int_0^\infty \arctan\left(\frac{2rx}{1+x^2}\right)\operatorname{cosech}(\pi x)\,dx = \log\left\{\frac{\Gamma\left(\frac{\sqrt{r^2+1}-r+2}{2}\right)\Gamma\left(\frac{\sqrt{r^2+1}+r+1}{2}\right)}{\Gamma\left(\frac{\sqrt{r^2+1}+r+2}{2}\right)\Gamma\left(\frac{\sqrt{r^2+1}-r+1}{2}\right)}\right\} + \operatorname{arcsinh}(r), \tag{4.1}$$

holds.

PROOF. Let $I(r)$ denote the left-hand side of (4.1). Applying (1.3) to the integral, a change in the order of integration produces

$$I(r) = \int_{e^{-\alpha}}^{e^\alpha}\int_0^\infty \frac{x}{t^2+x^2}\operatorname{cosech}(\pi x)\,dx\,dt. \tag{4.2}$$

The inner integral appears as entry **3.522.2** in [**2**]. Here

$$\int_0^\infty \frac{x}{t^2+x^2}\operatorname{cosech}(\pi x)\,dx = \frac{1}{2t} - \frac{1}{2}\left[\psi\left(\frac{t+2}{2}\right) - \psi\left(\frac{t+1}{2}\right)\right].$$

Making use of this result, the double integral in (4.2) reduces to

$$\begin{aligned} I(r) &= \frac{1}{2}\int_{e^{-\alpha}}^{e^\alpha}\left[\frac{1}{t} - \psi\left(\frac{t+2}{2}\right) + \psi\left(\frac{t+1}{2}\right)\right]dt \\ &= \left[\frac{1}{2}\log(t) - \log\Gamma\left(\frac{t+2}{2}\right) + \log\Gamma\left(\frac{t+1}{2}\right)\right]_{e^{-\alpha}}^{e^\alpha} \\ &= \alpha + \log\left\{\frac{\Gamma\left(\frac{e^{-\alpha}+2}{2}\right)\Gamma\left(\frac{e^\alpha+1}{2}\right)}{\Gamma\left(\frac{e^\alpha+2}{2}\right)\Gamma\left(\frac{e^{-\alpha}+1}{2}\right)}\right\}. \end{aligned}$$

The desired result then follows on substituting in the value for α. □

REMARK 4.1. For the special case when r is a positive integer, from the functional property for the gamma function, namely $\Gamma(x+1) = x\Gamma(x)$, it can be used to eliminate the four gamma functions appearing in the argument of the logarithmic term found in (4.1). Here we find

$$\int_0^\infty \arctan\left(\frac{2rx}{1+x^2}\right)\operatorname{cosech}(\pi x)\,dx = \log\left\{\frac{\prod_{k=1}^r\left(\sqrt{r^2+1}-r+2k+1\right)}{\prod_{k=1}^r\left(\sqrt{r^2+1}-r+2k\right)}\right\} + \operatorname{arcsinh}(r). \tag{4.3}$$

EXAMPLE 4.1. A particularly pretty example occurs when setting $r = 2$ in (4.3). In this case

$$\int_0^\infty \arctan\left(\frac{4x}{1+x^2}\right)\operatorname{cosech}(\pi x)\,dx = \log\left(\frac{4}{\sqrt{5}}\right). \tag{4.4}$$

REMARK 4.2. Setting $a = 2\pi$ and $b = 1$ in (3.33) and subtracting twice the result from (4.1) produces

$$\begin{aligned}\int_0^\infty \arctan\left(\frac{2rx}{1+x^2}\right)\exp\left(-\frac{\pi x}{2}\right)\operatorname{sech}\left(\frac{\pi x}{2}\right)\,dx &= 2r + 2\left(1-\sqrt{r^2+1}\right)\operatorname{arcsinh}(r)\\ &\quad + \log\left\{\frac{\Gamma\left(\sqrt{r^2+1}+r\right)}{\Gamma\left(\sqrt{r^2+1}-r\right)}\right\}\\ &\quad + \log\left\{\frac{\Gamma\left(\frac{\sqrt{r^2+1}-r+2}{2}\right)\Gamma\left(\frac{\sqrt{r^2+1}+r+1}{2}\right)}{\Gamma\left(\frac{\sqrt{r^2+1}+r+2}{2}\right)\Gamma\left(\frac{\sqrt{r^2+1}-r+1}{2}\right)}\right\},\end{aligned} \tag{4.5}$$

and is valid for $r \in \mathbb{R}$. If r is a positive integer the gamma functions appearing in the arguments for the logarithmic terms of (4.5) can be eliminated by employing the functional relation for Γ. Here we find

$$\begin{aligned}\int_0^\infty \arctan\left(\frac{2rx}{1+x^2}\right)\exp\left(-\frac{\pi x}{2}\right)\operatorname{sech}\left(\frac{\pi x}{2}\right)\,dx &= 2r + 2\left(1-\sqrt{r^2+1}\right)\operatorname{arcsinh}(r)\\ &\quad + \log\left\{\prod_{k=1}^{2r}\left(\sqrt{r^2+1}-r+k-1\right)\right\}\\ &\quad + \log\left\{\frac{\prod_{k=1}^{r}\left(\sqrt{r^2+1}-r+2k+1\right)}{\prod_{k=1}^{r}\left(\sqrt{r^2+1}-r+2k\right)}\right\}.\end{aligned} \tag{4.6}$$

EXAMPLE 4.2. A particularly pleasing special case occurs when setting $r = 2$ in (4.6). In this case

$$\int_0^\infty \arctan\left(\frac{4x}{1+x^2}\right)\exp\left(-\frac{\pi x}{2}\right)\operatorname{sech}\left(\frac{\pi x}{2}\right)\,dx = 4\left(1+\log(2)\right) - 6\sqrt{5}\log(\varphi), \tag{4.7}$$

where φ is the golden ratio.

PROPOSITION 4.2. *For $r \in \mathbb{R}$ the identity*

$$\int_0^\infty \arctan\left(\frac{2rx}{1+x^2}\right)\frac{\arctan(x)}{x}\,dx = \frac{\pi^2}{4}\operatorname{arcsinh}(r), \tag{4.8}$$

holds.

PROOF. Let $I(r)$ denote the left-hand side of (4.8). Enforcing a substitution of $x \mapsto \frac{1}{x}$ we find

$$I(r) = \int_0^\infty \arctan\left(\frac{2rx}{1+x^2}\right)\arctan\left(\frac{1}{x}\right)\frac{dx}{x}. \tag{4.9}$$

From the identity (1.5) the above integral may be rewritten as

$$I(r) = \frac{\pi}{2} \int_0^\infty \arctan\left(\frac{2rx}{1+x^2}\right) \frac{dx}{x} - I(r). \tag{4.10}$$

The value for the remaining integral in (4.10) is known. It is (2.14). The desired result then follows and completes the proof. □

Example 4.3. Setting $r = 1$ gives the attractive looking result of

$$\int_0^\infty \arctan\left(\frac{2x}{1+x^2}\right) \frac{\arctan(x)}{x} \, dx = \frac{\pi^2}{4} \log\left(1+\sqrt{2}\right). \tag{4.11}$$

Remark 4.3. Differentiating (4.8) with respect to the parameter r gives

$$\int_0^\infty \frac{(x^2+1)\arctan(x)}{x^4 + x^2(4r^2+2)+1} \, dx = \frac{\pi^2}{8\sqrt{1+r^2}}. \tag{4.12}$$

Setting $r = 0$ in (4.12) we immediately obtain the elementary result of

$$\int_0^\infty \frac{\arctan(x)}{1+x^2} \, dx = \frac{\pi^2}{8}, \tag{4.13}$$

while setting $r = 1$ gives the result for the far more difficult integral

$$\int_0^\infty \frac{(x^2+1)\arctan(x)}{x^4+6x^2+1} \, dx = \frac{\pi^2}{8\sqrt{2}}. \tag{4.14}$$

Finally, let $I(r)$ denote the left-hand side of (4.12). Enforcing a substitution of $x \mapsto \frac{1}{x}$ before applying identity (1.5) to the arctan $\left(\frac{1}{x}\right)$ term that results, we find

$$I(r) = \frac{\pi}{2} \int_0^\infty \frac{1+x^2}{x^4+2x^2(2r^2+1)+1} \, dx - I(r), \tag{4.15}$$

or

$$\int_0^\infty \frac{1+x^2}{x^4+2x^2(2r^2+1)+1} \, dx = \frac{\pi}{2\sqrt{r^2+1}}, \tag{4.16}$$

and is valid for $r \geqslant 0$. This result can also be obtained directly by elementary means.

We now find four definite integrals for the case where the function f in (1.2) is a product between sine and cosine functions involving powers with unequal arguments.

Proposition 4.3. *For $m \in \mathbb{N}$ and $r \in \mathbb{R}$ the identity*

$$\int_0^\infty \arctan\left(\frac{2rx}{1+x^2}\right) \cos^{m-1}(x) \sin[(m+1)x] \, dx = \frac{\pi}{2^m} \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{1}{k+1} \times \exp\left(-2(k+1)\sqrt{r^2+1}\right) \sinh[2r(k+1)], \tag{4.17}$$

holds.

Proof. Let $I_m(r)$ denote the left-hand side of (4.17) and set $r = \sinh\alpha$ where $\alpha \in \mathbb{R}$. Applying (1.3) to the integral, a change in the order of the integration produces

$$I_m(r) = \int_{e^{-\alpha}}^{e^{\alpha}} \int_0^{\infty} \frac{x\cos^{m-1}(x)\sin[(m+1)x]}{t^2+x^2}\,dx\,dx \tag{4.18}$$

The inner integral appears as entry **3.832.24** in [**2**]. Here

$$\int_0^{\infty} \frac{x\cos^{m-1}(x)\sin[(m+1)x]}{t^2+x^2}\,dx = \frac{\pi}{2^m}e^{-2t}(1+e^{-2t})^{m-1}, \quad t>0. \tag{4.19}$$

Making use of this result, the double integral in (4.18) reduces to

$$I_m(r) = \frac{\pi}{2^m}\int_{e^{-\alpha}}^{e^{\alpha}} e^{-2t}(1+e^{-2t})^{m-1}\,dt, \tag{4.20}$$

or

$$I_m(r) = -\frac{\pi}{2^{m+1}}\int_{e^{-2e^{-\alpha}}}^{e^{-2e^{\alpha}}} (1+u)^{m-1}\,du, \tag{4.21}$$

after a substitution of $u = e^{-2t}$ has been made. The integral can be found on application of the binomial theorem. Here

$$\begin{aligned} I_m(r) &= -\frac{\pi}{2^{m+1}}\sum_{k=0}^{m-1}\binom{m-1}{k}\int_{e^{-2e^{-\alpha}}}^{e^{-2e^{\alpha}}} u^k\,du \\ &= -\frac{\pi}{2^{m+1}}\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{1}{k+1}\left(e^{-2(k+1)e^{\alpha}} - e^{-2(k+1)e^{-\alpha}}\right). \end{aligned} \tag{4.22}$$

The desired result then follows on substituting the value for α into (4.22) and rearranging algebraically. This completes the proof. ☐

Remark 4.4. Note that setting $m = 1$ in (4.17) reproduces (1.1) for the special case when $a = 2$ and $b = 1$.

Proposition 4.4. *For $m \in \mathbb{N}\cup\{0\}$ and $r \in \mathbb{R}$ the identity*

$$\begin{aligned} \int_0^{\infty} \arctan\left(\frac{2rx}{1+x^2}\right)\cos^m(x)\sin[(m+1)x]\,dx &= \frac{\pi}{2^m}\sum_{k=0}^{m}\binom{m}{k}\frac{1}{2k+1} \\ &\times \exp\left(-(2k+1)\sqrt{r^2+1}\right)\sinh[(2k+1)r], \end{aligned} \tag{4.23}$$

holds.

Proof. Let $I_m(r)$ denote the left-hand side of (4.23) and set $r = \sinh\alpha$ where $\alpha \in \mathbb{R}$. Applying (1.3) to the integral, a change in the order of the integration produces

$$I_m(r) = \int_{e^{-\alpha}}^{e^{\alpha}} \int_0^{\infty} \frac{x\cos^{m}(x)\sin[(m+1)x]}{t^2+x^2}\,dx\,dx \tag{4.24}$$

The inner integral appears as entry **3.832.25** in [**2**]. Here

$$\int_0^{\infty} \frac{x\cos^{m}(x)\sin[(m+1)x]}{t^2+x^2}\,dx = \frac{\pi}{2^{m+1}}e^{-t}(1+e^{-2t})^{m}, \quad t>0. \tag{4.25}$$

The remainder of the proof then proceeds in a manner similar to that used in the proof of proposition 4.3. This completes the proof. □

REMARK 4.5. Note that setting $m = 0$ in (4.23) reproduces (1.1) for the special case when $a = b = 1$.

PROPOSITION 4.5. *For* $m \in \mathbb{N} \cup \{0\}$ *and* $r \in \mathbb{R}$ *the identity*

$$\int_0^\infty \arctan\left(\frac{2rx}{1+x^2}\right) \sin^{2m}(x) \sin[(2m+2)x]\, dx = \frac{(-1)^m \pi}{2^{2m+1}} \sum_{k=0}^{2m} \binom{2m}{k} \frac{(-1)^k}{k+1} \times \exp\left(-2(k+1)\sqrt{r^2+1}\right) \sinh[2r(k+1)], \tag{4.26}$$

holds.

PROOF. Let $I_m(r)$ denote the left-hand side of (4.26) and set $r = \sinh \alpha$ where $\alpha \in \mathbb{R}$. Applying (1.3) to the integral, a change in the order of the integration produces

$$I_m(r) = \int_{e^{-\alpha}}^{e^\alpha} \int_0^\infty \frac{x \sin^{2m}(x) \sin[(2m+2)x]}{t^2 + x^2}\, dx\, dx \tag{4.27}$$

The inner integral appears as entry **3.832.8** in [**2**]. Here

$$\int_0^\infty \frac{x \sin^{2m}(x) \sin[(2m+2)x]}{t^2 + x^2}\, dx = \frac{(-1)^m \pi}{2^{2m+1}} e^{-2t} (1 - e^{-2t})^{2m}, \quad t > 0. \tag{4.28}$$

The remainder of the proof then proceeds in a manner similar to that used in the proof of proposition 4.3. This completes the proof. □

Our fourth and final result for the function f in (1.2) consisting of a product between sine and cosine functions involving powers and unequal arguments is given in the next proposition.

PROPOSITION 4.6. *For* $m \in \mathbb{N}$ *and* $r \in \mathbb{R}$ *the identity*

$$\int_0^\infty \arctan\left(\frac{2rx}{1+x^2}\right) \sin^{2m-1}(x) \cos[(2m+1)x]\, dx = \frac{(-1)^m \pi}{2^{2m}} \sum_{k=0}^{2m-1} \binom{2m-1}{k} \frac{(-1)^k}{k+1} \times \exp\left(-2(k+1)\sqrt{r^2+1}\right) \sinh[2r(k+1)], \tag{4.29}$$

holds.

PROOF. Let $I_m(r)$ denote the left-hand side of (4.29) and set $r = \sinh \alpha$ where $\alpha \in \mathbb{R}$. Applying (1.3) to the integral, a change in the order of the integration produces

$$I_m(r) = \int_{e^{-\alpha}}^{e^\alpha} \int_0^\infty \frac{x \sin^{2m-1}(x) \cos[(2m+1)x]}{t^2 + x^2}\, dx\, dx \tag{4.30}$$

The inner integral appears as entry **3.832.19** in [**2**]. Here

$$\int_0^\infty \frac{x \sin^{2m-1}(x) \cos[(2m+1)x]}{t^2 + x^2}\, dx = \frac{(-1)^m \pi}{2^{2m}} e^{-2t} (1 - e^{-2t})^{2m-1}, \quad t > 0. \tag{4.31}$$

The remainder of the proof then proceeds in a manner similar to that used in the proof of proposition 4.3. This completes the proof. □

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