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The integrals in Gradshteyn and Ryzhik. Part 35: Fixing the confusion of Φ and `erf` in the table

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Abstract. The table of integrals by Gradshteyn and Ryzhik uses a large variety of special function. Some of them have very similar definitions. The different names have been kept because different users are accustomed to one name or the other. In occasion this leads to errors in the entries of the table. The error function, defined as the normalized primitive of $\exp(-x^2)$, and a function denoted by Φ have very similar definitions. The goal of this note is to fix some entries that have appeared incorrectly in the current edition of the table.

1. Introduction

The table of integrals by Gradshteyn and Ryzhik, called GR from this point on, is one of the most cited collection of integral evaluations. One of the earliest version, by I. M. Ryzhik alone [**21**] appeared in Russian. The oldest edition available to the authors is [**13**]. In view of the magnitude of the tasks of collecting all these evaluations, these tables are bound to have some errors in them. This is not special for this table. One of the original sources for the entries in GR is the table by Bierens de Haan [**3**] and the revised edition [**4**]. Some of the `errors` in this table were appeared in [**17**]. Perhaps one can say that `there is no table of integrals without errors`.

Some errors come from forgetting the conditions on the parameters. For instance, entry 3.257 in [**14**] appeared as

$$\int_0^\infty \left[(ax+b/x)^2+c\right]^{-p-1}dx = \frac{B(p+1/2,1/2)}{2ac^{p+1/2}}2ac^{p+1/2} \tag{1.1}$$

coming from a paper by Liouville [**18, 19**]. This is only valid for $b<0$. For $b>0$ the correct value is

$$\int_0^\infty \left[\frac{x^2}{a^2x^4+(2ab+c)x^2+b^2}\right]^{p+1}dx = \frac{B(p+1/2,1/2)}{2a[2a(b+|b|)+c]^{p+1/2}}. \tag{1.2}$$

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This was discussed in [**6**], one of the earliest works on the second author in the evaluation of integrals.

Motivated by this work, the authors of [**6**] began the investigation of the so-called `quartic integral`

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} \tag{1.3}$$

and shows that

$$N_{0,4}(a,m) = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a), \tag{1.4}$$

where $P_m(a)$ is a polynomial of degree m. Initially it was not clear that the coefficients of P_m were positive. The original proof of this fact, presented in [**7**], was based on the expansion

$$\sqrt{a+\sqrt{1+c}} = \sqrt{a+1} + \frac{1}{\pi\sqrt{2}} \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} N_{0,4}(a;k-1)c^k. \tag{1.5}$$

This was a surprising expansion, but it lead us to search for entries in GR containing double square roots. Then we found entry **3.248.5** in [**15**] stating that

$$\int_0^\infty \frac{1}{\sqrt{\varphi(x)+\sqrt{\varphi(x)}}} \frac{dx}{(x^2+1)^{3/2}} = \frac{\pi}{2\sqrt{6}} \tag{1.6}$$

with

$$\varphi(x) = 1 + \frac{4x^2}{3(x^2+1)^2}. \tag{1.7}$$

It turns out that is incorrect. The error is of a different nature. Entries in the table need to be typed. During this process someone forgot a **3**: the correct entry should have been

$$\int_0^\infty \frac{1}{\sqrt{\varphi(x)+\sqrt{\varphi^{\mathbf{3}}(x)}}} \frac{dx}{(x^2+1)^{3/2}} = \frac{\pi}{2\sqrt{6}} \tag{1.8}$$

with the same function φ. A discussion of this problem appears in [**2**], [**9**], and [**5**].

The third type of mistakes leading to errors in tables of integrals comes from using different names for the same function. This is not uncommon, since different authors use different normalization for classical functions. The example discussed here come from the `error function` defined by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \tag{1.9}$$

and the `probabilty integral` defined by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt. \tag{1.10}$$

A simple scaling shows that

$$\Phi(x) = \frac{1}{2}\text{erf}\left(\frac{x}{\sqrt{2}}\right), \tag{1.11}$$

and not

$$\Phi(x) = \text{erf}(x) \tag{1.12}$$

as it appears in **8.250.1**.

The goal of this note is to present correct statements of all the integrals appearing in [**16**] containing Φ and erf.

2. Some simple examples

Section **3.321** is part of a group of entries under the title `exponentials of more complicated arguments`. Entry **3.321.1** contains the error $\Phi(u) = \texttt{erf}(u)$ discussed above. This section also contain three entries with the same error: the correct versions should be

$$\textbf{3.321.2} \qquad \int_0^u e^{-q^2x^2}\,dx = \frac{\sqrt{\pi}}{2q}\text{erf}(qu), \tag{2.1}$$

the second example is

$$\textbf{3.321.5} \qquad \int_0^u x^2 e^{-q^2x^2}\,dx = \frac{1}{2q^3}\left[\frac{\sqrt{\pi}}{2}\text{erf}(qu) - que^{-q^2u^2}\right], \tag{2.2}$$

and finally

$$\textbf{3.321.7} \qquad \int_0^u x^4 e^{-q^2x^2}\,dx = \frac{1}{2q^5}\left[\frac{3\sqrt{\pi}}{4}\text{erf}(qu) - \left(\frac{3}{2} + q^2u^2\right)que^{-q^2u^2}\right]. \tag{2.3}$$

These entries admit simple proofs: start by making the change of variables $t = qx$ and then integrate by parts.

3. Section 3.361: Combinations of exponentials and algebraic functions

This section contains a couple of entries containing the probability integral in the answer. These are `not correct` and the elementary modifications to produce the correct versions are given below.

Example 3.1. Entry **3.361.1** states that

$$\int_0^u \frac{e^{-qx}}{\sqrt{x}}dx = \sqrt{\frac{\pi}{q}}\Phi(\sqrt{qu}). \tag{3.1}$$

In order to evaluate the integral start with the change of variables $t = qx$ to produce

$$\int_0^u \frac{e^{-qx}}{\sqrt{x}}dx = \frac{1}{\sqrt{q}}\int_0^{qu}\frac{e^{-t}}{\sqrt{t}}dt. \tag{3.2}$$

The change of variables $t = s^2$ now gives

$$\frac{1}{\sqrt{q}}\int_0^{qu}\frac{e^{-t}}{\sqrt{t}}dt = \frac{2}{\sqrt{q}}\int_0^{\sqrt{qu}} e^{-s^2}ds, \tag{3.3}$$

so that the corrected version of **3.361.1** is

$$\int_0^u \frac{e^{-qx}}{\sqrt{x}} dx = \sqrt{\frac{\pi}{q}} \texttt{erf}(\sqrt{qu}), \tag{3.4}$$

that is, simply replace Φ by `erf` in (3.1).

EXAMPLE 3.2. Entry **3.362.2** states that

$$\int_0^\infty \frac{e^{-\mu x} dx}{\sqrt{x+b}} = \sqrt{\frac{\pi}{\mu}} e^{b\mu} \left[1 - \Phi(\sqrt{b\mu}) \right]. \tag{3.5}$$

The parameters μ, b are complex, with $\text{Re}\, \mu > 0$ and $|\arg b| < \pi$. These are conditions required for the convergence of the integral. In the arguments presented below, we assume μ, b to be real and the passage to complex parameters comes from a standard analytic continuation procedure.

To evaluate the original integral, let $t = x + b$ to obtain

$$\int_0^\infty \frac{e^{-\mu x} \, dx}{\sqrt{x+b}} = e^{\mu b} \int_b^\infty \frac{e^{-\mu t} \, dt}{\sqrt{t}}. \tag{3.6}$$

The condition $\mu > 0$ is required for convergence. Now assume $b > 0$ and make the change of variables $t = \frac{1}{\mu} s^2$ to obtain

$$\begin{aligned} \int_b^\infty \frac{e^{-\mu t} \, dt}{\sqrt{t}} &= \frac{2}{\sqrt{\mu}} \int_{\sqrt{b\mu}}^\infty e^{-s^2} \, ds \\ &= \frac{2}{\sqrt{\mu}} \left[\int_0^\infty e^{-s^2} \, ds - \int_0^{\sqrt{b\mu}} e^{-s^2} \, ds \right] \\ &= \sqrt{\frac{\pi}{\mu}} \left(1 - \text{erf}(\sqrt{b\mu}) \right). \end{aligned} \tag{3.7}$$

As in the previous example, the corrected version simply comes by replacing Φ by `erf` in (3.5).

The function appearing on the last line of the computation above is called the `complementary error function`, defined by

$$\texttt{erfc}(x) = 1 - \texttt{erf}(x). \tag{3.8}$$

Then the corrected version of **3.362.2** can be written as

$$\int_0^\infty \frac{e^{-\mu x} \, dx}{\sqrt{x+b}} = \sqrt{\frac{\pi}{b}} e^{b\mu} \texttt{erfc}(\sqrt{b\mu}). \tag{3.9}$$

4. An earlier paper in the collection

Motivated by the error found in GR, the second author of the current paper started the project to obtain independent proofs of all the entries. The idea was to produce short papers with the complete evaluation of a selected group of formulas in GR. The first one was [**20**]. The magnitude of the project became clear almost immediately. The author has looked for help everywhere. Colleagues, graduate and undergraduate students have participated in it. Part 19 of this collection [**1**] contains a selection of entries involving the error function. In this section we list those appearing there containing the probability integral Φ, now written in terms of the error function $\texttt{erf}$. Complete proofs may be found in [**1**].

3.321.2 $$\int_0^u e^{-q^2x^2}\,dx = \frac{\sqrt{\pi}}{2q}\texttt{erf}(qu)$$

3.321.5 $$\int_0^u x^2e^{-q^2x^2}\,dx = \frac{1}{2q^3}\left[\frac{\sqrt{\pi}}{2}\texttt{erf}(qu) - que^{-q^2u^2}\right]$$

3.321.7 $$\int_0^u x^4e^{-q^2x^2}\,dx = \frac{1}{2q^5}\left[\frac{3\sqrt{\pi}}{4}\texttt{erf}(qu) - \left(\frac{3}{2}+q^2u^2\right)que^{-q^2u^2}\right]$$

3.361.1 $$\int_0^u \frac{e^{-qx}\,dx}{\sqrt{x}} = \sqrt{\frac{\pi}{q}}\texttt{erf}(\sqrt{qu})$$

3.362.2 $$\int_0^\infty \frac{e^{-\mu x}\,dx}{\sqrt{x+b}} = \sqrt{\frac{\pi}{\mu}}e^{b\mu}\left[1-\texttt{erf}(\sqrt{b\mu})\right]$$

3.363.1 $$\int_u^\infty \frac{\sqrt{x-u}}{x}e^{-\mu x}dx = \sqrt{\frac{\pi}{\mu}}e^{-u\mu} - \pi\sqrt{\mu}(1-\texttt{erf}(\sqrt{u\mu})$$

3.363.2 $$\int_u^\infty \frac{e^{-\mu x}\,dx}{x\sqrt{x-u}} = \frac{\pi}{\sqrt{u}}\left[1-\texttt{erf}(\sqrt{u\mu})\right]$$

3.369 $$\int_0^\infty \frac{e^{-\mu x}}{\sqrt{(x+a)^3}} = \frac{2}{\sqrt{a}} - 2\sqrt{\pi\mu}e^{a\mu}(1-\texttt{erf}(\sqrt{a\mu})$$

3.461.5 $$\int_u^\infty e^{-\mu x^2}\frac{dx}{x^2} = \frac{1}{u}e^{-\mu u^2} - \sqrt{\mu\pi}(1-\texttt{erf}(u\sqrt{\mu}))$$

3.466.1 $$\int_0^\infty \frac{e^{-\mu^2x^2}\,dx}{x^2+b^2} = (1-\texttt{erf}(b\mu))\,\frac{\pi}{2b}e^{b^2\mu^2}$$

3.466.2 $$\int_0^\infty \frac{x^2e^{-\mu^2x^2}}{x^2+b^2}\,dx = \frac{\sqrt{\pi}}{2\mu} - \frac{\pi b}{2}e^{\mu^2b^2}\,(1-\texttt{erf}(b\mu))$$

6.281.1 $$\int_0^\infty \left[1-\texttt{erf}(px)\right]x^{2q-1}\,dx = \frac{\Gamma(q+\frac{1}{2})}{2\sqrt{\pi}qp^{2q}}$$

6.282.1 $$\int_0^\infty \texttt{erf}(qt)e^{-pt}\,dt = \frac{1}{p}\left[1-\texttt{erf}\left(\frac{p}{2q}\right)\right]\exp\left(\frac{p^2}{4q^2}\right)$$

6.282.2 $$\int_0^\infty \left[\texttt{erf}\left(x+\tfrac{1}{2}\right)-\texttt{erf}\left(\tfrac{1}{2}\right)\right]e^{-\mu x+\frac{1}{4}}\,dx = \frac{1}{\mu}\exp\frac{(\mu+1)^2}{4}\left[1-\texttt{erf}\left(\frac{\mu+1}{2}\right)\right]$$

6.283.1 $$\int_0^\infty e^{\beta x}\left[1-\texttt{erf}(\sqrt{\alpha x})\right]\,dx = \frac{1}{\beta}\left[\frac{\sqrt{\alpha}}{\sqrt{\alpha-\beta}}-1\right]$$

6.283.2 $$\int_0^\infty \texttt{erf}(\sqrt{qt})e^{-pt}\,dt = \frac{\sqrt{q}}{p}\frac{1}{\sqrt{p+q}}$$

There are some entries in GR involving the Φ function where the answer involves more advanced functions. These have been checked numerically using `Mathematica` and the proofs will appear in a future publications. These include entry **6.281.2**

$$\int_0^\infty \left[1 - \texttt{erf}\left(at^\alpha \pm \frac{b}{t^\alpha} \right) \right] dt = \frac{2b}{\sqrt{\pi}} \left(\frac{b}{a} \right)^{\frac{1-\alpha}{2\alpha}} \left[K_{\frac{1+\alpha}{2\alpha}}(2ab) \pm K_{\frac{1-\alpha}{2\alpha}}(2ab) \right] e^{\pm 2ab}$$

and entry **6.295.1**:

$$\int_0^\infty \left[1 - \texttt{erf}\left(\frac{1}{x} \right) \right] \exp\left(-\mu^2 x^2 + \frac{1}{x^2} \right) dx = \frac{1}{\sqrt{\pi}\mu} \left[\sin 2\mu \;\; \texttt{ci}(2\mu) - \cos 2\mu \;\; \texttt{si}(2\mu) \right].$$

5. Section 8.250

Section **8.25** `The probability integral` $\Phi(x)$`, the Fresnel integrals` $S(x)$, $C(x)$`, the error function` $\texttt{erf}(x)$`, and the complementary error function` $\texttt{erfc}(x)$ contains a collection of entries involving the function $\Phi(x)$. The goal of this section is to give a correct version and present a proof of them. Section **8.250** contains five entries. Four of the are evaluated here. Entry **8.250.6** requires a different technique and its evaluation is presented in the next section.

5.1. Entry 8.250.5. The 8^{th} edition of GR contains the entry

$$\begin{aligned} \int_0^\infty \frac{e^{-(p+x)y}}{\pi(p+x)} \sin(a\sqrt{x})\, dx = & -\sinh(a\sqrt{p}) \\ & + \frac{1}{2} e^{-a\sqrt{p}} \Phi\left(\frac{a}{2\sqrt{y}} - \sqrt{py} \right) + \frac{1}{2} e^{a\sqrt{p}} \Phi\left(\frac{a}{2\sqrt{y}} + \sqrt{py} \right) \end{aligned} \tag{5.1}$$

Testing this entry numerically shows that this is incorrect. The correct version is

$$\begin{aligned} \int_0^\infty \frac{e^{-(p+x)y}}{\pi(p+x)} \sin(a\sqrt{x})\, dx = & -\sinh(a\sqrt{p}) \\ & + \frac{1}{2} e^{-a\sqrt{p}} \texttt{erf}\left(\frac{a}{2\sqrt{y}} - \sqrt{py} \right) + \frac{1}{2} e^{a\sqrt{p}} \texttt{erf}\left(\frac{a}{2\sqrt{y}} + \sqrt{py} \right) \end{aligned} \tag{5.2}$$

`The proof`. We first check that the derivatives of both sides with respect to y agree. The derivative of the left-hand side is

$$\frac{\partial}{\partial y} \text{lhs} = -\frac{1}{\pi} \int_0^\infty e^{-(p+x)y} \sin(a\sqrt{x}) dx \tag{5.3}$$

and differentiating the right-hand side gives, after some simplification,

$$\frac{\partial}{\partial y} \text{rhs} = -\frac{a}{2\sqrt{\pi} y^{3/2}} e^{-a^2/4y - py}. \tag{5.4}$$

Therefore the matching of the derivatives with respect to y of the entry is equivalent to the identity

$$\int_0^\infty e^{-xy} \sin(a\sqrt{x})\, dx = \frac{a\sqrt{\pi}}{2y^{3/2}} e^{-a^2/4y}. \tag{5.5}$$

To prove this, let $t = a\sqrt{x}$ and define $b = y/a^2$ to convert (5.5) into

$$\int_0^\infty t \sin t \, e^{-bt^2} \, dt = \frac{\sqrt{\pi}}{4b\sqrt{b}} e^{-1/4b}. \tag{5.6}$$

To evaluate (5.6), integrate by parts to produce

$$\begin{aligned} \int_0^\infty t \sin t \, e^{-bt^2} \, dt &= -\frac{1}{2b} \int_0^\infty \sin t \, \frac{d}{dt} \left(e^{-bt^2} \right) \, dt \\ &= \frac{1}{2b} \int_0^\infty e^{-bt^2} \cos t \, dt. \end{aligned} \tag{5.7}$$

The change of variables $x = \sqrt{b}t$ shows that we need to prove

$$\int_0^\infty e^{-bt^2} \cos t \, dt = \frac{1}{\sqrt{b}} I \left(\frac{1}{\sqrt{b}} \right), \tag{5.8}$$

with

$$I(\alpha) = \int_0^\infty e^{-x^2} \cos(\alpha x) \, dx. \tag{5.9}$$

To evaluate $I(\alpha)$, differentiate with respect to α and integrate by parts to obtain

$$\begin{aligned} I'(\alpha) &= -\int_0^\infty e^{-x^2} \sin(\alpha x) \, dx \\ &= -\frac{\alpha}{2} \int_0^\infty e^{-x^2} \cos(\alpha x) \, dx \\ &= -\frac{\alpha}{2} I(\alpha), \end{aligned} \tag{5.10}$$

and this gives $I(\alpha) = I(0) e^{-\alpha^2/4}$. The initial value is

$$I(0) = \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} \tag{5.11}$$

so that

$$I(\alpha) = \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}. \tag{5.12}$$

This explicit formula verifies that the derivatives of (5.1) with respect to y agree.

The limit as $y \to \infty$ of the left-hand side is 0. The limit of the right-hand side is

$$-\sinh(a\sqrt{p}) + \frac{1}{2} e^{-a\sqrt{p}} \text{erf}(+\infty) + \frac{1}{2} e^{a\sqrt{p}} \text{erf}(+\infty) \tag{5.13}$$

and this is 0. The proof is complete.

5.2. Entry 8.250.7. The 8^{th} edition of GR contains the entry

$$\int_0^p \exp(-x^2) \Phi(p-x) \, dx = \int_0^p \exp(-x^2) \texttt{erf}(p-x) = \frac{\sqrt{\pi}}{2} \left[\Phi \left(\frac{p}{\sqrt{2}} \right) \right]^2 \tag{5.14}$$

Since $\Phi \neq \texttt{erf}$ this entry is not valid. The correct formulation is

$$\int_0^p \exp(-x^2) \texttt{erf}(p-x) = \frac{\sqrt{\pi}}{2} \left[\texttt{erf} \left(\frac{p}{\sqrt{2}} \right) \right]^2 \tag{5.15}$$

`The proof`. Introduce the notation

$$A_1(p) = \int_0^p \exp(-x^2)\texttt{erf}(p-x)\, dx \tag{5.16}$$

and

$$A_2(p) = \frac{\sqrt{\pi}}{2}\left[\texttt{erf}\left(\frac{p}{\sqrt{2}}\right)\right]^2. \tag{5.17}$$

Since $A_1(0) = A_2(0)$, it suffices to prove that $A_1'(p) = A_2'(p)$ to establish (5.15).

Write

$$A_1(p) = \int_0^\infty e^{-x^2}\chi_{[0,p]}(x)\texttt{erf}(p-x)\, dx \tag{5.18}$$

and then

$$A_1'(p) \quad = \quad \int_0^\infty e^{-x^2}\left[\chi_{[0,p]}(x)\frac{d}{dp}\texttt{erf}(p-x) + \texttt{erf}(p-x)\frac{d}{dp}\chi_{[0,p]}(x)\right] dx, \tag{5.19}$$

where $\chi_{[0,p]}(x)$ is the characteristic function of the interval $[0,p]$, given by

$$\chi_{[0,p]}(x) = \begin{cases} 1 & \text{if } x \in [0,p] \\ 0 & \text{otherwise.} \end{cases} \tag{5.20}$$

Now

$$\frac{d}{dp}\chi_{[0,p]}(x) = \delta(p-x) \quad \text{and} \quad \frac{d}{dp}\texttt{erf}(p-x) = \frac{2}{\sqrt{\pi}}e^{-(p-x)^2}. \tag{5.21}$$

with δ being the Dirac delta function. Therefore

$$A_1'(p) = \frac{2}{\sqrt{\pi}}\int_0^p e^{-x^2-(p-x)^2}\, dx + \int_0^\infty \delta(p-x)e^{-x^2}\texttt{erf}(p-x)\, dx. \tag{5.22}$$

The second integral vanishes in view of $\texttt{erf}(0) = 0$. The first one is

$$\begin{aligned} \frac{2}{\sqrt{\pi}}\int_0^p e^{-2x^2+2px-p^2}\, dx \quad &= \quad \frac{2}{\sqrt{\pi}}e^{-p^2/2}\int_0^p e^{-2(x-p/2)^2}\, dx \\ &= \quad \frac{2}{\sqrt{2\pi}}e^{-p^2/2}\int_{-p/\sqrt{2}}^{p/\sqrt{2}} e^{-t^2} \\ &= \quad \frac{4}{\sqrt{2\pi}}e^{-p^2/2}\int_0^{p/\sqrt{2}} e^{-t^2} \\ &= \quad \sqrt{2}e^{-p^2/2}\texttt{erf}\left(\frac{p}{\sqrt{2}}\right). \end{aligned} \tag{5.23}$$

A direct calculation shows that $A_2(p)$ is given by the same expression. The proof is complete.

5.3. Entry 8.250.8. The 8^{th} edition of GR contains the statement

$$\int_0^p x^2 \exp(-x^2)\Phi(p-x)\,dx = \int_0^p x^2\exp(-x^2)\text{erf}(p-x)dx = \frac{\sqrt{\pi}}{4}\left[\Phi\left(\frac{p}{\sqrt{2}}\right)\right]^2 - \frac{p}{2\sqrt{2}}\Phi\left(-\frac{p^2}{2}\right)\text{erf}\left(\frac{p}{\sqrt{2}}\right). \tag{5.24}$$

This is not valid. The correct formula is

$$\int_0^p x^2 e^{-x^2}\text{erf}(p-x)\,dx = \frac{\sqrt{\pi}}{4}\left[\text{erf}\left(\frac{p}{\sqrt{2}}\right)\right]^2 - \frac{1}{2\sqrt{2}}pe^{-p^2/2}\text{erf}\left(\frac{p}{\sqrt{2}}\right). \tag{5.25}$$

`The proof`. Let

$$\begin{aligned} A(p) &= \int_0^p x^2e^{-x^2}\text{erf}(p-x)\,dx \\ &= \int_0^\infty \chi_{[0,p]}(x)x^2e^{-x^2}\text{erf}(p-x)\,dx. \end{aligned} \tag{5.26}$$

Differentiating (5.26) produces

$$\begin{aligned} A'(p) &= \int_0^\infty \delta(p-x)x^2e^{-x^2}\text{erf}(p-x)\,dx \\ &+\frac{2}{\sqrt{\pi}}\int_0^\infty \chi_{[0,p]}(x)x^2e^{-x^2}e^{-(p-x)^2}\,dx, \end{aligned} \tag{5.27}$$

where δ is the Dirac function.

The first integral is

$$\int_0^\infty \delta(p-x)x^2e^{-x^2}\text{erf}(p-x)\,dx = x^2e^{-x^2}\text{erf}(p-x) \text{ at } x=p. \tag{5.28}$$

This vanishes since $\text{erf}(0)=0$. Completing the square, It follows that

$$\begin{aligned} A'(p) &= \frac{2}{\sqrt{\pi}}\int_0^p x^2e^{-x^2}e^{-(p-x)^2}\,dx \\ &= \frac{2}{\sqrt{\pi}}e^{-p^2/2}\int_0^p x^2e^{-2(x-p/2)^2}\,dx. \end{aligned} \tag{5.29}$$

The change of variables $t = x - p/2$ gives

$$A'(p) = \frac{2}{\sqrt{\pi}}e^{-p^2/2}\int_{-p/2}^{p/2}\left[t^2+pt+p^2/4\right]e^{-2t^2}\,dt. \tag{5.30}$$

The part of the integrand with te^{-2t^2} is odd, so the integral vanishes. This yields

$$A'(p) == \frac{4}{\sqrt{\pi}}e^{-p^2/2}\int_0^{p/2}\left[t^2+p^2/4\right]e^{-2t^2}\,dt, \tag{5.31}$$

using the symmetry of the integrand.

The change of variables $\sigma = \sqrt{2}t$ gives

$$A'(p) = \sqrt{\frac{2}{\pi}}e^{-p^2/2}\int_0^{p/\sqrt{2}}\sigma^2e^{-\sigma^2}d\sigma + \frac{p^2}{\sqrt{2\pi}}e^{-p^2/2}\int_0^{p/\sqrt{2}}e^{-\sigma^2}d\sigma. \tag{5.32}$$

The primitives

$$\int e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\text{erf } x, \quad \int x^2 e^{-x^2}\,dx = -\frac{x}{2}e^{-x^2} + \frac{1}{2}\frac{\sqrt{\pi}}{2}\text{erf } x, \tag{5.33}$$

yield

$$\begin{aligned} A'(p) &= \sqrt{\frac{2}{\pi}}\left(-\frac{x}{2}e^{-x^2} + \frac{\sqrt{\pi}}{4}\text{erf } x\right) + \frac{p^2}{\sqrt{2\pi}}e^{-p^2/2}\frac{\sqrt{\pi}}{2}\text{erf } x\Big|_{x=0}^{x=p/\sqrt{2}} \\ &= -\frac{pe^{-p^2}}{2\sqrt{\pi}} + \frac{(1+p^2)}{2\sqrt{2}}e^{-p^2/2}\text{erf}\left(\frac{p}{\sqrt{2}}\right). \end{aligned} \tag{5.34}$$

Now we integrate to produce $A(p)$. Recall that $A(0) = 0$ takes care of the constant of integration. Start with

$$\begin{aligned} A(p) &= -\frac{1}{2\sqrt{\pi}}\int_0^p xe^{-x^2}\,dx \\ &\quad +\frac{1}{2\sqrt{2}}\int_0^p e^{-x^2/2}\text{erf}\left(\frac{x}{\sqrt{2}}\right)\,dx + \frac{1}{2\sqrt{2}}\int_0^p x^2e^{-x^2/2}\text{erf}\left(\frac{x}{\sqrt{2}}\right)\,dx \\ &:= I_1(p) + I_2(p) + I_3(p). \end{aligned}$$

The integral I_1 is elementary with value

$$I_1(p) = \frac{\sqrt{\pi}}{4}e^{-p^2}. \tag{5.35}$$

The integral I_2 is also elementary:

$$I_2(p) = \frac{\sqrt{\pi}}{8}\left[\text{erf}\left(\frac{p}{\sqrt{2}}\right)\right]^2. \tag{5.36}$$

Finally,

$$I_3(p) = \frac{1}{2\sqrt{2}}\int_0^p x^2e^{-x^2/2}\text{erf}\left(\frac{x}{\sqrt{2}}\right)\,dx. \tag{5.37}$$

The change of variables $x = \sqrt{2}t$ and integration by parts gives

$$I_3(p) = \frac{\sqrt{\pi}}{8}p^2\left[\text{erf}\left(\frac{p}{\sqrt{2}}\right)\right]^2 - \frac{\sqrt{\pi}}{2}H(p). \tag{5.38}$$

with

$$H(p) = \int_0^{p/\sqrt{2}} t\,(\text{erf } t)^2\,dt, \tag{5.39}$$

Integrate by parts with $u = (\text{erf } t)^2$ and $dv = t\,dt$ to obtain

$$H(p) = \frac{2}{\sqrt{\pi}}t^2e^{-t^2}\ \text{erf } t\Big|_{t=0}^{p/\sqrt{2}} - 2\int_0^{p/\sqrt{2}} t^2e^{-t^2}\ \text{erf } t\,dt. \tag{5.40}$$

The last integral is evaluated by integration by parts to produce

$$I_3(p) = \frac{1}{4\sqrt{\pi}} - \frac{e^{-p^2}}{4\sqrt{\pi}} - \frac{1}{2\sqrt{2}}e^{-p^2/2}\text{erf}\left(\frac{p}{\sqrt{2}}\right) + \frac{\sqrt{\pi}}{8}\text{erf}^2\left(\frac{p}{\sqrt{2}}\right). \tag{5.41}$$

Combining the values of $I_j(p)$, $1 \leqslant j \leqslant 3$, shows that $A(p)$ has the value (5.25).

5.4. Entry 8.250.9. As before, the entry appearing in GR is not correct. The correct version is

$$\int_{(a+b)/\sqrt{2}}^{(b-a)/\sqrt{2}} \exp(-x^2)\text{erf}(b\sqrt{2}-x)\,dx + \int_{(a+b)/\sqrt{2}}^{(a-b)/\sqrt{2}} \exp(-x^2)\text{erf}(a\sqrt{2}-x)\,dx = -\sqrt{\pi}\,\text{erf}(a)\,\text{erf}(b) \tag{5.42}$$

`The proof`. The first part of the proof is to differentiate both sides of (5.42) with respect to b. If the derivatives match, then the identity follows since both sides vanish at $b=0$.

Denote the integrals in (5.42) as $I_1(a,b)$ and $I_2(a,b)$.

`Differentiation of` $I_1(a,b)$. A direct differentiation yields

$$\frac{\partial I_1}{\partial b} = \frac{2\sqrt{2}}{\sqrt{\pi}} \int_{\frac{a+b}{\sqrt{2}}}^{\frac{a-b}{\sqrt{2}}} e^{-x^2} e^{-(b\sqrt{2}-x)^2}\,dx - \frac{1}{\sqrt{2}} \exp\left(-\left(\frac{a+b}{\sqrt{2}}\right)^2\right) \text{erf}\left(\frac{b-a}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} \exp\left(-\left(\frac{b-a}{\sqrt{2}}\right)^2\right) \left(\frac{b+a}{\sqrt{2}}\right). \tag{5.43}$$

`Differentiation of` $I_2(a,b)$. Similarly

$$\frac{\partial I_2}{\partial b} = -\frac{1}{\sqrt{2}} \exp\left(-\left(\frac{a+b}{\sqrt{2}}\right)^2\right) \text{erf}\left(\frac{a-b}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}} \exp\left(-\left(\frac{a-b}{\sqrt{2}}\right)^2\right) \text{erf}\left(\frac{a+b}{\sqrt{2}}\right). \tag{5.44}$$

To compute the derivative of $I = I_1 + I_2$ observe that the terms without the integral cancel and use the identity

$$-x^2 - (b\sqrt{2}-x)^2 = -b^2 - 2\left(x - b/\sqrt{2}\right)^2 \tag{5.45}$$

to verify that

$$\begin{aligned} \frac{\partial I(a,b)}{\partial b} &= \frac{\partial I_1(a,b)}{\partial b} + \frac{\partial I_2(a,b)}{\partial b} \\ &= -2e^{-b^2}\text{erf}(a). \end{aligned} \tag{5.46}$$

This matches with

$$\frac{\partial}{\partial b}\left(-\sqrt{\pi}\,\text{erf}(a)\,\text{erf}(b)\right) \tag{5.47}$$

and, as stated at the beginning, this completes the proof.

6. The last example

This section discusses entry **8.250.6** in the 8^{th} edition of GR. It states that

$$\int_0^\infty \frac{e^{-(p+x)y}}{\pi(p+x)} \cos(a\sqrt{x})\, dx = \frac{1}{\sqrt{\pi y}} \exp\left(-\frac{a^2}{4y} - py\right) - \frac{\sqrt{p}}{2} e^{-a\sqrt{p}} \Phi\left(\frac{a}{a\sqrt{y}} - \sqrt{py}\right) + \frac{\sqrt{p}}{2} e^{\sqrt{p}} \Phi\left(\frac{a}{2\sqrt{y}} + \sqrt{py}\right) - \sqrt{p}\cosh(a\sqrt{p}). \tag{6.1}$$

There are some clear typos seen just by looking at this expression. The first attempt to correct this expression is to write it as

$$\int_0^\infty \frac{e^{-(p+x)y}}{\pi(p+x)} \cos(a\sqrt{x})\, dx = \frac{1}{\sqrt{\pi y}} \exp\left(-\frac{a^2}{4y} - py\right) - \frac{\sqrt{p}}{2} e^{-a\sqrt{p}} \Phi\left(\frac{a}{\mathbf{2}\sqrt{y}} - \sqrt{py}\right) + \frac{\sqrt{p}}{2} e^{\mathbf{a}\sqrt{p}} \Phi\left(\frac{a}{2\sqrt{y}} + \sqrt{py}\right) - \sqrt{p}\cosh(a\sqrt{p}). \tag{6.2}$$

A numerical test of this expression shows that it is not correct. Following the changes presented in the previous section, we change Φ by `erf` to produce

$$\int_0^\infty \frac{e^{-(p+x)y}}{\pi(p+x)} \cos(a\sqrt{x})\, dx = \frac{1}{\sqrt{\pi y}} \exp\left(-\frac{a^2}{4y} - py\right) - \frac{\sqrt{p}}{2} e^{-a\sqrt{p}} \texttt{erf}\left(\frac{a}{2\sqrt{y}} - \sqrt{py}\right) + \frac{\sqrt{p}}{2} e^{a\sqrt{p}} \texttt{erf}\left(\frac{a}{2\sqrt{y}} + \sqrt{py}\right) - \sqrt{p}\cosh(a\sqrt{p}). \tag{6.3}$$

A numerical evaluation shows that these formulas do not match. For example, for $p = 1/7$, $y = 1/10$, $a = 1/2$, the left-hand side is 0.870561 and the right-hand side is 0.636928. A different approach is required.

The goal of this section is to evaluate the integral

$$\begin{aligned} I(a,p) &= \int_0^\infty \frac{e^{-(p+x)y}}{\pi(p+x)} \cos(a\sqrt{x})\, dx \\ &= \frac{e^{-py}}{\pi} H(a,p), \end{aligned} \tag{6.4}$$

with

$$H(a,p) = \int_0^\infty \frac{e^{-xy}\cos(a\sqrt{x})}{p+x}\, dx. \tag{6.5}$$

Make the change of variables $x = pt$ and introduce the new parameters

$$A = a\sqrt{p}, \;\; B = py, \tag{6.6}$$

to obtain

$$J(A,B) = \int_0^\infty \frac{e^{-Bt}\,\cos(A\sqrt{t})}{1+t}\, dt. \tag{6.7}$$

(The integral $J(A,B)$ is $H(a,p)$ in the new parameters A, B).

6.1. The method of brackets. The proof of the identity (6.4) is based on a new method of integration developed by the first author in [**10**]. Given a function f admitting an expansion of the form

$$f(x) = \sum_{n=0}^{\infty} \phi_n C(n) x^{\alpha n + \beta - 1}, \tag{6.8}$$

where $\phi_n = (-1)^n/n!, C(n) \in \mathbb{C}$ and $\alpha,\, \beta \in \mathbb{R}$. The method introduces the notion of `bracket`, defined by the divergent integral

$$\langle a \rangle = \int_0^{\infty} x^{a-1}\, dx \tag{6.9}$$

and formal integration gives the `bracket series`

$$\int_0^{\infty} f(x) dx = \sum_{n=0}^{\infty} \phi_n C(n) \langle \alpha n + \beta \rangle. \tag{6.10}$$

A small number of operational rules gives the value of the integral.

In principle these rules are heuristic. A rigorous justification of all the rules for brackets is in preparation [**8**].

6.2. Rules for the method of brackets. The method of brackets is based on the method of negative dimension. For background, the reader will find in [**23**] a detailed calculation of an integral corresponding to the two-loop massless sunset diagram illustrating the method of negative dimension and in [**12**] an introduction to the method of brackets and its applications.

The expression (6.10) shows, at a formal level, that the evaluation of the definite integral I is given by a bracket series. The goal of this section is to enumerate a small list of rules to evaluate this series. For reasons that will become immediately clear, the symbol C_n is written as $C(n)$.

`Rule` E_1. The bracket series in (6.10) is assigned a number via

$$\sum_n \phi_n C(n) \langle \alpha n + \beta \rangle \mapsto \frac{1}{|\alpha|} C(n^*) \Gamma(-n^*), \tag{6.11}$$

where n^* is the unique solution of the equation $\alpha n + \beta = 0$; that is, $n^* = -\beta/\alpha$.

REMARK 6.1. An important point is that in Rule E_1, the value of the bracket series is given by $C(n^*)$. Initially the function C is defined only at the positive integers and since, in general $n^* \notin \mathbb{N}$, an extension of C is needed for the application of this rule.

In all the bracket series arising from integration one obtains multi-dimensional sums where the number of sums is at least the number of brackets. The difference is called the `index of the representation`, or simply the `index of the sum`; that is,

$$\text{index of a sum} = \text{number of sums} - \text{number of brackets}.$$

For example, the bracket series in Rule E_1 has index 0.

The second rule is about the evaluation of a multi-dimensional bracket series of index 0.

`Rule` E_2. The multi-dimensional bracket series of index 0, with the notation $\phi_{n_1\cdots n_k} = \phi_{n_1}\cdots\phi_{n_k}$,

$$\sum_{n_1,\cdots,n_k} \phi_{n_1\cdots n_k} C(n_1,\cdots,n_k)\langle \alpha_{11}n_1+\cdots+\alpha_{1k}n_k+\beta_1\rangle\cdots\langle \alpha_{k1}n_1+\cdots+\alpha_{kk}n_k+\beta_k\rangle,$$

is assigned the value

$$\frac{1}{|\det(A)|} C(n_1^*,\cdots,n_k^*)\prod_{j=1}^{k}\Gamma(-n_j^*). \tag{6.12}$$

Here A is the $k\times k$ matrix with entries α_{ij} and $(n_1^*,\ldots,n_k^*)$ is the unique solution of the linear system $A\vec{n}+\vec{\beta}=0$, indicating the vanishing of the brackets. The method does not apply if the matrix A is not invertible.

The next rule deals with the situation of a brackets series of positive index.

`Rule` E_3. Assume a bracket series has k series and b brackets, with positive index $i=k-b$. Among the k indices n_j, every choice of i of them produces a brackets series with $k-i=b$ sums and b brackets. This gives a collection of bracket series of index zero. These will be called `basis series`.

The value of the integral is obtained by summing the basis series which converge in a common region.

The next rule is of a different type: it allows to convert a multinomial term appearing in the integrand into a bracket series.

`Rule` E_4. `Multinomial expansion rule`. For $\alpha\in\mathbb{R}$, the expression

$$(a_1+a_2+\cdots+a_r)^{-\alpha} \tag{6.13}$$

is assigned the brackets series

$$\sum_{n_1\cdots n_r}\phi_{1,2,\cdots,r}\, a_1^{n_1}\cdots a_r^{n_r}\frac{\langle\alpha+n_1+\cdots+n_r\rangle}{\Gamma(\alpha)}, \tag{6.14}$$

where $\phi_{1,2,\cdots,r}$ is a short-hand notation for the product $\phi_{n_1}\cdots\phi_{n_r}$. Therefore the sum of r terms in (6.13) is assigned an r-dimensional bracket series.

An interesting connection with the method of brackets and Mellin transforms was established in [**11**]:

THEOREM 6.1. *Let*

$$\varphi(s)=\int_0^\infty x^{s-1}f(x)\,dx \tag{6.15}$$

be the Mellin transform of a function $f(x)$*. Then, for any choice of* $\alpha,\beta\in\mathbb{R}$*, the function* f *admits an expansion of the form*

$$f(x)=\sum_{n=0}^{\infty}\phi_n C(n)x^{\alpha n+\beta} \tag{6.16}$$

where the coefficient $C(n)$ *is given by*

$$C(n)=\frac{|\alpha|\varphi(-(\alpha n+\beta))}{\Gamma(-n)}. \tag{6.17}$$

In order to evaluate (6.4), start with entry **3.383.10** in [**16**]:

$$\varphi(s) = \int_0^\infty x^{s-1} \frac{e^{-\mu x}}{x+b} dx = b^{s-1} e^{b\mu} \Gamma(s) \Gamma(1-s, b\mu). \tag{6.18}$$

Theorem 6.1, with the choice $\alpha = 1$, $\beta = 0$, gives the representation

$$\frac{e^{-\mu x}}{b+x} = \frac{\exp(b\mu)}{b} \sum_{n=0}^\infty \phi_n b^{-n} \Gamma(n+1, b\mu) x^n. \tag{6.19}$$

To evaluate (6.4), take $\mu = B$ and $b = 1$, to produce the bracket series expansion

$$\frac{\exp(-Bt)}{1+t} = e^B \sum_{n=0}^\infty \phi_n \Gamma(n+1, B) t^n \tag{6.20}$$

and

$$\cos(a\sqrt{x}) = \sqrt{\pi} \sum_{m=0}^\infty \phi_m \frac{a^{2m}}{4^m \Gamma(m+\frac{1}{2})} x^m. \tag{6.21}$$

Replacing in (6.7) gives the bracket series

$$J(A,B) = \sqrt{\pi} e^B \sum_n \sum_m \phi_{n,m} \frac{\Gamma(n+1,B) A^{2m}}{2^{2m} \Gamma(m+\frac{1}{2})} \langle n+m+1 \rangle. \tag{6.22}$$

The method of brackets now produces

$$J_1 = \sqrt{\pi} e^B \sum_{m=0}^\infty \frac{\Gamma(-m,B)}{\Gamma(m+\frac{1}{2})} \left(-\frac{A^2}{4} \right)^m \tag{6.23}$$

and

$$J_2 = \frac{4\sqrt{\pi} e^B}{A^2} \sum_{n=0}^\infty \frac{\Gamma(n+1,B)}{\Gamma(-n-\frac{1}{2})} \left(-\frac{4}{A^2} \right)^n. \tag{6.24}$$

These expressions are now simplified using the formulas

$$\Gamma(-m,B) = \frac{(-1)^{m+1}}{m!} \left[\frac{e^{-B}}{B} \sum_{k=0}^{n-1} \frac{k!}{(-B)^k} + \text{Ei}(-B) \right] \tag{6.25}$$

and

$$\Gamma(n+1,B) = n! e^{-B} \sum_{k=0}^n \frac{B^k}{k!}. \tag{6.26}$$

These appear in Chapter 45 of Spanier-Oldham [**22**].

The simplified form of J_1 is

$$J_1 = -\frac{1}{B} \sum_{m=0}^\infty \frac{A^{2m}}{(2m)!} \sum_{j=0}^{m-1} \frac{j!}{(-B)^j} - e^B \text{Ei}(-B) \cosh A \tag{6.27}$$

and

$$J_2 = -\frac{2}{A^2}\sum_{n=0}^{\infty}\frac{(2n+1)!}{A^{2n}}\sum_{k=0}^{n}\frac{B^k}{k!}. \tag{6.28}$$

As $n \to \infty$, the estimate

$$\sum_{k=0}^{n}\frac{B^k}{k!} \geqslant \frac{1}{2}e^B \tag{6.29}$$

shows that J_2 diverges.

The conclusion is that

$$\int_0^\infty \frac{e^{-Bt}\cos(A\sqrt{t})}{1+t}dt = -\frac{1}{B}\sum_{m=0}^{\infty}\frac{A^{2m}}{(2m)!}\sum_{j=0}^{m-1}\frac{j!}{(-B)^j} - e^B\text{Ei}(-B)\cosh A \tag{6.30}$$

Therefore, the original integral in Entry 8.250.6 is given by

$$\int_0^\infty \frac{e^{-(p+x)y}}{\pi(p+x)}\cos(a\sqrt{x})\,dx = -\frac{1}{\pi}\frac{e^{-py}}{py}\sum_{m=0}^{\infty}\frac{(a^2p)^m}{(2m)!}\sum_{j=0}^{m-1}\frac{j!}{(-py)^j} - \frac{1}{\pi}\text{Ei}(-py)\cosh(a\sqrt{p}). \tag{6.31}$$

This completes the discussion.

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