SCIENTIA
Series A: *Mathematical Sciences*, Vol. 37 (2026), 45–68
e-ISSN 2810-7519
DOI: https://doi.org/10.71712/spxv-gv83

The integrals in Gradshteyn and Ryzhik. Part 36: Integrals containing hyperbolic functions

Julian Huddell and Victor H. Moll

Abstract. This paper is part of a series created with the goal of providing proofs for all the entries in the table of integrals by I. S. Gradshteyn and I. M. Ryzhik. Entries containing hyperbolic functions in the integrand are considered here.

1. Introduction

The hyperbolic functions form part of the class of elementary functions. The work presented here is part of the project consisting of proving all entries in the table by I. S. Gradshteyn and I. M. Ryzhik [**4**]. In the current paper we provide proofs of some entries containing hyperbolic functions in the integrand. This is a continuation of [**2**] where some of these evaluations were discussed.

The two fundamental functions, hyperbolic sine and cosine are defined by

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \quad \text{and} \quad \cosh x = \frac{1}{2}(e^x + e^{-x}). \tag{1.1}$$

In particular this suggests that the hyperbolic part of the integrand in any of the entries discussed here can be expressed purely in terms of the exponential functions.

2. Some reduction formulas

The first entry appears as **2.411**:

2000 *Mathematics Subject Classification.* Primary 33.
Key words and phrases. Integrals, hyperbolic functions.

$$\begin{aligned}
\int \sinh^p x \cosh^q x\, dx &= \frac{\sinh^{p+1} x \cosh^{q-1} x}{p+q} + \frac{q-1}{p+q} \int \sinh^p x \cosh^{q-2}\, dx \\
&= \frac{\sinh^{p-1} x \cosh^{q+1} x}{p+q} - \frac{p-1}{p+q} \int \sinh^{p-2} x \cosh^{q}\, dx \\
&= \frac{\sinh^{p-1} x \cosh^{q+1} x}{q+1} - \frac{p-1}{q+1} \int \sinh^{p-2} x \cosh^{q+2}\, dx \\
&= \frac{\sinh^{p+1} x \cosh^{q-1} x}{p+1} - \frac{q-1}{p+1} \int \sinh^{p+2} x \cosh^{q-2}\, dx \\
&= \frac{\sinh^{p+1} x \cosh^{q+1} x}{p+1} - \frac{p+q+2}{p+1} \int \sinh^{p+2} x \cosh^{q}\, dx \\
&= -\frac{\sinh^{p+1} x \cosh^{q+1} x}{q+1} + \frac{p+q+2}{q+1} \int \sinh^{p} x \cosh^{q+2}\, dx.
\end{aligned}$$

Proof. `First identity`. Begin with the expression

$$\frac{d}{dx}(\sinh^{p+1} x \cosh^{q-1} x) + (q-1)\sinh^p x \cosh^{q-2} x \tag{2.1}$$

that simplifies to

$$\begin{aligned}
&(p+1)\sinh^p x \cosh^q x + (q-1)\sinh^{p+2} x \cosh^{q-2} x + (q-1)\sinh^p x \cosh^{q-2} x \\
&= (p+1)\sinh^p x \cosh^q x + (q-1)\sinh^p x \cosh^{q-2} x(\sinh^2 x + 1) \\
&= (p+1)\sinh^p x \cosh^q x + (q-1)\sinh^p x \cosh^q x \\
&= (p+q)\sinh^p x \cosh^q x.
\end{aligned} \tag{2.2}$$

This leads to the identity

$$(p+q)\sinh^p x \cosh^q x = \frac{d}{dx}(\sinh^{p+1} x \cosh^{q-1} x) + (q-1)\sinh^p x \cosh^{q-2} x. \tag{2.3}$$

The first identity above now follows by integration.

`Second identity`. The proof is similar as the one above, this time starting with

$$\frac{d}{dx}(\sinh^{p-1} x \cosh^{q+1} x) - (p-1)\sinh^{p-2} x \cosh^q x = (p+q)\sinh^p x \cosh^q x \tag{2.4}$$

and then integrating.

`Third identity`. The result follows by integration by parts with $u = \sinh^{p-1} x$ and $v = \frac{1}{q+1}\cosh^{q+1} x$.

`Fourth identity`. The result follows by integration by parts with $u = \cosh^{q-1} x$ and $v = \frac{1}{p+1}\sinh^{q+1} x$.

`Fifth identity`. This follows by integrating the identity

$$\frac{d}{dx}(\sinh^{p+1} x \cosh^{q+1} x) = (p+1)\sinh^p x \cosh^{q+1} x + (p+q+2)\sinh^{p+2} x \cosh^q x. \tag{2.5}$$

`Sixth identity`. This follows by integrating the identity

$$\frac{d}{dx}(\sinh^{p+1} x \cosh^{q+1} x) = (q+1)\sinh^p x \cosh^q x + (p+q+2)\sinh^p x \cosh^{q+2} x. \tag{2.6}$$

□

3. Combination of three terms

3.1. Entry 3.511.5.

$$\int_0^\infty \frac{\sinh ax \, \cosh bx}{\sinh x} \, dx = \frac{\pi}{2} \frac{\sin \pi a}{\cos \pi a + \cos \pi b}, \tag{3.1}$$

provided $|a| + |b| < 1$.

Proof. Rewrite the integrand in terms of exponentials gives

$$I = \int_0^\infty \frac{\sinh ax \, \cosh bx}{\sinh x} \, dx = \frac{1}{2} \int_0^\infty \frac{e^{-x}(e^{ax} - e^{-ax})(e^{bx} + e^{-bx})}{1 - e^{-2x}} \, dx. \tag{3.2}$$

Interpret the denominator as the sum of a geometric series of ratio $r = e^{-2x}$ to produce

$$2I = \int_0^\infty \left(-e^{-(a+b+2n+1)x} + e^{-(-a+b+2n+1)x} - e^{-(a-b+2n+1)x} + e^{-(-a-b+2n+1)x} \right) dx$$

where the series converges uniformly if $|a| + |b| < 1$. Evaluating the integrals yields

$$2I = \sum_{n=0}^\infty \left(-\frac{1}{a+b+(2n-1)} - \frac{1}{a+b-(2n+1)} \right) + \sum_{n=0}^\infty \left(-\frac{1}{a-b+(2n-1)} - \frac{1}{a-b-(2n+1)} \right)$$

and the partial decomposition of tangent

$$\frac{\pi}{2} \tan\left(\frac{\pi z}{2} \right) = \sum_{n=0}^\infty \left(-\frac{1}{z+(2n+1)} - \frac{1}{z-(2n+1)} \right) \tag{3.3}$$

gives the result. □

3.2. Entry 3.511.6.

$$\int_0^\infty \frac{\cosh ax \cosh bx}{\cosh x} \, dx = \pi \frac{\cos \frac{\pi a}{2} \, \cos \frac{\pi b}{2}}{\cos \pi a + \cos \pi b} \tag{3.4}$$

provided $|a| + |b| < 1$.

`Note`. The entry in Edition 8^{th} has an error. It was stated as

$$\int_0^\infty \frac{\cosh ax \cosh bx}{\cosh x} \, dx = \pi \frac{\sin \frac{\pi a}{2} \, \cos \frac{\pi b}{2}}{\cos \pi a + \cos \pi b}. \tag{3.5}$$

This was corrected in edition 9^{th}.

Proof. Write the integrand in terms of exponentials as

$$\begin{aligned} I &= \frac{1}{2}\int_0^\infty \frac{(e^{ax}+e^{-ax})(e^{bx}+e^{-bx})}{e^x+e^{-x}}\,dx \\ &= \frac{1}{2}\int_0^\infty e^{-x}\frac{(e^{ax}+e^{-ax})(e^{bx}+e^{-bx})}{1+e^{-2x}}\,dx. \end{aligned} \tag{3.6}$$

Now identify the integrand as the sum of geometric series with ratio $r = -e^{-2x}$ to obtain

$$\begin{aligned} I &= \frac{1}{2}\int_0^\infty \sum_{n=0}^\infty (-1)^n e^{-x}(e^{ax}+e^{-ax})(e^{bx}+e^{-bx})e^{-2nx}\,dx \\ &= \frac{1}{2}\int_0^\infty \sum_{n=0}^\infty \left(e^{-(a+b+2n+1)x}+e^{-(-a+b+2n+1)x}+e^{-(a-b+2n+1)x}+e^{-(-a-b+2n+1)x}\right)\,dx. \end{aligned} \tag{3.7}$$

The condition $|a|+|b|<1$ guarantees that the integral and the series can be exchanged leading to

$$\begin{aligned} I &= \frac{1}{2}\sum_{n=0}^\infty (-1)^n \int_0^\infty \left(e^{-(a+b+2n+1)x}+e^{-(-a+b+2n+1)x}+e^{-(a-b+2n+1)x}+e^{-(-a-b+2n+1)x}\right)\,dx \\ &= \frac{1}{2}\sum_{n=0}^\infty (-1)^n \left(\frac{1}{a+b+2n+1}+\frac{1}{-a+b+2n+1}+\frac{1}{a-b+2n+1}+\frac{1}{-a-b+2n+1}\right) \\ &= \frac{1}{2}\sum_{n=0}^\infty (-1)^n \left(\frac{1}{a+b+(2n+1)}-\frac{1}{a+b-(2n+1)}\right) \\ &\quad + \frac{1}{2}\sum_{n=0}^\infty (-1)^n \left(\frac{1}{a-b+(2n+1)}-\frac{1}{a-b-(2n+1)}\right). \end{aligned} \tag{3.8}$$

Now use the identity

$$\frac{\pi}{2}\sec\left(\frac{\pi z}{2}\right) = \sum_{n=0}^\infty (-1)^n \left(\frac{1}{z+(2n+1)}-\frac{1}{z-(2n+1)}\right) \tag{3.9}$$

to obtain

$$I = \frac{\pi}{4}\left[\sec\left(\frac{\pi(a+b)}{2}\right)+\sec\left(\frac{\pi(a-b)}{2}\right)\right]. \tag{3.10}$$

In order to simplify this, let

$$u = \tfrac{\pi}{2}(a+b), \quad \text{and } v = \tfrac{\pi}{2}(a-b) \tag{3.11}$$

to write

$$\begin{aligned} \frac{\pi}{4}(\sec u + \sec v) &= \frac{\pi}{4}\left(\frac{1}{\cos u} + \frac{1}{\cos v} \right) \\ &= \frac{\pi}{4} \frac{\cos u + \cos v}{\cos u \cos v} \\ &= \frac{\pi}{4} \frac{\cos(\frac{\pi a}{2}) + \cos(\frac{\pi a}{2} - \frac{\pi b}{2})}{\cos u \cos v} \\ &= \frac{\pi \cos(\frac{\pi a}{2}) \cos(\frac{\pi b}{2})}{\cos(\pi a) + \cos(\pi b)}. \end{aligned} \tag{3.12}$$

□

3.3. Entry 3.511.7.

$$\int_0^\infty \frac{\sinh ax \sinh bx}{\cosh x} \, dx = \pi \frac{\sin \frac{\pi a}{2} \, \sin \frac{\pi b}{2}}{\cos \pi a + \cos \pi b} \tag{3.13}$$

provided $|a| + |b| < 1$.

PROOF. Write the integrand in terms of exponentials as

$$\begin{aligned} I &= \frac{1}{2} \int_0^\infty \frac{(e^{ax} - e^{-ax})(e^{bx} - e^{-bx})}{e^x + e^{-x}} \, dx \\ &= \frac{1}{2} \int_0^\infty e^{-x} \frac{(e^{ax} - e^{-ax})(e^{bx} - e^{-bx})}{1 + e^{-2x}} \, dx. \end{aligned} \tag{3.14}$$

Now identify the integrand as the sum of geometric series with ratio $r = -e^{-2x}$ to obtain

$$I = \frac{1}{2} \int_0^\infty \sum_{n=0}^\infty (-1)^n e^{-x} (e^{ax} - e^{-ax})(e^{bx} - e^{-bx}) e^{-2nx} \, dx \tag{3.15}$$

$$= \frac{1}{2} \int_0^\infty \sum_{n=0}^\infty \left(e^{-(a+b+2n+1)x} - e^{-(-a+b+2n+1)x} - e^{-(a-b+2n+1)x} + e^{-(-a-b+2n+1)x} \right) dx.$$

The condition $|a|+|b|<1$ guarantees that the integral and the series can be exchanged leading to

$$
\begin{aligned}
I &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \int_0^{\infty} \left(e^{-(a+b+2n+1)x} - e^{-(-a+b+2n+1)x} - e^{-(a-b+2n+1)x} + e^{-(-a-b+2n+1)x} \right) dx \\
&= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \left(\frac{1}{a+b+2n+1} - \frac{1}{-a+b+2n+1} - \frac{1}{a-b+2n+1} + \frac{1}{-a-b+2n+1} \right) \\
&= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \left(\frac{1}{a+b+(2n+1)} - \frac{1}{a+b-(2n+1)} \right) \\
&\quad - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \left(\frac{1}{a-b+(2n+1)} - \frac{1}{a-b-(2n+1)} \right).
\end{aligned} \tag{3.16}
$$

Now use the identity

$$\frac{\pi}{2}\sec\left(\frac{\pi z}{2}\right) = \sum_{n=0}^{\infty}(-1)^n \left(\frac{1}{z+(2n+1)} - \frac{1}{z-(2n+1)} \right) \tag{3.17}$$

to obtain

$$I = \frac{\pi}{4}\left[\sec\left(\frac{\pi(a+b)}{2}\right) + \sec\left(\frac{\pi(a-b)}{2}\right) \right]. \tag{3.18}$$

In order to simplify this, let

$$u = \tfrac{\pi}{2}(a+b), \quad \text{and } \ v = \tfrac{\pi}{2}(a-b) \tag{3.19}$$

to write

$$
\begin{aligned}
\frac{\pi}{4}(\sec u + \sec v) &= \frac{\pi}{4}\left(\frac{1}{\cos u} + \frac{1}{\cos v} \right) \\
&= \frac{\pi}{4}\, \frac{\cos u + \cos v}{\cos u \cos v} \\
&= \frac{\pi}{4}\, \frac{\cos(\frac{\pi a}{2}) + \cos(\frac{\pi a}{2} - \frac{\pi b}{2})}{\cos u \cos v} \\
&= \frac{\pi \cos(\frac{\pi a}{2})\cos(\frac{\pi b}{2})}{\cos(\pi a) + \cos(\pi b)}.
\end{aligned} \tag{3.20}
$$

□

4. Rational linear expressions

This section contain integrands that are quotients of linear expressions in $\sinh x$ and $\cosh x$.

4.1. Entry 3.513.1.

$$\int_0^\infty \frac{dx}{a+b\sinh x} = \frac{1}{\sqrt{a^2+b^2}} \ln \frac{a+b+\sqrt{a^2+b^2}}{a+b-\sqrt{a^2+b^2}} \tag{4.1}$$

provided $ab \neq 0$.

PROOF. The classical Weierstrass substitution, $u = \tan x/2$, used to transform rational functions of $\sin x$, $\cos x$ into rational functions of u, has a hyperbolic analogue. Make the change of variables

$$t = \tanh \frac{x}{2} \tag{4.2}$$

and observe that

$$\sinh x = \frac{2t}{1-t^2}, \quad \text{and} \quad dx = \frac{2\,dt}{1-t^2}. \tag{4.3}$$

This yields

$$\int_0^\infty \frac{dx}{a+b\sinh x} = -2\int_0^1 \frac{dt}{at^2-2bt-a}. \tag{4.4}$$

Integrating the partial fraction decomposition

$$\frac{-2}{at^2-2bt-a} = \frac{1}{c}\left(\frac{1}{t-\frac{b-c}{a}} - \frac{1}{t-\frac{b+c}{a}} \right) \tag{4.5}$$

with $c = \sqrt{a^2+b^2}$ gives the result. □

5. Combinations of hyperbolic functions and algebraic functions

The evaluation of the next entries uses a standard method in complex analysis to evaluate real integrals. The original integral is over a part of the real axis, usually a half-line, it is then extended by symmetry of the integrand to the whole line. A contour on the complex plane is now constructed containing a finite part of the real axis and closed by lines to form a rectangle of finite size. The theorem of residues is then used to evaluate the integral over the boundary of the contour. Passing to the limit to recover the real line is done by using estimates on the integrand. Complete details are given only in the first example in this section.

5.1. Entry 3.522.5.

$$\int_0^\infty \frac{x\,dx}{(1+x^2)\sinh \pi x} = \ln 2 - \frac{1}{2} \tag{5.1}$$

PROOF. The integrand is even and the evaluation of the entry will come by integrating the function

$$f(z) = \frac{z}{(1+z^2)\sinh \pi z} \tag{5.2}$$

around the contour $\Gamma(N)$ being the rectangle with vertices at $\pm\sqrt{N}$ and $\pm\sqrt{N} + \left(N+\frac{1}{2}\right)i$, where N is a positive integer.

To use the residue theorem, we identify the poles of the integrand: these come from the zeros of $\sinh \pi z$, located at $z = ni$, with $n \in \mathbb{Z}$ and the zeros of $1 + z^2$ occurring at $z = \pm i$. It follows that the integrand has simple poles at $z_n = ni$ with $n \in \mathbb{Z}$ and $n \neq \pm 1$ and double poles at $z = \pm i$. As $N \to \infty$, the contour will enclose the double pole at $z = i$ and the simple poles at $z = ni$, for $n = 2, 3, \cdots$. The residue theorem yields

$$\lim_{N \to \infty} \int_{\Gamma(N)} \frac{z\,dz}{(1+z^2)\sinh \pi z} = 2\pi i \left(\text{Res}(f(z), z = i) + \sum_{n=2}^{\infty} \text{Res}(f(z), z = ni) \right). \tag{5.3}$$

To evaluate the residue at $z = i$, let $w = z - i$ and consider the expansion of the function $g(w) = f(w + i)$ at $w = 0$. This is

$$\begin{aligned} a_{-2}w^{-2} + a_{-1}w^{-1} + \cdots &= \frac{w+i}{(2iw + w^2)\sinh(\pi i + \pi w)} \\ &= \frac{-i-w}{(2iw+w^2)\sinh(\pi w)} \\ &= \frac{-i-w}{w^2(2i+w)\left[\pi - \mathcal{O}(w^2)\right]}. \end{aligned} \tag{5.4}$$

From here $a_{-2} = -1/2\pi$ and $a_{-1} = i/4\pi$. This gives

$$\text{Res}[f(z), i] = \frac{i}{4\pi}. \tag{5.5}$$

For $n \geqslant 2$, f has a simple pole at $z = ni$. Recall that, for a simple pole at $z = z_0$,

$$\text{Res}\left(\frac{g(z)}{h(z)}, z_0 \right) = \frac{g(z_0)}{h'(z_0)}. \tag{5.6}$$

This gives

$$\text{Res}(f(z), ni) = \frac{(-1)^n ni}{\pi(1-n^2)}. \tag{5.7}$$

The right-hand side of (5.3) becomes

$$2\pi i \left(\frac{i}{4\pi} + \sum_{n=2}^{\infty} \frac{(-1)^n ni}{\pi(1-n^2)} \right) = -2 \left(\frac{1}{4} + \sum_{n=2}^{\infty} \frac{(-1)^n n}{1-n^2} \right). \tag{5.8}$$

Now we examine the left-hand side of (5.3). The vertical segments are located at $x = \pm\sqrt{N}$ and have length $(N + \frac{1}{2})$. Observe that

$$|\sinh(x+iy)| = \sqrt{\sinh^2 x + \sin^2 y} \geqslant |\sinh x|, \tag{5.9}$$

and integrating over the vertical segment with $x = \sqrt{N}$, $0 \leqslant y \leqslant (N + \frac{1}{2})$ and $z = x + iy$, one has the bound

$$\left| \int \frac{z\,dz}{(1+z^2)\sinh \pi z} \right| \leqslant \frac{(N+\frac{1}{2})}{\sinh(\pi\sqrt{N})} \max\left\{ \left| \frac{z}{1+z^2} \right| : z = \sqrt{N} + iy, \, 0 \leqslant y \leqslant (N + \tfrac{1}{2}) \right\}$$

and this goes to 0 as $N \to \infty$. The same analysis works on the vertical segment over $x = -\sqrt{N}$. For the estimate on the top of the rectangle, observe that if $\text{Im}\, z = N + \frac{1}{2}$, then $|\sinh \pi z| \geqslant 1$ and for z on top of the rectangle $|z| \leqslant 2N$ which implies $|1+z^2| \leqslant 4N^2+1$. The length of the segment is $2\sqrt{N}$. Therefore the integral over the top segment is bounded by $4N^{3/2}/(4N^2+1)$, so it vanishes as $N \to \infty$.

The identity (5.3) now produces

$$2\int_0^\infty \frac{x\,dx}{(1+x^2)\sinh \pi x} = -2\left(\frac{1}{4} + \sum_{n=2}^\infty \frac{(-1)^n n}{1-n^2}\right), \tag{5.10}$$

and evaluating the series by partial fractions completes the proof. □

5.2. Entry 3.522.6.

$$\int_0^\infty \frac{dx}{(1+x^2)\,\cosh \pi x} = 2 - \frac{\pi}{2} \tag{5.11}$$

Proof. Let $f(z) = 1/(1+z^2)\cosh \pi z$. As in the previous example construct the rectangle with vertices $\pm N$ and $\pm N + Ni$. The integrand now has simple poles at $z = i$ and $z = \frac{2n+1}{2}i$, for $n \in \mathbb{Z}$. A direct calculation gives

$$\text{Res}[f(z), i] = \frac{i}{2}, \quad \text{and} \quad \text{Res}[f(z), \tfrac{2n+1}{2}i] = \frac{-i(-1)^n}{\pi\left(1 - (\frac{2n+1}{2})^2\right)}. \tag{5.12}$$

This produces

$$\int_0^\infty \frac{dx}{(1+x^2)\cosh \pi x} = -\frac{\pi}{2} + \sum_{n=0}^\infty \frac{(-1)^n}{1 - \frac{(2n+1)^2}{4}}. \tag{5.13}$$

Using partial fractions on the summand gives a telescoping series that gives the final result. □

5.3. Entry 3.522.7.

$$\int_0^\infty \frac{x\,dx}{(1+x^2)\,\sinh \frac{\pi x}{2}} = \frac{\pi}{2} - 1 \tag{5.14}$$

Proof. The proof proceeds as in the previous two examples. Let

$$f(z) = \frac{z}{(1+z^2)\sinh \frac{\pi z}{2}} \tag{5.15}$$

and compute the residues at the poles of f (all are simple)

$$\text{Res}[f(z), i] = -\frac{i}{2}, \quad \text{Res}[f(z), 2ni] = (-1)^n \frac{4ni}{\pi(1-4n^2)}. \tag{5.16}$$

The series appearing from the residue theorem is again a telescoping sum. This completes the proof. □

5.4. Entry 3.522.8.

$$\int_0^\infty \frac{dx}{(1+x^2)\,\cosh\frac{\pi x}{2}} = \ln 2 \tag{5.17}$$

Proof. The proof proceeds as in the previous examples. Now let

$$f(z) = \frac{z}{(1+z^2)\cosh\frac{\pi z}{2}}. \tag{5.18}$$

This function has a double pole at $z = 0$ and simple poles at $(2n+1)i$. A direct calculation produces

$$\text{Res}[f(z), i] = -\frac{i}{2\pi} \text{ and } \text{Res}[f(z),(2n+1)i] = -\frac{2i}{\pi}\frac{(-1)^n}{1-(2n+1)^2}. \tag{5.19}$$

The residue theorem produces a series that can be identified to $\ln 2$. This completes the evaluation. ☐

5.5. Entry 3.522.9.

$$\int_0^\infty \frac{x\,dx}{(1+x^2)\,\sinh\frac{\pi x}{4}} = \frac{1}{\sqrt{2}}\left[\pi + 2\ln(\sqrt{2}+1)\right] - 2 \tag{5.20}$$

Proof. Proceed as before creating a rectangle with vertices $\pm\sqrt{N}$ and $\pm\sqrt{N} + (4N+2)i$, with $N \in \mathbb{N}$. The function

$$f(z) = \frac{z}{(1+z^2)\sinh\frac{\pi z}{4}} \tag{5.21}$$

hass poles at $z = i$ and $z = 4ni$, with $n \in \mathbb{Z}$. These poles are all simple with residues

$$\text{Res}[f(z), i] = -\frac{i}{\sqrt{2}}, \quad \text{and } \text{Res}[f(z), 4ni] = (-1)^n \frac{16ni}{\pi(1-16n^2)}. \tag{5.22}$$

Now use the residue theorem and let $N \to \infty$ to obtain

$$\int_0^\infty \frac{x\,dx}{(1+x^2)\sinh\frac{\pi x}{4}} = \frac{\pi}{\sqrt{2}} + 2\sum_{n=1}^\infty (-1)^n \left(\frac{1}{4n-1} + \frac{1}{4n+1} \right). \tag{5.23}$$

The evaluation of the series is obtained by the analysis of the function

$$f_{\pm}(x) = \sum_{n=1}^\infty \frac{x^n}{4n \pm 1}. \tag{5.24}$$

To simplify the computations, replace x by x^4 and then

$$f_-(x) = \sum_{n=1}^\infty \frac{x^{4n}}{4n+1} = \int_0^x \sum_{n=1}^\infty x^{4n}\,dx. \tag{5.25}$$

The last series is a geometric series. Elementary methods now yield

$$f_-(x) = \frac{x}{2}\left(-\text{arctan}(x) + \text{arctanh}(x)\right). \tag{5.26}$$

Similar arguments for $f_-(x)$ give the final evaluation. ☐

5.6. Entry 3.522.10.

$$\int_0^\infty \frac{dx}{(1+x^2)\cosh\frac{\pi x}{4}} = \frac{1}{\sqrt{2}}\left[\pi - 2\ln(\sqrt{2}+1)\right] \tag{5.27}$$

Proof. The proof is similar as the use presented in Entry 3.522.9 above. □

6. Combinations of hyperbolic functions and rational functions

6.1. Entry 3.526.1.

$$\int_0^\infty \frac{\sinh ax \cosh bx}{\cosh cx}\frac{dx}{x} = \frac{1}{2}\ln\left\{\tan\frac{(a+b+c)\pi}{4c}\cot\frac{(b+c-a)\pi}{4c}\right\} \tag{6.1}$$

provided $c > |a| + |b|$.

Proof. Let $I = I(a,b,c)$ be the integral to be evaluated. Then

$$\frac{\partial}{\partial b}I = \int_0^\infty \frac{\sinh ax \sinh bx}{\cosh cx}\,dx. \tag{6.2}$$

The next step is to use Entry 3.511.7,

$$\int_0^\infty \frac{\sinh ax \sinh bx}{\cosh x}\,dx = \frac{\pi\sin\frac{\pi a}{2}\sin\frac{\pi b}{2}}{\cos\pi a + \cos\pi b}. \tag{6.3}$$

To prove this identity use the representation of the incomplete beta function

$$\beta(a) = \int_0^1 \frac{x^{a-1}\,dx}{1+x} \tag{6.4}$$

appearing as entry 3.541.6 and proved in [**1**]:

$$\int_0^\infty \frac{e^{-ax}\,dx}{\cosh x} = \beta\left(\frac{a+1}{2}\right). \tag{6.5}$$

Expressing the integrand in (6.2) in terms of exponentials, using the change of variables $t = cx$ and the identity

$$\beta(a) + \beta(1-a) = \frac{\pi}{\sin\pi a} \tag{6.6}$$

gives

$$\frac{\partial I}{\partial b} = \frac{A\sin\frac{\pi b}{2a}}{B + \cos\frac{\pi b}{c}} \tag{6.7}$$

with the notation

$$A = \frac{\pi}{c}\sin\frac{\pi a}{2c} \quad \text{and} \quad B = \cos\frac{\pi a}{c}. \tag{6.8}$$

Now integrate with respect to b to obtain

$$I(a,b,c) = \int \frac{A\sin\frac{\pi b}{2c}}{B + \cos\frac{\pi b}{c}}\,db + \omega(a,c), \tag{6.9}$$

where $\omega = \omega(a,c)$ is the constant of integration.

To evaluate the integral, call it $K = K(a,b,c)$, let $r = \pi b/2c$ and make the change of variables $u = \cos r$, to produce

$$K = -\frac{1}{2} \int \frac{du}{u^2 - C^2} \tag{6.10}$$

with $C = \sqrt{(1-B)/2}$ (recall that $|B| \leqslant 1$). Use the method of partial fractions to evaluate this form of K, this yields

$$I(a,b,c) = \frac{1}{2} \ln \left(\frac{\cos\frac{\pi b}{2c} + \sin\frac{\pi a}{2c}}{\cos\frac{\pi b}{2c} - \sin\frac{\pi a}{2c}} \right) + \omega(a,c). \tag{6.11}$$

To evaluate the constant $\omega(a,c)$, take $b = 0$ to obtain

$$\int_0^\infty \frac{\sinh ax}{\cosh cx} \frac{dx}{x} = \frac{1}{2} \ln \left(\frac{1 + \sin\frac{\pi a}{2c}}{1 - \sin\frac{\pi a}{2c}} \right) + \omega(a,c) \tag{6.12}$$

and use Entry 3.524.23, proved in [**2**],

$$\int_0^\infty \frac{\sinh ax}{\cosh cx} \frac{dx}{x} = \ln \tan \left(\frac{\pi a}{4c} + \frac{\pi}{4} \right) \tag{6.13}$$

and this determines $\omega(a,c)$. From here it follows that

$$\begin{aligned} I(a,b,c) \quad = \quad & \frac{1}{2} \ln \left(\frac{\cos\frac{\pi b}{2c} + \sin\frac{\pi a}{2c}}{\cos\frac{\pi b}{2c} - \sin\frac{\pi a}{2c}} \right) \\ & + \ln \tan \left(\frac{\pi a}{4b} + \frac{\pi}{4} \right) - \frac{1}{2} \ln \left(\frac{1 + \sin\frac{\pi a}{2c}}{1 - \sin\frac{\pi a}{2c}} \right). \end{aligned} \tag{6.14}$$

Therefore, the proof of the statement in Entry 3.526.1 has been reduced to a trigonometric identity. Introduce the angles

$$\alpha = \frac{\pi b}{2c}, \quad \text{and} \quad \beta = \frac{\pi a}{2c}, \tag{6.15}$$

then we need to prove that

$$\begin{aligned} & \left\{ \frac{\cos\alpha + \sin\beta}{\cos\alpha - \sin\beta} \right\} \times \tan^2 \left(\frac{\beta}{2} + \frac{\pi}{4} \right) = \\ & \qquad \left\{ \frac{1 + \sin\beta}{1 - \sin\beta} \right\} \times \frac{\sin[\frac{1}{2}(\frac{\pi}{2} + \alpha + \beta)] \cdot \cos[\frac{1}{2}(\frac{\pi}{2} + \alpha - \beta)]}{\cos[\frac{1}{2}(\frac{\pi}{2} + \alpha + \beta)] \cdot \sin[\frac{1}{2}(\frac{\pi}{2} + \alpha - \beta)]} \end{aligned} \tag{6.16}$$

The identity

$$\sin x \cos y = \frac{\sin(x+y) + \sin(x-y)}{2} \tag{6.17}$$

transforms the required identity to

$$\tan^2 \left(\frac{\beta}{2} + \frac{\pi}{4} \right) = \frac{1 + \sin\beta}{1 - \sin\beta}. \tag{6.18}$$

This follows directly from the addition theorem for the tangent function. The proof is complete. □

Several entries in the table can be evaluated by the methods presented in this section. Here is a small sample:

6.2. Entry 3.529.1.

$$\int_0^\infty \left(\frac{1}{\sinh x} - \frac{1}{x} \right) \frac{dx}{x} = -\ln 2 \tag{6.19}$$

6.3. Entry 3.529.2.

$$\int_0^\infty \frac{\cosh ax - 1}{\sinh bx} \frac{dx}{x} = -\ln \cos \frac{\pi a}{2b} \tag{6.20}$$

6.4. Entry 3.529.3.

$$\int_0^\infty \left(\frac{a}{\sinh ax} - \frac{b}{\sinh bx} \right) \frac{dx}{x} = (b-a)\ln 2 \tag{6.21}$$

7. Combinations of hyperbolic functions and exponential functions

7.1. Entry 3.545.1.

$$\int_0^\infty \frac{\sinh ax}{e^{px}+1}\, dx = \frac{\pi}{2p} \text{cosec}\, \frac{\pi a}{p} - \frac{1}{2a} \tag{7.1}$$

provided $p > a$, $p > 0$.

Proof. Write the hyperbolic sine in terms of exponentials, expand the term $1/(e^{px}+1)$ as the sum of a geometric series and integrate term by term to produce

$$\int_0^\infty \frac{\sinh ax}{e^{px}+1}\, dx = \frac{1}{2p} \sum_{k=0}^\infty \frac{(-1)^k}{k+1-a/p} - \frac{1}{2p} \sum_{k=0}^\infty \frac{(-1)^k}{k+1+a/p}. \tag{7.2}$$

The next step is to transform the series above in order to use the expansion

$$\frac{\pi}{2} \sec\left(\frac{\pi z}{2} \right) = \sum_{n=0}^\infty (-1)^n \left(\frac{1}{z+(2n-1)} - \frac{1}{z-(2n-1)} \right), \tag{7.3}$$

appearing as entry 1.422.1 in [**4**].

Some elementary arguments yield

$$\int_0^\infty \frac{\sinh ax}{e^{px}+1}\, dx = -\frac{1}{p} \sum_{k=0}^\infty \frac{(-1)^k}{\left(1+\frac{2a}{p}\right)+(2k+1)} - \frac{1}{p} \sum_{k=0}^\infty \frac{(-1)^k}{\left(1+\frac{2a}{p}\right)-[2(k+1)+1]}$$

and from the second sum one sees that the change $k+1 \mapsto j$ is a natural one. This gives

$$\begin{aligned} \int_0^\infty \frac{\sinh ax}{e^{px}+1}\, dx &= -\frac{1}{p} \sum_{k=0}^\infty \frac{(-1)^k}{\left(1+\frac{2a}{p}\right)+(2k+1)} + \frac{1}{p} \sum_{j=1}^\infty \frac{(-1)^j}{\left(1+\frac{2a}{p}\right)-(2j+1)} \\ &= -\frac{1}{p} \left(\sum_{k=0}^\infty \frac{(-1)^k}{\left(1+\frac{2a}{p}\right)+(2k+1)} - \sum_{j=1}^\infty \frac{(-1)^j}{\left(1+\frac{2a}{p}\right)-(2j+1)} \right). \end{aligned}$$

The only remaining thing to fix is to start the second series above at $j = 0$, instead of $j = 1$. This extra term gives the extra $-1/2a$ appearing in the formula. ☐

7.2. Entry 3.545.2.

$$\int_0^\infty \frac{\sinh ax}{e^{px}-1}\,dx = \frac{1}{2a} - \frac{\pi}{2p}\cot\frac{\pi a}{p} \tag{7.4}$$

provided $p > a$, $p > 0$.

Proof. The proof is very similar to that given for Entry 3.545.1. The identity

$$\frac{\pi}{2}\tan\left(\frac{\pi}{2}z\right) = \sum_{n=0}^{\infty}\left(-\frac{1}{z+(2n+1)} - \frac{1}{z-(2n+1)}\right) \tag{7.5}$$

is used instead of (7.3). The details are left to the reader. ☐

8. Combinations of hyperbolic functions and quadratic exponential functions

8.1. Entry 3.546.1.

$$\int_0^\infty e^{-\beta x^2}\sinh ax\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}\exp\frac{a^2}{4\beta}\operatorname{erf}\left(\frac{a}{2\sqrt{\beta}}\right) \tag{8.1}$$

provided $\operatorname{Re}\beta > 0$.

Proof. Write the integral as

$$\int_0^\infty e^{-\beta x^2}\sinh ax\,dx = \frac{1}{2}\int_0^\infty e^{-\beta x^2+ax}\,dx - \frac{1}{2}\int_0^\infty e^{-\beta x^2-ax}\,dx. \tag{8.2}$$

Introduce the notation

$$J(\beta,a) = \int_0^\infty e^{-\beta x^2+ax}\,dx, \tag{8.3}$$

then the entry states that

$$\int_0^\infty e^{-\beta x^2}\sinh ax\,dx = \frac{1}{2}(J(\beta,a) - J(\beta,-a)). \tag{8.4}$$

Evaluate $J(\beta,a)$ by completing squares as

$$\begin{aligned} -\beta x^2 + ax &= -\beta\left[\left(x-\frac{a}{2\beta}\right)^2 - \frac{a^2}{4\beta^2}\right] \\ &= -\beta\left(x-\frac{a}{2\beta}\right)^2 + \frac{a^2}{4\beta}, \end{aligned} \tag{8.5}$$

so that

$$\begin{aligned}
J(\beta,a) &= e^{a^2/4\beta}\int_0^\infty e^{-\beta(x-a/2\beta)^2}\,dx \\
&= e^{a^2/4\beta}\int_{-a/2\beta}^\infty e^{-\beta t^2}\,dt \\
&= e^{a^2/4\beta}\frac{1}{\sqrt{\beta}}\int_{-a/2\sqrt{\beta}}^\infty e^{-u^2}\,du \\
&= e^{a^2/4\beta}\frac{1}{\sqrt{\beta}}\left(\frac{\sqrt{\pi}}{2}+\int_0^{a/2\sqrt{\beta}} e^{-u^2}\,du\right) \\
&= e^{a^2/4\beta}\frac{\sqrt{\pi}}{2\sqrt{\beta}}\left(1+\text{erf}\,\left(\frac{a}{2\sqrt{\beta}}\right)\right).
\end{aligned}$$

Therefore

$$J(\beta,a) = e^{a^2/4\beta}\frac{\sqrt{\pi}}{2\sqrt{\beta}}\left(1+\text{erf}\,\left(\frac{a}{2\sqrt{\beta}}\right)\right) \tag{8.6}$$

Also,

$$J(\beta,-a) = e^{a^2/4\beta}\frac{\sqrt{\pi}}{2\sqrt{\beta}}\left(1-\text{erf}\,\left(\frac{a}{2\sqrt{\beta}}\right)\right) \tag{8.7}$$

and this gives the result. □

8.2. Entry 3.546.2.

$$\int_0^\infty e^{-\beta x^2}\cosh ax\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}\exp\frac{a^2}{4\beta} \tag{8.8}$$

provided $\text{Re}\,\beta > 0$.

Proof. This entry is $J(\beta,a)+J(\beta,-a)$, with the notation in (8.3). The result follows directly from (8.6) and (8.7). □

8.3. Entry 3.546.3.

$$\int_0^\infty e^{-\beta x^2}\sinh^2 ax\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left[\exp\left(\frac{a^2}{\beta}\right)-1\right] \tag{8.9}$$

provided $\text{Re}\,\beta > 0$.

Proof. The relation

$$\begin{aligned}
\sinh^2 ax &= \left(\frac{e^{ax}-e^{-ax}}{2}\right)^2 \\
&= \frac{1}{4}e^{2ax}-\frac{1}{2}+\frac{1}{4}e^{-2ax}
\end{aligned} \tag{8.10}$$

gives, with the notation in (8.3)

$$\int_0^\infty e^{-\beta x^2}\sinh^2 ax\,dx = \frac{1}{4}J(\beta,2a)-\frac{1}{2}\int_0^\infty e^{-\beta x^2}\,dx+\frac{1}{4}J(\beta,-2a). \tag{8.11}$$

The result now follows from (8.6), (8.7), and the gaussian integral

$$\int_0^\infty e^{-\beta x^2}\,dx = \frac{\sqrt{\pi}}{2\sqrt{\beta}}. \tag{8.12}$$

□

8.4. Entry 3.546.4.

$$\int_0^\infty e^{-\beta x^2}\cosh^2 ax\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left[\exp\left(\frac{a^2}{\beta}\right)+1\right] \tag{8.13}$$

provided $\text{Re}\,\beta > 0$.

Proof. The proof is the same as that in Entry 3.546.3, but now using

$$\begin{aligned}\cosh^2 ax &= \left(\frac{e^{ax}+e^{-ax}}{2}\right)^2 \\ &= \frac{1}{4}e^{2ax}+\frac{1}{2}+\frac{1}{4}e^{-2ax}\end{aligned} \tag{8.14}$$

□

8.5. Entry 3.562.3.

$$\int_0^\infty xe^{-\beta x^2}\sinh\gamma x\,dx = \frac{\gamma}{4\beta}\sqrt{\frac{\pi}{\beta}}\exp\left(\frac{\gamma^2}{4\beta}\right) \tag{8.15}$$

Proof. Expressing the hyperbolic function as exponentials and completing the square in the new exponents gives

$$\begin{aligned} I &= \frac{1}{2}e^{\gamma^2/4\beta}\int_0^\infty xe^{-\beta(x-\gamma/2\beta)^2}\,dx - \frac{1}{2}e^{\gamma^2/4\beta}\int_0^\infty xe^{-\beta(x+\gamma/2\beta)^2}\,dx \\ &= \frac{1}{2}e^{\gamma^2/4\beta}\int_{-\gamma/2\beta}^\infty\left(t+\frac{\gamma}{2\beta}\right)e^{-\beta t^2}\,dt - \frac{1}{2}e^{\gamma^2/4\beta}\int_{\gamma/2\beta}^\infty\left(t-\frac{\gamma}{2\beta}\right)e^{-\beta t^2}\,dt \\ &= \frac{1}{2}e^{\gamma^2/4\beta}\int_{-\gamma/2\beta}^{\gamma/2\beta} te^{-\beta t^2}\,dt + \frac{\gamma}{4\beta}e^{\gamma^2/4\beta}\int_{-\infty}^\infty e^{-\beta t^2}\,dt. \end{aligned}$$

This completes the proof. The first integral vanishes since the integrand is odd. The second integral is evaluated using the change of variables $\sqrt{\beta}t = r$. □

8.6. Entry 3.562.4.

$$\int_0^\infty xe^{-\beta x^2}\cosh\gamma x\,dx = \frac{\gamma}{4\beta}\sqrt{\frac{\pi}{\beta}}\exp\left(\frac{\gamma^2}{4\beta}\right)\Phi\left(\frac{\gamma}{2\sqrt{\beta}}\right)+\frac{1}{2\beta} \tag{8.16}$$

Proof. Starting as in the proof of Entry 3.562.3 one obtains

$$I = \frac{1}{2}e^{\gamma^2/4\beta}\int_{-\gamma/2\beta}^\infty\left(t+\frac{\gamma}{2\beta}\right)e^{-\beta t^2}\,dt + \frac{1}{2}e^{\gamma^2/4\beta}\int_{\gamma/2\beta}^\infty\left(t-\frac{\gamma}{2\beta}\right)e^{-\beta t^2}\,dt.$$

The terms with the integrand $t\exp(-t^2)$ contribute

$$\frac{1}{2}e^{\gamma^2/4\beta}\left(\int_{-\gamma/2\beta}^\infty te^{-\beta t^2}\,dt + \int_{\gamma/2\beta}^\infty te^{-\beta t^2}\,dt\right) \tag{8.17}$$

and this evaluates to $1/2\beta$ since the integrand admits a simple antiderivative.

The other two terms contribute

$$\frac{\gamma}{4\beta}e^{\gamma^2/2\beta}\int_{-\gamma/2\beta}^{\infty} e^{-\beta t^2}\,dt - \frac{\gamma}{4\beta}e^{\gamma^2/2\beta}\int_{\gamma/2\beta}^{\infty} e^{-\beta t^2}\,dt = \frac{\gamma}{2\beta}e^{\gamma^2/2\beta}\int_{0}^{\gamma/2\beta} e^{-\beta t^2}\,dt$$

The change of variables $\sqrt{\beta}t = r$ and the expression

$$\texttt{erf}\ z = \frac{2}{\sqrt{\pi}}\int_0^z e^{-r^2}\,dr \tag{8.18}$$

now gives

$$I = \frac{\gamma}{4\beta}\sqrt{\frac{\pi}{\beta}}\texttt{erf}\left(\frac{\gamma}{2\sqrt{\beta}}\right). \tag{8.19}$$

This is the answer stated in the table, provided one replaces Φ by `erf`. This replacement issue has been addressed in [**3**]. □

Similar arguments yield the evaluation of the next two entries.

8.7. Entry 3.562.5.

$$\int_0^\infty x^2 e^{-\beta x^2}\sinh\gamma x\,dx = \frac{\sqrt{\pi}(2\beta+\gamma^2)}{8\beta^2\sqrt{\beta}}\exp\left(\frac{\gamma^2}{4\beta}\right)\Phi\left(\frac{\gamma}{2\sqrt{\beta}}\right) + \frac{\gamma}{4\beta^2} \tag{8.20}$$

8.8. Entry 3.562.6.

$$\int_0^\infty x^2 e^{-\beta x^2}\cosh\gamma x\,dx = \frac{\sqrt{\pi}(2\beta+\gamma^2)}{8\beta^2\sqrt{\beta}}\exp\left(\frac{\gamma^2}{4\beta}\right) \tag{8.21}$$

9. Combinations of hyperbolic functions, exponentials and powers

9.1. Entry 3.551.1.

$$\int_0^\infty x^{\mu-1}e^{-\beta x}\sinh\gamma x\,dx = \tfrac{1}{2}\Gamma(\mu)\left[(\beta-\gamma)^{-\mu} - (\beta+\gamma)^{-\mu}\right] \tag{9.1}$$

provided $\text{Re}\,\mu > -1$, $\text{Re}\,\beta > |\,\text{Re}\,\gamma|$.

Proof. Expressing the hyperbolic function in terms of exponentials gives

$$I = \frac{1}{2}\int_0^\infty x^{\mu-1}e^{-(\beta-\gamma)x}\,dx - \frac{1}{2}\int_0^\infty x^{\mu-1}e^{-(\beta+\gamma)x}\,dx. \tag{9.2}$$

Now make the change $t = (\beta-\mu)x$ in the first integral and $t = (\beta+\mu)x$ in the second one to obtain

$$I = \frac{1}{2(\beta-\gamma)^\mu}\int_0^\infty t^{\mu-1}e^{-t}dt - \frac{1}{2(\beta+\gamma)^\mu}\int_0^\infty t^{\mu-1}e^{-t}dt. \tag{9.3}$$

The proof concludes by identifying the integral in the last line as $\Gamma(\mu)$. □

9.2. Entry 3.551.2.

$$\int_0^\infty x^{\mu-1}e^{-\beta x}\cosh\gamma x\,dx = \tfrac{1}{2}\Gamma(\mu)\left[(\beta-\gamma)^{-\mu}+(\beta+\gamma)^{-\mu}\right] \tag{9.4}$$

provided $\operatorname{Re}\mu > -1$, $\operatorname{Re}\beta > |\operatorname{Re}\gamma|$.

Proof. The evaluation of this entry is identical as the one given for Entry 3.551.1 with the only change that now $\cosh\gamma x = \frac{1}{2}(e^{\gamma x}+e^{-\gamma x})$. ☐

9.3. Entry 3.551.4.

$$\int_0^\infty x^n e^{-(p+mq)x}\sinh^m qx\,dx = 2^{-m}n!\sum_{k=0}^m \binom{m}{k}\frac{(-1)^k}{(p+2kq)^{n+1}} \tag{9.5}$$

provided $p>0$, $q>0$, $m<p+qm$.

Proof. Write the hyperbolic sine in terms of exponentials to obtain

$$\begin{aligned} I &= \frac{1}{2^n}\int_0^\infty x^n e^{-px}\left(1-e^{-2qx}\right)^m\,dx \\ &= \frac{1}{2^m}\sum_{k=0}^m (-1)^k\binom{m}{k}\int_0^\infty x^n e^{-(p+2qk)x} \\ &= \frac{1}{2^m}\sum_{k=0}^m (-1)^k\binom{m}{k}\frac{1}{(p+2qk)^{n+1}}\int_0^\infty t^n e^{-t}\,dt, \end{aligned} \tag{9.6}$$

using the change of variables $t=(p+2qk)x$ to get the last step.

The gamma function, defined by

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}\,dt, \qquad \text{for } \operatorname{Re}a>0, \tag{9.7}$$

interpolates factorials in the form

$$\Gamma(n) = (n-1)!, \qquad \text{if } a\in\mathbb{N}. \tag{9.8}$$

This completes the proof. ☐

9.4. Entry 3.551.6.

$$\int_0^\infty \frac{e^{-\beta x}}{x}\sinh\gamma x\,dx = \frac{1}{2}\ln\frac{\beta+\gamma}{\beta-\gamma} \tag{9.9}$$

provided $\operatorname{Re}\beta > |\operatorname{Re}\gamma|$.

Proof. Introduce the notation

$$I(\beta,\gamma) = \int_0^\infty \frac{e^{-\beta x}}{x}\sinh\gamma x\,dx \tag{9.10}$$

and differentiate with respect to β to obtain

$$\frac{\partial I}{\partial\beta} = -\int_0^\infty e^{-\beta x}\sinh\gamma x\,dx. \tag{9.11}$$

This last integral is elementary: simply express the hyperbolic factor in terms of the exponentials to produce

$$\begin{aligned} \frac{\partial I}{\partial \beta} &= -\frac{1}{2}\int_0^\infty e^{-(\beta-\gamma)x}\,dx + \frac{1}{2}\int_0^\infty e^{-(\beta+\gamma)x}\,dx \\ &= -\frac{1}{2(\beta-\gamma)} + \frac{1}{2(\beta+\gamma)}. \end{aligned} \tag{9.12}$$

Integrate back with respect to β and use the fact that $I(\beta,\gamma) \to 0$ when $\beta \to \infty$ to confirm that the constant of integration vanishes. This completes the proof. ☐

9.5. Entry 3.551.8.

$$\int_0^\infty xe^{-x}\coth x\,dx = \frac{\pi^2}{4} - 1 \tag{9.13}$$

Proof. Use $\coth x = \cosh x/\sinh x = (e^x+e^{-x})/(e^x-e^{-x})$ to produce

$$e^{-x}\coth x = \frac{1+e^{-2x}}{e^x-e^{-x}} = \frac{e^{-x}+e^{-3x}}{1-e^{-2x}} \tag{9.14}$$

and then

$$\int_0^\infty xe^{-x}\coth x\,dx = \int_0^\infty x\left(\frac{e^{-x}+e^{-3x}}{1-e^{-2x}}\right)dx. \tag{9.15}$$

Expand the denominator of the previous integrand in a geometric series to obtain

$$\int_0^\infty xe^{-x}\coth x\,dx = \int_0^\infty \left(\sum_{k=0}^\infty xe^{-(2k+1)x} + \sum_{k=0}^\infty xe^{-(2k+3)x}\right)dx. \tag{9.16}$$

Integration by parts yields

$$\int_0^\infty xe^{-ax}\,dx = \frac{1}{a^2}, \quad \text{for } a > 0, \tag{9.17}$$

and replacing in (9.16) gives

$$\begin{aligned} \int_0^\infty xe^{-x}\coth x\,dx &= \sum_{k=0}^\infty \frac{1}{(2k+1)^2} + \sum_{k=0}^\infty \frac{1}{(2k+3)^2} \\ &= 2\sum_{k=0}^\infty \frac{1}{(2k+1)^2} - 1. \end{aligned} \tag{9.18}$$

The evaluation now follows from the classical value

$$\sum_{k=0}^\infty \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}. \tag{9.19}$$

☐

10. Some entries producing finite sums

10.1. Entry 3.558.1.

$$\int_0^\infty x \frac{1-e^{-nx}}{\sinh^2 \frac{x}{2}} \, dx = \frac{2n\pi^2}{3} - 4 \sum_{k=1}^{n-1} \frac{n-k}{k^2} \tag{10.1}$$

Proof. The change of variables $t = x/2$ and writing the hyperbolic function in terms of exponentials give

$$I = \int_0^\infty \frac{16te^{-2t}(1-e^{-2nt})\, dt}{(1-e^{-2t})^2}. \tag{10.2}$$

Now use the power series expansion

$$\frac{1}{(1-u)^2} = \sum_{j=1}^\infty j u^{j-1} \tag{10.3}$$

to produce

$$\begin{aligned} I &= \int_0^\infty 16te^{-2t}(1-e^{-2nt}) \sum_{j=1}^\infty j e^{-2(j-1)t} \, dt \\ &= \sum_{j=1}^\infty 16j \int_0^\infty te^{-2jt} \, dt - \sum_{j=1}^\infty 16j \int_0^\infty 16je^{-(n+j)t} \, dt \\ &= \sum_{j=1}^\infty 16j \int_0^\infty te^{-2jt} \, dt - \sum_{j=n+1}^\infty 16(j-n) \int_0^\infty te^{-2jt} \, dt \\ &= \sum_{j=1}^n 16j \int_0^\infty te^{-2jt} \, dt + 16n \sum_{j=n+1}^\infty \int_0^\infty te^{-2jt} \, dt. \end{aligned} \tag{10.4}$$

(`Note`. A rigorous proof requires to cut the sum at $j = N \gg n$ and then pass to the limit as $N \to \infty$. The reader is invited to check the details). Integration by parts gives

$$\int_0^\infty te^{-2jt} dt = \frac{1}{4j^2}, \tag{10.5}$$

and this gives

$$\begin{aligned} I &= 4 \sum_{j=1}^n \frac{1}{j} + 4n \sum_{j=n+1}^\infty \frac{1}{j^2} \\ &= 4 \sum_{j=1}^n \frac{1}{j} + 4n \left(\sum_{j=1}^\infty \frac{1}{j^2} - \sum_{j=1}^n \frac{1}{j^2} \right) \\ &= 4n \cdot \frac{\pi^2}{6} - 4 \sum_{j=1}^n \left(\frac{n-j}{j^2} \right). \end{aligned}$$

This completes the proof. □

10.2. Entry 3.558.2.

$$\int_0^\infty x \frac{1-(-1)^n e^{-nx}}{\cosh^2 \frac{x}{2}} \, dx = \frac{n\pi^2}{3} + 4 \sum_{k=1}^{n-1} (-1)^k \frac{n-k}{k^2} \tag{10.6}$$

Proof. Make the change of variables $t = x/2$ and write the hyperbolic function in terms of exponentials to write

$$I = 16 \int_0^\infty \frac{te^{-2t} \left[1 - (-1)^n e^{-2nt} \right]}{(1+e^{-2t})^2} \, dt. \tag{10.7}$$

Now use the expansion

$$\frac{1}{(1+u)^2} = \sum_{j=1}^\infty (-1)^{j-1} u^{j-1} \tag{10.8}$$

to produce

$$I = \int_0^\infty te^{-2t} \left[1 - (-1)^n e^{-2nt} \right] \sum_{j=1}^\infty (-1)^{j-1} j e^{-2(j-1)t} \, dt. \tag{10.9}$$

This impies

$$I = \sum_{j=1}^\infty 16j(-1)^{j-1} \int_0^\infty te^{-2jt} \, dt - \sum_{j=1}^\infty 16j(-1)^{n+j-1} \int_0^\infty te^{-2(n+j)t} \, dt, \tag{10.10}$$

This can be written as

$$I = \sum_{j=1}^\infty 16j(-1)^{j-1} \int_0^\infty te^{-2jt} \, dt - \sum_{k=n+1}^\infty 16(-1)^{k-1}(k-n) \int_0^\infty te^{-2kt}$$

and this yields

$$\begin{aligned} I = \sum_{j=1}^\infty 16j(-1)^{j-1} & \int_0^\infty te^{-2jt} \, dt \\ & - \sum_{k=n+1}^\infty 16k(-1)^{k-1} \int_0^\infty te^{-2kt} \, dt \\ & \qquad + 16n \sum_{k=n+1}^\infty (-1)^{k-1} \int_0^\infty te^{-2kt} \, dt. \end{aligned} \tag{10.11}$$

The values

$$\int_0^\infty te^{-2kt} \, dt = \frac{1}{4k^2} \tag{10.12}$$

and

$$\sum_{j=1}^\infty \frac{(-1)^{j-1}}{j^2} = \frac{\pi^2}{12}, \tag{10.13}$$

give the evaluation of this entry. □

10.3. Entry 3.558.3.

$$\int_0^\infty x^2 \frac{1-e^{-nx}}{\sinh^2 \frac{x}{2}} \, dx = 8n\zeta(3) - 8 \sum_{k=1}^{n-1} \frac{n-k}{k^3} \tag{10.14}$$

Proof. The change of variables $t = x/2$ and writing the hyperbolic function in terms of exponentials give

$$I = \int_0^\infty 32t^2 e^{-2t} \left(\frac{1-e^{-2nt}}{(1-e^{-2t})^2} \right) dt. \tag{10.15}$$

Using the expansion (10.3) produces

$$\begin{aligned} I &= \int_0^\infty 32t^2(1-e^{-2nt}) \sum_{j=1}^\infty j e^{-2jt} \, dt \\ &= 32 \sum_{j=1}^\infty j \int_0^\infty t^2 e^{-2jt} \, dt - 32 \sum_{k=n+1}^\infty (k-n) \int_0^\infty t^2 e^{-2kt} \, dt. \end{aligned}$$

Now

$$\int_0^\infty t^2 e^{2jt} \, dt = \frac{1}{4j^3} \tag{10.16}$$

yields

$$\begin{aligned} I &= 8 \sum_{j=1}^\infty \frac{1}{j^2} - 8 \sum_{k=n+1}^\infty \frac{(k-n)}{k^3} \\ &= 8 \sum_{j=1}^n \frac{1}{j^2} + 8n \sum_{k=n+1}^\infty \frac{1}{k^3} \\ &= 8 \sum_{j=1}^n \frac{1}{j^2} + 8n \sum_{k=1}^\infty \frac{1}{k^3} - 8n \sum_{k=1}^n \frac{1}{k^3} \\ &= 8 \sum_{j=1}^{n-1} \frac{(j-n)}{j^3} + 8n \sum_{k=1}^\infty \frac{1}{k^3} \\ &= 8 \sum_{j=1}^{n-1} \frac{(j-n)}{j^3} + 8n\zeta(3). \end{aligned} \tag{10.17}$$

This completes the proof. □

The proof of the next four entries are similar to the one given for Entry 3.558.3. The details are left to the reader.

10.4. Entry 3.558.4.

$$\int_0^\infty x^2 e^x \frac{(1-e^{-2nx})}{\sinh^2 x} \, dx = 8n \sum_{k=1}^\infty \frac{1}{(2k-1)^3} - 8 \sum_{k=1}^{n-1} \frac{n-k}{(2k-1)^3} \tag{10.18}$$

10.5. Entry 3.558.5.

$$\int_0^\infty x^2 \frac{1+(-1)^n e^{-nx}}{\cosh^2 \frac{x}{2}} \, dx = 6n\zeta(3) - 8 \sum_{k=1}^{n-1} \frac{n-k}{k^3} \tag{10.19}$$

10.6. Entry 3.558.6.

$$\int_0^\infty x^3 \frac{1-e^{-nx}}{\sinh^2 \frac{x}{2}} \, dx = \frac{4}{15} n\pi^4 - 24 \sum_{k=1}^{n-1} \frac{n-k}{k^4} \tag{10.20}$$

10.7. Entry 3.558.6.

$$\int_0^\infty x^3 \frac{1-e^{-nx}}{\sinh^2 \frac{x}{2}} \, dx = \frac{4}{15} n\pi^4 - 24 \sum_{k=1}^{n-1} \frac{n-k}{k^4} \tag{10.21}$$

10.8. Entry 3.558.7.

$$\int_0^\infty x^3 \frac{[1+(-1)^n e^{-nx}]}{\cosh^2 \frac{x}{2}} \, dx = \frac{7}{30} n\pi^4 + 24 \sum_{k=1}^{n-1} (-1)^k \frac{n-k}{k^4} \tag{10.22}$$

Remark 10.1. The entry **3.561**

$$\int_0^\infty \frac{e^{-2x} \tanh \frac{x}{2}}{x \cosh x} \, dx = 2 \ln \frac{\pi}{2\sqrt{2}} \tag{10.23}$$

can be evaluated with the methods used in this work. This will be included in a future publication dealing with the generalization

$$A(n,m) = \int_0^\infty \frac{e^{-nx} \tanh \frac{x}{m}}{x \cosh x} \, dx. \tag{10.24}$$

Acknowledgments. The work presented here began as part of an undergraduate seminar on the evaluation of integrals. This was partially funded by Tulane University. The authors acknowledge their support.

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Received 25 01 2026, revised 30 05 2026

Department of Mathematics,
Tulane University, New Orleans,
LA 70118,
USA.
Email address: `jhuddell@tulane.edu`
Email address: `vhm@math.tulane.edu`