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The valuation of Catalan numbers

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Abstract. Elementary recurrences for the Catalan numbers are used to determine divisibility properties.

1. Introduction

The Catalan numbers C_n appear in combinatorics as counting the number of paths, with northeast (NE) and southeast (SE) steps, which start and end on the horizontal axis without ever dipping below it. The total number of steps clearly must be even. Denote by C_n the number of such paths ending at $(2n, 0)$. Concentrating on the first time the path hits the horizontal axes easily leads to the recurrence

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}, \quad \text{for} \quad n \geqslant 1, \tag{1.1}$$

with the initial condition $C_0 = 1$. The goal of this note is to present some arithmetic properties of C_n which can be obtained by elementary means. The reader will find in [**2**] a wealth of information on these numbers.

The parity of C_n can be determined directly from this recurrence: if n is even, the sum is

$$C_n = 2\sum_{k=0}^{r} C_k C_{n-1-k}, \quad \text{with } r = \frac{n}{2} - 1, \tag{1.2}$$

showing that C_n is even.

On the other hand, if n is odd, write

$$C_n = 2\sum_{k=0}^{s-1} C_k C_{n-k} + C_s^2, \quad \text{with } s = \frac{n-1}{2}. \tag{1.3}$$

Since the parity of a number and its square always agree, it follows that C_n and C_s have the same parity. Now observe that if $n = 1a_1a_2 \cdots a_r1$ is the binary expansion of (the odd number) n, then $s = \frac{1}{2}(n-1) = 1a_1 \cdots a_r$. Thus, if n contains a non-zero

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digit in its binary expansion, iteration of this procedure will produce an index n^* with 0 as the last digit, that is, n^* is even. Therefore C_{n^*} is even and since C_{n^*} and C_n have the same parity, so is C_n.

The conclusion is that C_n is odd precisely when n has only 1's in its binary expansion; that is, $n = 2^a - 1$.

2. The 2-adic valuation of Catalan numbers

The exact power of 2 dividing a positive integer x is called the 2-adic valuation of x and is denoted by $\nu_2(x)$. The value of C_n does not seem to follow directly from (1.1), but it can be obtained from the explicit formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{2.1}$$

This expression gives the second recurrence $C_{n+1} = \frac{2(2n+1)}{n+2}C_n$. A remarkable proof of (2.1) was obtained by André using his *reflection principle*: Catalan numbers count the number of paths joining $(0,0)$ to (n,n) that stay below the diagonal line, with steps north (N) or east (E). Define a *bad path* as one that crosses the diagonal. Any such path $\mathcal{C}$ contains a vertex with coordinates $(k, k+1)$. Now reflect the portion of $\mathcal{C}$ after this crossing about the line $y = x + 1$. This yields a path joining $(0,0)$ and $(n-1, n+1)$. It is easy to see that this is a bijection, proving that there are $\binom{2n}{n+1}$ bad paths. The formula (2.1) now follows from here. The reader will find in [**1**] an alternative proof of this result.

It follows from (2.1) that

$$\nu_2(C_n) = \nu_2((2n)!) - 2\nu_2(n!) - \nu_2(n+1), \tag{2.2}$$

and it remains to find an expression for the 2-adic valuation of factorials. Starting with the binary expansion of n, the classical formula of Legendre

$$\nu_2(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{2^k} \right\rfloor \tag{2.3}$$

reduces to $\nu_2(n!) = n - S_2(n)$, where $S_2(n)$ is the sum of the binary digits of n. Therefore (2.2) becomes

$$\nu_2(C_n) = S_2(n) - \nu_2(n+1) = n - \nu_2((n+1)!) = S_2(n+1) - 1. \tag{2.4}$$

The form $\nu_2(C_n) = n - \nu_2((n+1)!)$ can be proven directly: define a new sequence of rational numbers by $u_n = (n+1)!C_n/2^n$ and the second recurrence for C_n yields $u_{n+1} = (2n+1)u_n$. Then with the initial value $u_1 = 1$, it is clear that u_n is an odd integer. The direct proof alluded to above is now complete.

The result in Section 1 now follows directly from (2.4): C_n is odd precisely when $S_2(n+1) = 1$; that is, when $n+1$ is a power of 2. Naturally (2.4) gives more: for instance, C_n is exactly divisible by 2; that is, by 2 and not by 4; precisely when n has the form $2^a + 2^b - 1$.

References

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