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A Note on an Arctangent Integral by Fritz Oberhettinger

Robert Reynolds

Abstract. An exact expression for an Arctangent definite integral is derived and evaluated using contour integration. A new closed form expression for this definite integral is given in terms of the Hurwitz-Lerch zeta function. Special cases of this definite are evaluated in terms of special functions and fundamental constants. A short table summarizing interesting results is produced.

1. Significance statement

The Fourier transform is a mathematical operation that decomposes a function into its constituent frequencies. This transform is widely used in signal processing, physics, engineering, and other disciplines to analyze and manipulate functions in the frequency domain. The book by Oberhettinger [**6**] places an emphasis on providing a comprehensive collection of integrals involving the Fourier transforms of functions involving sine, cosine, and exponential terms. Such a compendium is a valuable resource for researchers, engineers, and scientists who use Fourier transform methods in their work. In this current article contour integration is applied to the cosine Fourier transform of an the arctangent function to derive a closed form solution in terms of special functions. This approach is an extension of the method applied to previous definite integrals which added to other mathematical tables such Mellin transform and exponential transform tables. The contour integral method used in this article has been adopted in previous work [**5**] to derive other definite integrals, infinite and finite sums and products involving special functions. We will be applying this method to other integral forms to derive other formulae to add to current literature.

2. Introduction

The arctangent function and its definite integral find significant applications across diverse fields of mathematics, physics, engineering, and scientific research. Its role in trigonometric substitution simplifies complex integrals involving square roots and

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trigonometric expressions. In complex analysis, the arctangent contributes to defining complex logarithms and exponential functions. In trigonometry, it serves as an essential inverse trigonometric function, aiding in solving equations and angle determinations. Moreover, this function appears in control systems, aiding phase analysis, and in physics for angle calculations in projectile motion. Its utility extends to computer graphics, probability distributions, robotics for inverse kinematics, and even optics for angle of incidence calculations. The arctangent function's versatility underscores its significance as a mathematical tool that underpins various theoretical frameworks and practical applications. The definite integral of the arctangent function is widely used physics. Some areas of study utilizing such integrals are in the theory of localized magnetic moments in metals [**1**], theory of localized magnetic states in metals [**2**], localized magnetic moments in dilute metallic alloys: correlation effects [**3**] and the interaction between localized states in metals in [**4**].

The definite integral derived in this manuscript is given by

$$\begin{aligned}\int_0^\infty e^{-imx}\left(e^{2imx}(\log(a)+ix)^{k-1}\right.&\\ +\frac{m\left(e^{2imx}(\log(a)+ix)^k+(\log(a)-ix)^k\right)}{k}&\\ \left.+(\log(a)-ix)^{k-1}\right)\tan^{-1}(\sinh(\alpha)\operatorname{sech}(bx))dx&\end{aligned} \tag{2.1}$$

where the parameters k, a, m are general complex numbers with $Re(\alpha) > 0, Re(b) > 0$. Since definite integrals of this general form do not appear in current literature we use our contour integral method to derive and evaluate this integral and provide some interesting material in the form of special cases. The derivation of the definite integral follows the method used by us in [**5**] which involves Cauchy's integral formula. The generalized Cauchy's integral formula is given by

$$\frac{y^k}{\Gamma(k+1)}=\frac{1}{2\pi i}\int_C\frac{e^{wy}}{w^{k+1}}dw. \tag{2.2}$$

where C is in general an open contour in the complex plane where the bilinear concomitant [**5**] has the same value at the end points of the contour. A bilinear concomitant is a bilinear form (a function of two variables that is linear in each) that arises naturally when integrating the product of two functions by parts over a given interval. The method in [**5**] involves using a form of equation (2.2) then multiply both sides by a function, then take a definite integral of both sides. This yields a definite integral in terms of a contour integral. A second contour integral is derived by multiplying equation (2.2) by a function and performing some substitutions so that the contour integrals are the same.

3. Definite integral of the contour integral

We use the method in [**5**]. The variable of integration in the contour integral is $z = w + m$. The cut and contour are in the first quadrant of the complex z-plane.

The cut approaches the origin from the interior of the first quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using equation (2.2) we replace y by $ix+\log(a)$ then multiply by e^{mxi} then replace $x \to -x$ to form a second equation and add both equations. Next we multiply both sides by $\frac{1}{2}m\tan^{-1}(\sinh(\alpha)\text{sech}(bx))$ take the finite integral over $x \in [0,\infty)$. We form the second equation by replacing $k \to k-1$ and divide by m and add to get;

(3.1)
$$\begin{aligned}
&\frac{1}{\Gamma(k+1)}\int_0^\infty e^{-imx}\left(e^{2imx}(\log(a)+ix)^{k-1}+\frac{m\left(e^{2imx}(\log(a)+ix)^k+(\log(a)-ix)^k\right)}{k}\right.\\
&\qquad\qquad\left.+(\log(a)-ix)^{k-1}\right)\tan^{-1}(\sinh(\alpha)\text{sech}(bx))dx\\
&=\frac{1}{2\pi i}\int_0^\infty\int_C a^w w^{-k-1}(m+w)\cos(x(m+w))\tan^{-1}(\sinh(\alpha)\text{sech}(bx))dwdx\\
&=\frac{1}{2\pi i}\int_C\int_0^\infty a^w w^{-k-1}(m+w)\cos(x(m+w))\tan^{-1}(\sinh(\alpha)\text{sech}(bx))dxdw\\
&=\frac{1}{2\pi i}\int_C \frac{1}{2}\pi a^w w^{-k-1}\text{sech}\left(\frac{\pi(m+w)}{2b}\right)\sin\left(\frac{\alpha(m+w)}{b}\right)dw
\end{aligned}$$

from equation (1.7.7.122) in [**6**] where $Re\left(\frac{\pi(w+m)}{2b}\right) > 0, Re(\alpha) > 0, Re(b) > 0$, given by;

$$\int_0^\infty \cos(xy)\tan^{-1}(\sinh(a\text{sech}(bx)))\,dx = \frac{\pi\sin\left(\frac{ay}{b}\right)\text{sech}\left(\frac{\pi y}{2b}\right)}{2y} \tag{3.2}$$

The logarithmic function is given for example in section (4.2) in [**7**]. We are able to switch the order of integration over $z = w+m$ and x using Fubini's theorem since the integrand is of bounded measure over the space $\mathbb{C}\times[0,\infty)$.

4. The Hurwitz-Lerch zeta function

We use equation (25.14) in [**7**] where $\Phi(z,s,v)$ is the Lerch function which is a generalization of the Hurwitz zeta $\zeta(s,v)$ and Polylogarithm functions $Li_n(z)$. The Lerch function has a series representation given by

$$\Phi(z,s,v) = \sum_{n=0}^\infty (v+n)^{-s}z^n \tag{4.1}$$

where $|z| < 1, v \neq 0, -1, ..$ and is continued analytically by its integral representation,

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t-z}dt, \tag{4.2}$$

where $Re(v) > 0$, and either $|z| \leqslant 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

5. Infinite sum of the contour integral

In this section we will again use Cauchy's integral formula (2.2) and take the infinite sum to derive equivalent sum representations for the contour integrals. We proceed using equation (2.2) and replace y by $y+\frac{i\alpha}{b}$ and multiply both sides by $e^{\frac{i\alpha m}{b}}$ then replace $\alpha \to -\alpha$ to form a second equation and take their difference. Next replace $y \to \log(a)+\frac{\pi(2y+1)}{2b}$ and multiply by $\frac{1}{2}i\pi(-1)^y e^{\frac{\pi m(2y+1)}{2b}}$ and take the infinite sum over $y \in [0,\infty)$ simplifying in terms of the Lerch function to get

$$\begin{aligned}(5.1)\quad &\frac{1}{2\Gamma(k+1)} i\pi^{k+1}\left(\frac{1}{b}\right)^k e^{\frac{(\pi-2i\alpha)m}{2b}}\left(\Phi\left(-e^{\frac{m\pi}{b}},-k,\frac{-2i\alpha+2b\log(a)+\pi}{2\pi}\right)\right.\\ &\left.-e^{\frac{2i\alpha m}{b}}\Phi\left(-e^{\frac{m\pi}{b}},-k,\frac{2i\alpha+2b\log(a)+\pi}{2\pi}\right)\right)\\ &=\frac{1}{2\pi}\sum_{y=0}^{\infty}\int_C \pi(-1)^y a^w w^{-k-1} e^{\frac{\pi(2y+1)(m+w)}{2b}}\sin\left(\frac{\alpha(m+w)}{b}\right)dw\\ &=\frac{1}{2\pi}\int_C\sum_{y=0}^{\infty} \pi(-1)^y a^w w^{-k-1} e^{\frac{\pi(2y+1)(m+w)}{2b}}\sin\left(\frac{\alpha(m+w)}{b}\right)dw\\ &=\frac{1}{2\pi i}\int_C \frac{1}{2}\pi a^w w^{-k-1}\operatorname{sech}\left(\frac{\pi(m+w)}{2b}\right)\sin\left(\frac{\alpha(m+w)}{b}\right)dw\end{aligned}$$

from equation (1.232.2) in [**8**] where $Im(w+m)>0$ in order for the sum to converge.

6. Definite integral in terms of the Hurwitz-Lerch zeta function

In this section we will evaluate the definite integral involving the arctangent and Hurwitz-Lerch zeta functions in terms of special functions and fundamental constants.

THEOREM 6.1. *For all $k,a\in\mathbb{C}, Re(\alpha)>0, Re(b)>0, Re(m)>0$,*

$$\begin{aligned}(6.1)\quad &\int_0^\infty e^{-imx}\left(e^{2imx}(\log(a)+ix)^{k-1}\right.\\ &+\frac{m\left(e^{2imx}(\log(a)+ix)^k+(\log(a)-ix)^k\right)}{k}\\ &\left.+(\log(a)-ix)^{k-1}\right)\tan^{-1}(\sinh(\alpha)\operatorname{sech}(bx))dx\\ &=\frac{1}{k}i\pi^{k+1}\left(\frac{1}{b}\right)^k e^{\frac{(\pi-2i\alpha)m}{2b}}\left(\Phi\left(-e^{\frac{m\pi}{b}},-k,\frac{-2i\alpha+2b\log(a)+\pi}{2\pi}\right)\right.\\ &\left.-e^{\frac{2i\alpha m}{b}}\Phi\left(-e^{\frac{m\pi}{b}},-k,\frac{2i\alpha+2b\log(a)+\pi}{2\pi}\right)\right)\end{aligned}$$

PROOF. multline Observe that the right-hand side of equations (3.1) and (5.1) are equal so we may equate the left-hand sides and simplify the gamma function to yield the stated result. □

EXAMPLE 6.1. The degenerate case

$$\int_0^\infty \cos(mx)\tan^{-1}(\sinh(\alpha)\text{sech}(bx))dx = \frac{\pi\text{sech}\left(\frac{\pi m}{2b}\right)\sin\left(\frac{\alpha m}{b}\right)}{2m} \tag{6.2}$$

PROOF. Use equation (6.1) and set $k = 0$ and simplify using entry (2) in table below (64:12:7) in [**9**]. □

EXAMPLE 6.2. A Mellin transform. For all $Re(s) > 0, Re(\alpha) > 0, Re(b) > 0$,

$$\int_0^\infty x^{s-1}\tan^{-1}(\sinh(\alpha)\text{sech}(bx))dx \tag{6.3}$$

$$= \frac{1}{s} i 2^{s-1}\pi^{s+1}\left(\frac{1}{b}\right)^s \csc\left(\frac{\pi s}{2}\right)$$

$$\left(\zeta\left(-s, \frac{\pi - 2i\alpha}{4\pi}\right) - \zeta\left(-s, \frac{2i\alpha + \pi}{4\pi}\right) - \zeta\left(-s, \frac{3}{4} - \frac{i\alpha}{2\pi}\right) + \zeta\left(-s, \frac{i\alpha}{2\pi} + \frac{3}{4}\right)\right)$$

PROOF. Use equation (6.1) and set $a = 1, m = 0$ and simplify in terms of the Hurwitz zeta function $\zeta(s, a)$ using entry (4) in table below (64:12:7) in [**9**]. □

EXAMPLE 6.3. The Hurwitz zeta function. For all $Re(b) > Re(\alpha)$ then,

$$\int_0^\infty \left((\log(a) - ix)^{k-1} + (\log(a) + ix)^{k-1}\right)\tan^{-1}(\sinh(\alpha)\,\text{sech}(bx))dx \tag{6.4}$$

$$= \frac{1}{k} i 2^k \pi^{k+1}\left(\frac{1}{b}\right)^k \left(\zeta\left(-k, \frac{-2i\alpha + 2b\log(a) + \pi}{4\pi}\right) - \zeta\left(-k, \frac{-2i\alpha + 2b\log(a) + 3\pi}{4\pi}\right)\right.$$
$$\left. -\zeta\left(-k, \frac{2i\alpha + 2b\log(a) + \pi}{4\pi}\right) + \zeta\left(-k, \frac{2i\alpha + 2b\log(a) + 3\pi}{4\pi}\right)\right)$$

PROOF. Use equation (6.1) and set $m = 0$ and simplify in terms of the Hurwitz zeta function $\zeta(s, a)$ using entry (4) in table below (64:12:7) in [**9**]. □

EXAMPLE 6.4. The digamma function.

$$\int_0^\infty \left(\frac{1}{(\log(a) + ix)^2} + \frac{1}{(\log(a) - ix)^2}\right)\tan^{-1}(\sinh(\alpha)\,\text{sech}(bx))dx \tag{6.5}$$
$$= \frac{1}{2} ib\left(\psi^{(0)}\left(\frac{-2i\alpha + 2b\log(a) + \pi}{4\pi}\right) - \psi^{(0)}\left(\frac{-2i\alpha + 2b\log(a) + 3\pi}{4\pi}\right)\right.$$
$$\left. -\psi^{(0)}\left(\frac{2i\alpha + 2b\log(a) + \pi}{4\pi}\right) + \psi^{(0)}\left(\frac{2i\alpha + 2b\log(a) + 3\pi}{4\pi}\right)\right)$$

PROOF. Use equation (6.4) and apply l'Hopital's rule as $k \to -1$ and simplify using equation (64:4:1) in [**9**], given by

$$\zeta(n, u) = \frac{(-1)^n \psi^{(n-1)}(u)}{(n-1)!} \tag{6.6}$$

□

EXAMPLE 6.5. A special case of equation (6.5) in terms of log(2).

$$\int_0^\infty \frac{\left(x^2-1\right)\tanh^{-1}(\operatorname{sech}(\pi x))}{\left(x^2+1\right)^2}dx = -\frac{1}{2}\pi(\log(4)-1) \tag{6.7}$$

PROOF. Use equation (6.5) and set $a = e, b = \pi, \alpha = \pi i/2$ and simplify using entry (1) in Table below (44:7:1) in [**9**]. □

EXAMPLE 6.6. Catalan's constant C.

$$\int_0^\infty \frac{\left(12x^2-1\right)\tanh^{-1}(\operatorname{sech}(\pi x))}{\left(4x^2+1\right)^3}dx = \frac{1}{8}(\pi-2\pi C) \tag{6.8}$$

PROOF. Use equation (6.4) and set $k = -2, a = e^{1/2}, b = \pi, \alpha = \pi i/2$ and simplify in terms of Catalan's constant C using equation (64:4:1) in [**9**] and equation (2.2.1.2.7) in [**10**]. □

EXAMPLE 6.7. Apéry's constant $\zeta(3)$.

$$\int_0^\infty \frac{\left(x^4-6x^2+1\right)\tanh^{-1}(\operatorname{sech}(\pi x))}{\left(x^2+1\right)^4}dx = \frac{1}{12}\pi(3\zeta(3)-2) \tag{6.9}$$

PROOF. Use equation (6.4) and set $k = -3, a = e, b = \pi, \alpha = \pi i/2$ and simplify in terms of Apréy's constant $\zeta(3)$ using entry (2) in Table below (64:7) in [**9**]. □

EXAMPLE 6.8. The Riemann zeta function.

$$\int_0^\infty \left(\frac{1}{(1+ix)^6} - \frac{1}{(x+i)^6}\right)\tanh^{-1}(\operatorname{sech}(\pi x))dx = \frac{1}{40}\pi(15\zeta(5)-8) \tag{6.10}$$

PROOF. Use equation (6.4) and set $k = -5, a = e, b = \pi, \alpha = \pi i/2$ and simplify in terms of the constant $\zeta(5)$ using entry (2) in Table below (64:7) in [**9**]. □

EXAMPLE 6.9. The Riemann zeta function.

$$\int_0^\infty \left(\frac{1}{(1+ix)^{3/2}} + \frac{1}{(1-ix)^{3/2}}\right)\tanh^{-1}(\operatorname{sech}(\pi x))dx = -4\left(\sqrt{2}-1\right)\pi\zeta\left(\frac{1}{2}\right) - 2\pi \tag{6.11}$$

PROOF. Use equation (6.4) and set $k = -1/2, a = e, b = \pi, \alpha = \pi i/2$ and simplify in terms of the constant $\zeta(1/2)$ using entry (2) in Table below (64:7) in [**9**]. □

EXAMPLE 6.10. The Riemann zeta function.

$$\int_0^\infty \left(\frac{1}{\sqrt{1+ix}} + \frac{1}{\sqrt{1-ix}}\right)\tanh^{-1}(\operatorname{sech}(\pi x))dx = \left(1-2\sqrt{2}\right)\zeta\left(\frac{3}{2}\right) + 2\pi \tag{6.12}$$

PROOF. Use equation (6.4) and set $k = 1/2, a = e, b = \pi, \alpha = \pi i/2$ and simplify in terms of the constant $\zeta(3/2)$ using entry (2) in Table below (64:7) in [**9**]. □

EXAMPLE 6.11. Second derivative of the Hurwitz zeta function and log-gamma functions.

$$\begin{aligned}(6.13)\quad &\int_0^\infty \left(\frac{\log(\log(a)-ix)}{\log(a)-ix}+\frac{\log(\log(a)+ix)}{\log(a)+ix}\right)\tan^{-1}(\sinh(\alpha)\text{sech}(bx))dx \\ &= \frac{1}{2}i\pi\left(\zeta''\left(0,\frac{2b\log(a)-2i\alpha+\pi}{4\pi}\right)-\zeta''\left(0,\frac{2b\log(a)-2i\alpha+3\pi}{4\pi}\right)\right. \\ &-\zeta''\left(0,\frac{2b\log(a)+2i\alpha+\pi}{4\pi}\right)+\zeta''\left(0,\frac{2b\log(a)+2i\alpha+3\pi}{4\pi}\right) \\ &+2\log\left(\frac{2\pi}{b}\right)(\log(-4(-2b\log(a)+2i\alpha+\pi))+\log(2b\log(a)+2i\alpha-3\pi) \\ &-\log(2b\log(a)+2i\alpha-\pi)-\log(8b\log(a)-8i\alpha-12\pi) \\ &-\log\Gamma\left(-\frac{-2i\alpha-2b\log(a)+\pi}{4\pi}\right)+\log\Gamma\left(-\frac{2i\alpha-2b\log(a)+\pi}{4\pi}\right) \\ &\left.-\log\Gamma\left(-\frac{2i\alpha-2b\log(a)+3\pi}{4\pi}\right)+\log\Gamma\left(\frac{2i\alpha+2b\log(a)-3\pi}{4\pi}\right)\right)\Bigg)\end{aligned}$$

PROOF. Use equation (6.4) and take the first partial derivative with respect to k and then set $k=0$ and simplify using equation (25.11.18) in [**7**]. ☐

EXAMPLE 6.12. The logarithm of the gamma function.

$$\begin{aligned}(6.14)\quad &\int_0^\infty \frac{\tan^{-1}(\sinh(\alpha)\text{sech}(bx))}{\log^2(a)+x^2}dx \\ &= -\frac{i\pi}{2\log(a)}\log\left(\frac{\Gamma\left(\frac{-2i\alpha+2b\log(a)+\pi}{4\pi}\right)\Gamma\left(\frac{2i\alpha+2b\log(a)+3\pi}{4\pi}\right)}{\Gamma\left(\frac{-2i\alpha+2b\log(a)+3\pi}{4\pi}\right)\Gamma\left(\frac{2i\alpha+2b\log(a)+\pi}{4\pi}\right)}\right)\end{aligned}$$

PROOF. Use equation (6.4) and apply l'Hopital's rule as $k\to 0$ and simplify using equation (25.11.18) in [**7**]. ☐

6.1. Special cases involving the log-gamma function. In this subsection we look at simply examples in terms of the log-gamma function.

$$(6.15)\quad \int_0^\infty \frac{\tanh^{-1}(\text{sech}(\pi x))}{x^2+1}dx = -\frac{1}{2}\pi\log\left(\frac{2}{\pi}\right)$$

$$(6.16)\quad \int_0^\infty \frac{\tanh^{-1}(\sin(\alpha)\text{sech}(bx))}{a^2+x^2}dx = -\frac{\pi\log\left(\frac{\Gamma\left(\frac{2ab-2\alpha+3\pi}{4\pi}\right)\Gamma\left(\frac{2ab+2\alpha+\pi}{4\pi}\right)}{\Gamma\left(\frac{2ab-2\alpha+\pi}{4\pi}\right)\Gamma\left(\frac{2ab+2\alpha+3\pi}{4\pi}\right)}\right)}{2a}$$

$$\int_0^\infty \frac{\tanh^{-1}(\sin(\alpha)\operatorname{sech}(bx))}{\left(a^2+x^2\right)^2}dx \tag{6.17}$$
$$= \frac{ab}{8a^3}\left(-\psi^{(0)}\left(\frac{2ab-2\alpha+\pi}{4\pi}\right)+\psi^{(0)}\left(\frac{2ab-2\alpha+3\pi}{4\pi}\right)+\psi^{(0)}\left(\frac{2ab+2\alpha+\pi}{4\pi}\right)\right.$$
$$\left.-\psi^{(0)}\left(\frac{2ab+2\alpha+3\pi}{4\pi}\right)\right)-2\pi\log\left(\frac{\Gamma\left(\frac{2ab-2\alpha+3\pi}{4\pi}\right)\Gamma\left(\frac{2ab+2\alpha+\pi}{4\pi}\right)}{\Gamma\left(\frac{2ab-2\alpha+\pi}{4\pi}\right)\Gamma\left(\frac{2ab+2\alpha+3\pi}{4\pi}\right)}\right)$$

$$\int_0^\infty \frac{\tanh^{-1}(\operatorname{sech}(x))}{\left(x^2+\pi^2\right)^2}dx = \frac{\log\left(\frac{2\pi}{e}\right)}{4\pi^2} \tag{6.18}$$

$$\int_0^\infty \frac{\coth^{-1}\left(\sqrt{2}\cosh(9\pi x)\right)}{x^2+4}dx = -\frac{1}{4}\pi\log\left(\frac{\Gamma\left(\frac{75}{8}\right)\Gamma\left(\frac{77}{8}\right)}{\Gamma\left(\frac{73}{8}\right)\Gamma\left(\frac{79}{8}\right)}\right) \tag{6.19}$$

(6.20)
$$\int_0^\infty \operatorname{sech}(t)\tanh^{-1}(\sin(\alpha)\operatorname{sech}(ab\sinh(t)))dt = -\frac{1}{2}\pi\log\left(\frac{\Gamma\left(\frac{2ab-2\alpha+3\pi}{4\pi}\right)\Gamma\left(\frac{2ab+2\alpha+\pi}{4\pi}\right)}{\Gamma\left(\frac{2ab-2\alpha+\pi}{4\pi}\right)\Gamma\left(\frac{2ab+2\alpha+3\pi}{4\pi}\right)}\right)$$

$$\int_0^\infty \operatorname{sech}(t)\tanh^{-1}\left(\operatorname{sech}\left(\frac{1}{4}\pi\sinh(t)\right)\right)dt = -\frac{1}{2}\pi\log\left(\frac{\Gamma\left(\frac{5}{8}\right)^2}{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{9}{8}\right)}\right) \tag{6.21}$$

Example 6.13. The ratio of Euler gamma functions.

$$\exp\left(\int_0^\infty \frac{2i\log(a)\tan^{-1}(\sinh(\alpha)\operatorname{sech}(bx))}{\pi\left(\log^2(a)+x^2\right)}dx\right) \tag{6.22}$$
$$= \frac{\Gamma\left(\frac{-2i\alpha+2b\log(a)+\pi}{4\pi}\right)\Gamma\left(\frac{2i\alpha+2b\log(a)+3\pi}{4\pi}\right)}{\Gamma\left(\frac{-2i\alpha+2b\log(a)+3\pi}{4\pi}\right)\Gamma\left(\frac{2i\alpha+2b\log(a)+\pi}{4\pi}\right)}$$

Proof. Use equation (6.14) and take the exponential function of both sides and simplify. □

Example 6.14. Bierens de Haan form.

(6.23)
$$\int_0^\infty \frac{2(mx\sin(mx)+\cos(mx)-rx\sin(rx)-\cos(rx))\tan^{-1}(\sinh(\alpha)\operatorname{sech}(bx))}{x^2}dx$$
$$= ib\left(e^{\frac{(\pi-2i\alpha)m}{2b}}\Phi\left(-e^{\frac{m\pi}{b}},1,\frac{1}{2}-\frac{i\alpha}{\pi}\right)-e^{\frac{(\pi+2i\alpha)m}{2b}}\Phi\left(-e^{\frac{m\pi}{b}},1,\frac{i\alpha}{\pi}+\frac{1}{2}\right)\right.$$
$$\left.-e^{\frac{(\pi-2i\alpha)r}{2b}}\Phi\left(-e^{\frac{\pi r}{b}},1,\frac{1}{2}-\frac{i\alpha}{\pi}\right)+e^{\frac{(\pi+2i\alpha)r}{2b}}\Phi\left(-e^{\frac{\pi r}{b}},1,\frac{i\alpha}{\pi}+\frac{1}{2}\right)\right)$$

Proof. Use equation (6.1) and form a second equation by replacing $m \to r$. Next take the difference of the second and first equation. Next set $k=-1, a=1$ and simplify. This form is recorded in equation (3.231.3) in [**8**]. □

7. Table of arctangent definite integrals

This section is a summary of integral formulae in this work produced for easy reading.

$f(x)$	$\int_0^\infty f(x)dx$
$\cos(mx)\tan^{-1}(\sinh(\alpha)\text{sech}(bx))$	$\frac{\pi\text{sech}\left(\frac{\pi m}{2b}\right)\sin\left(\frac{\alpha m}{b}\right)}{2m}$
$\frac{\left(x^2-1\right)\tanh^{-1}(\text{sech}(\pi x))}{(x^2+1)^2}$	$-\frac{1}{2}\pi(\log(4)-1)$
$\frac{\left(12x^2-1\right)\tanh^{-1}(\text{sech}(\pi x))}{(4x^2+1)^3}$	$\frac{1}{8}(\pi-2\pi C)$
$\frac{\left(x^4-6x^2+1\right)\tanh^{-1}(\text{sech}(\pi x))}{(x^2+1)^4}$	$\frac{1}{12}\pi(3\zeta(3)-2)$
$\left(\frac{1}{(1+ix)^6}-\frac{1}{(x+i)^6}\right)\tanh^{-1}(\text{sech}(\pi x))$	$\frac{1}{40}\pi(15\zeta(5)-8)$
$\left(\frac{1}{(1+ix)^{3/2}}+\frac{1}{(1-ix)^{3/2}}\right)\tanh^{-1}(\text{sech}(\pi x))$	$-4\left(\sqrt{2}-1\right)\pi\zeta\left(\frac{1}{2}\right)-2\pi$
$\left(\frac{1}{\sqrt{1+ix}}+\frac{1}{\sqrt{1-ix}}\right)\tanh^{-1}(\text{sech}(\pi x))$	$\left(1-2\sqrt{2}\right)\zeta\left(\frac{3}{2}\right)+2\pi$
$\frac{\tan^{-1}(\sinh(\alpha)\text{sech}(bx))}{\log^2(a)+x^2}$	$-\frac{i\pi\log\left(\frac{\Gamma\left(\frac{-2i\alpha+2b\log(a)+\pi}{4\pi}\right)\Gamma\left(\frac{2i\alpha+2b\log(a)+3\pi}{4\pi}\right)}{\Gamma\left(\frac{-2i\alpha+2b\log(a)+3\pi}{4\pi}\right)\Gamma\left(\frac{2i\alpha+2b\log(a)+\pi}{4\pi}\right)}\right)}{2\log(a)}$
$\frac{\tanh^{-1}(\text{sech}(\pi x))}{x^2+1}$	$-\frac{1}{2}\pi\log\left(\frac{2}{\pi}\right)$
$\frac{\tanh^{-1}(\sin(\alpha)\text{sech}(bx))}{a^2+x^2}$	$-\frac{\pi\log\left(\frac{\Gamma\left(\frac{2ab-2\alpha+3\pi}{4\pi}\right)\Gamma\left(\frac{2ab+2\alpha+\pi}{4\pi}\right)}{\Gamma\left(\frac{2ab-2\alpha+\pi}{4\pi}\right)\Gamma\left(\frac{2ab+2\alpha+3\pi}{4\pi}\right)}\right)}{2a}$
$\frac{\tanh^{-1}(\text{sech}(x))}{(x^2+\pi^2)^2}$	$\frac{\log\left(\frac{2\pi}{e}\right)}{4\pi^2}$
$\frac{\coth^{-1}\left(\sqrt{2}\cosh(9\pi x)\right)}{x^2+4}$	$-\frac{1}{4}\pi\log\left(\frac{\Gamma\left(\frac{75}{8}\right)\Gamma\left(\frac{77}{8}\right)}{\Gamma\left(\frac{73}{8}\right)\Gamma\left(\frac{79}{8}\right)}\right)$
$\text{sech}(t)\tanh^{-1}(\sin(\alpha)\text{sech}(ab\sinh(t)))$	$-\frac{1}{2}\pi\log\left(\frac{\Gamma\left(\frac{2ab-2\alpha+3\pi}{4\pi}\right)\Gamma\left(\frac{2ab+2\alpha+\pi}{4\pi}\right)}{\Gamma\left(\frac{2ab-2\alpha+\pi}{4\pi}\right)\Gamma\left(\frac{2ab+2\alpha+3\pi}{4\pi}\right)}\right)$
$\text{sech}(t)\tanh^{-1}\left(\text{sech}\left(\frac{1}{4}\pi\sinh(t)\right)\right)$	$-\frac{1}{2}\pi\log\left(\frac{\Gamma\left(\frac{5}{8}\right)^2}{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{9}{8}\right)}\right)$

8. Conflict of Interest

The author has no conflicts of interest to declare.

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10. Discussion

In this paper, we have presented a method for deriving a new Arctangent integral transform along with some interesting definite integrals similar to those published by Oberhettinger, using contour integration. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

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Department of Mathematics and Statistics
York University
Toronto, Ontario, M3J 1P3
Canada
Email address: milver73@gmail.com