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A note on the Laplace transform of $|\sin(x)/x|^p$.

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Abstract. This note concerns the derivation of several sum and integral identities relating to the Laplace transform

$$\int_0^\infty e^{-ax}\left|\frac{\sin(x)}{x}\right|^p dx$$

and related integrals.

1. Introduction

The function $\operatorname{sinc}(x) = \sin(x)/x$ occurs frequently in applications such as approximation theory and computer graphics , and, lately, interest in its purely mathematical character has been increasing [**1, 2**]. This note is devoted to a brief examination of the Laplace transform,

$$J(a,p) = \int_0^\infty e^{-ax}\,|\operatorname{sinc}(x)|^p\,dx = \int_0^\infty e^{-ax}\left|\frac{\sin(x)}{x}\right|^p \tag{1.1}$$

for $\operatorname{Re} p \geqslant 0$, for which only the cases $p = 0, 2$ appear to be known [**4**].

First we decompose the range of integration into the intervals $[n\pi, (n+1)\pi]$ $n = 0, 1, \ldots$ and in the n-th interval let $x \to x + n\pi$. Next we "exponentiate" the denominator by means of the integral representation for the Gamma function to get

$$J(a,p) = \frac{1}{\Gamma(p)}\sum_{n=0}^\infty \int_0^\infty ds\ s^{p-1}\int_0^\pi dx\ e^{-n\pi(a+s)}e^{-(a+s)x}\sin^p(x). \tag{1.2}$$

The next step is to sum the geometric series and introduce $s + a = u$. Then scaling a out of the $u-$integral gives us

$$J(a,p) = \frac{a^p}{\Gamma(p)}\int_1^\infty du\frac{(u-1)^{p-1}}{1-e^{-\pi au}}\int_0^\pi dx\ e^{-aux}\sin^p(x) \tag{1.3}$$

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However,

$$\int_0^\pi e^{-cx}\sin^p(x)dx = \frac{\pi}{2^p}\frac{\Gamma(p+1)e^{-\pi c/2}}{|\Gamma(1+\frac{1}{2}p+i\frac{c}{2})|^2} \tag{1.4}$$

yielding the desired representation

$$J(a,p) = \frac{\pi p}{2}\left(\frac{a}{2}\right)^p \int_1^\infty du \frac{(u-1)^{p-1}}{\sinh(\pi au/2)|\Gamma(1+\frac{1}{2}p+i\frac{au}{2})|^2}. \tag{1.5}$$

and

$$J(0,p) = \frac{\pi p}{2}\int_0^\infty dx \frac{x^{p-1}}{\sinh(\pi x)|\Gamma(1+\frac{1}{2}p+ix)|^2}. \tag{1.6}$$

As an application of (1.5) we shall derive a closed form expression for $p = 2n$, $n \in \mathbb{Z}^+$ for which only the case $n = 1$ appears in standard tables, such as [**4**]. By iterating the functional equation for the Gamma function and noting that $|\Gamma(1+ix)|^2 = \pi x/\sinh(\pi x)$, we find

$$B_n(a) = \int_0^\infty e^{-ax}\left(\frac{\sin(x)}{x}\right)^{2n} dx \tag{1.7}$$

$$= n\left(\frac{a}{2}\right)^{2n-1}\int_1^\infty \frac{du}{u}\frac{(u-1)^{2n-1}}{\prod_{k=1}^n (k^2+a^2u^2/4)} \tag{1.8}$$

The natural next step is to apply the partial fraction decomposition

$$\left[\prod_{k=1}^n (k^2+a^2u^2/4)\right]^{-1} = \sum_{k=1}^n \frac{A_k(n)}{a^2u^2/4+k^2} \tag{1.9}$$

where

$$A_k(n) = \prod_{j=1}^{n}{}' \frac{1}{j^2-k^2} \tag{1.10}$$

and the prime denotes $j \neq k$. Here one encounters a problem, however, since the individual integrals do not converge at infinity for $n > 1$. Therefore we introduce a finite upper limit and write

$$B_n(a) = n\left(\frac{a}{2}\right)^{2n-3}\lim_{g\to\infty}\sum_{k=1}^n A_k(n)\int_1^{g+1}\frac{du}{u}\frac{(u-1)^{2n-1}}{u^2+(2k/a)^2}. \tag{1.11}$$

The $u-$integral is elementary

$$(1.12)\quad b^2\int_1^{g+1}\frac{(u-1)^{2n-1}}{u^2+b^2}\frac{du}{u}=$$
$$\mathrm{Re}\left[(1-ib)^{2n-1}\{\mathrm{Log}\left(1+\frac{g}{1-ib}\right)+\sum_{l=1}^{2n-1}\frac{(-1)^l}{l}\frac{g^l}{(1-ib)^l}\}\right]$$
$$-\mathrm{Log}(1+g)-\sum_{l=1}^{2n-1}(-1)^l\frac{g^l}{l}.$$

Since the left hand side of (1.11) is finite, all terms on the right hand side of (1.11) containing positive powers of g must cancel out; this gives the identities $\sum_{k=1}^{n}k^{2p}A_k(n)=0,\ \ p=0,1,\dots,n-1$. With the divergent terms eliminated (1.12) becomes

$$(1.13)\quad B_n(a)=\frac{n}{2}\left(\frac{a}{2}\right)^{2n-1}\sum_{k=1}^{n}\frac{A_k(n)}{k^2}\left[s_1(a)\ln(1+4k^2/a^2)+2s_2(a)\tan^{-1}(2k/a)\right]$$

where

$$(1.14)\quad \begin{aligned} s_1(a) &= \sum_{l=0}^{n-1}(-1)^{l+1}\binom{2n-1}{2l}(2k/a)^{2l}\\ s_2(a) &= \sum_{l=1}^{n}(-1)^{l+1}\binom{2n-1}{2l-1}(2k/a)^{2l-1}.\end{aligned}$$

In particular, for $n=2$, (1.13) gives

$$(1.15)\quad B_2(a)=\frac{1}{96}[16(3a^2-4)\tan^{-1}(2/a)-8(3a^2-16)\tan^{-1}(4/a)-$$
$$4a(a^2-12)\ln(1+4/a^2)+a(a^2-48)\ln(1+16/a^2)].$$

For $a\to 0$, only the last term in $s_2(a)$ contributes to (1.12) yielding

$$(1.16)\quad \begin{aligned}\int_0^\infty dx\left(\frac{\sin\ x}{x}\right)^{2n} &= n\int_0^\infty\frac{x^{2(n-1)}}{\prod_{k=1}^{n}(x^2+k^2)}dx\\ &= (-1)^{n+1}\frac{\pi n}{2}\sum_{k=1}^{n}k^{2n-3}A_k(n).\end{aligned}$$

Thus we have the interesting identity [**2**]

$$(1.17)\quad \sum_{k=1}^{n}k^{2n-3}A_k(n)=(-1)^{n+1}\frac{2}{(2n)!}\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}(n-k)^{2n-1}.$$

For odd p, in a similar way one finds

$$(1.18)\quad J(a,2n+1)=(2n+1)a^{2n+1}\int_1^\infty du\frac{(u-1)^{2n}\coth(nau/2)}{\prod_{k=0}^{n}[(2k+1)^2+a^2u^2]}$$

which appears to be intractable even for $n=0$.

Finally, we examine the related integrals

$$C_n(y) = \int_0^\infty \cos(xy)\left(\frac{\sin\ x}{x}\right)^{2n} dx \tag{1.19}$$

$$D_n(x) = \int_0^\infty \sin(xy)\left(\frac{\sin\ x}{x}\right)^{2n} dx$$

the first of which is given incorrectly in [**3, 4**]. By setting $a = iy$ in (1.13), assuming y is real, and separating the real and imaginary parts, we find, θ denoting the unit step function,

$$C_n(y) = (-1)^{n+1}\frac{n\pi}{2}\left(\frac{y}{2}\right)^{2n-1}\sum_{k=1}^{n}\frac{A_k(n)}{k^2}\theta(2k-y) \left[\sum_{l=1}^{n}\binom{2n-1}{2l-1}(2k/y)^{2l-1} - \sum_{l=0}^{n-1}\binom{2n-1}{2l}(2k/y)^{2l}\right] \tag{1.20}$$

and

(1.21)

$$D_n(y) = (-1)^{n+1}\frac{n}{2}\left(\frac{y}{2}\right)^{2n-1}\sum_{k=1}^{n}\frac{A_k(n)}{k^2}\left[\ln|1-4k^2/y^2|\sum_{l=0}^{n-1}\binom{2n-1}{2l}(2k/y)^{2l} + \ln\left|\frac{y+2k}{y-2k}\right|\sum_{l=1}^{n}\binom{2n-1}{2l-1}(2k/y)^{2l-1}\right].$$

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