SCIENTIA
Series A: *Mathematical Sciences*, Vol. 30 (2020), 11–22
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

Inequalities of Jensen's Type for K-Bounded Modulus Convex Complex Functions

Silvestru Sever Dragomir[1,2]

Abstract. Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. We say that the function $f : D \subset \mathbb{C} \to \mathbb{C}$ is called *K-bounded modulus convex*, for the given $K > 0$, if it satisfies the condition

$$\left|(1-\lambda) f(x) + \lambda f(y) - f((1-\lambda) x + \lambda y)\right| \leq \frac{1}{2} K \lambda (1-\lambda) |x-y|^2$$

for any $x, y \in D$ and $\lambda \in [0,1]$.

In this paper we establish some new Jensen's type inequalities for the complex integral on γ, a smooth path from $\mathbb{C}$ and K-bounded modulus convex functions. Some examples for the complex exponential and complex logarithm are also given.

1. Introduction

Let $(X; \|\cdot\|_X)$ and $(Y; \|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$. Let C be a convex set in X. In the recent paper [**3**] we introduced the following class of functions:

Definition 1. *A mapping $F : C \subset X \to Y$ is called K-bounded norm convex, for some given $K > 0$, if it satisfies the condition*

$$\|(1-\lambda) F(x) + \lambda F(y) - F((1-\lambda) x + \lambda y)\|_Y \leq \frac{1}{2} K \lambda (1-\lambda) \|x-y\|_X^2 \tag{1.1}$$

for any $x, y \in C$ and $\lambda \in [0,1]$. For simplicity, we denote this by $F \in \mathcal{BN}_K(C)$.

We have from (1.1) for $\lambda = \frac{1}{2}$ the Jensen's inequality

$$\left\|\frac{F(x) + F(y)}{2} - F\left(\frac{x+y}{2}\right)\right\|_Y \leq \frac{1}{8} K \|x-y\|_X^2$$

for any $x, y \in C$.

We observe that $\mathcal{BN}_K(C)$ is a convex subset in the linear space of all functions defined on C and with values in Y.

1991 *Mathematics Subject Classification.* 26D15, 26D10, 30A10, 30A86.
Key words and phrases. Complex integral, Continuous functions, Holomorphic functions, Jensen's inequality, Hermite-Hadamard inequality, Midpoint inequality, Trapezoid inequality.

In the same paper [**3**], we obtained the following result which provides a large class of examples of such functions.

THEOREM 1. *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be two normed linear spaces,* C *an open convex subset of* X *and* $F : C \to Y$ *a twice-differentiable mapping on* C*. Then for any* x, $y \in C$ *and* $\lambda \in [0,1]$ *we have*

$$\|(1-\lambda) F(x) + \lambda F(y) - F((1-\lambda) x + \lambda y)\|_Y \leq \frac{1}{2} K\lambda (1-\lambda) \|y - x\|_X^2, \tag{1.2}$$

where

$$K_{F''} := \sup_{z \in C} \|F''(z)\|_{\mathcal{L}(X^2;Y)} \tag{1.3}$$

is assumed to be finite, namely $F \in \mathcal{BN}_{K_{F''}}(C)$.

We have the following inequalities of Hermite-Hadamard type [**3**]:

THEOREM 2. *Let* $(X; \|\cdot\|_X)$ *and* $(Y; \|\cdot\|_Y)$ *be two normed linear spaces over the complex number field* $\mathbb{C}$ *with* Y *complete. Assume that the mapping* $F : C \subset X \to Y$ *is continuous on the convex set* C *in the norm topology. If* $F \in \mathcal{BN}_K(C)$ *for some* $K > 0$, *then we have*

$$\left\| \frac{F(x) + F(y)}{2} - \int_0^1 F((1-\lambda) x + \lambda y) \, d\lambda \right\|_Y \leq \frac{1}{12} K \|x - y\|_X^2 \tag{1.4}$$

and

$$\left\| \int_0^1 F((1-\lambda) x + \lambda y) \, d\lambda - F\left(\frac{x+y}{2}\right) \right\|_Y \leq \frac{1}{24} K \|x - y\|_X^2 \tag{1.5}$$

for any x, $y \in C$.

The constants $\frac{1}{12}$ *and* $\frac{1}{24}$ *are best possible.*

Following [**1**, p. 59], let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces, Ω an open subset of X and $F : \Omega \to Y$. If $a \in \Omega$, $u \in X \setminus \{0\}$ and if the limit

$$\lim_{t \to 0} \frac{1}{t} [F(a + tu) - F(a)]$$

exists, then we denote this derivative $\partial_u F(a)$. It is called the directional derivative of F at a in the direction u. If the directional derivative is defined in all directions and there is a continuous linear mapping Φ from X into Y such that for all $u \in X$

$$\partial_u F(a) = \Phi(u),$$

then we say that F is Gâteaux-differentiable at a and that Φ is the Gâteaux differential of F at a. If a mapping F is differentiable at a point a, then clearly all its directional derivatives exist and we have

$$\partial_u F(a) = F'(a) u, \; u \in X.$$

Thus F is Gâteaux-differentiable at a. However, the Gâteaux differential may exist without the differential existing. The existence of directional derivatives at a point does not imply that the mapping is Gâteaux-differentiable. To distinguish the differential from the Gâteaux differential, the differential is often referred as the Fréchet differential.

In an earlier and more comprehensive version of [**3**], see [**2**], we also obtained the following Jensen's type discrete inequality:

Theorem 3. *Let $(X;\|\cdot\|_X)$ and $(Y;\|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$. Assume that the mapping $F : C \subset X \to Y$ is defined on the open convex set C and $F \in \mathcal{BN}_K(C)$ for some $K > 0$. If $x_k \in C$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^n p_k = 1$ and F is Gâteaux-differentiable at $\sum_{k=1}^n p_k x_k \in C$, then for any $y_j \in C$ and $q_j \geq 0$ for $j \in \{1, ..., m\}$ with $\sum_{j=1}^m q_j = 1$ and $\sum_{j=1}^m q_j y_j = \sum_{k=1}^n p_k x_k$ we have*

$$\left\|\sum_{j=1}^{m} q_j F(y_j) - F\left(\sum_{k=1}^{n} p_k x_k\right)\right\|_Y \leq \frac{1}{2} K \sum_{j=1}^{m} q_j \left\|y_j - \sum_{k=1}^{n} p_k x_k\right\|_X^2. \tag{1.6}$$

In particular, we have

$$\left\|\sum_{j=1}^{n} p_j F(x_j) - F\left(\sum_{k=1}^{n} p_k x_k\right)\right\|_Y \leq \frac{1}{2} K \sum_{j=1}^{n} p_j \left\|x_j - \sum_{k=1}^{n} p_k x_k\right\|_X^2. \tag{1.7}$$

If $(X; \langle\cdot,\cdot\rangle)$ is an inner product space, then

$$\sum_{j=1}^{n} p_j \left\|x_j - \sum_{k=1}^{n} p_k x_k\right\|_X^2 = \sum_{j=1}^{n} p_j \|x_j\|_X^2 - \left\|\sum_{k=1}^{n} p_k x_k\right\|_X^2$$

and by (1.7) we have

$$\left\|\sum_{j=1}^{n} p_j F(x_j) - F\left(\sum_{k=1}^{n} p_k x_k\right)\right\|_Y \leq \frac{1}{2} K \left[\sum_{j=1}^{n} p_j \|x_j\|_X^2 - \left\|\sum_{k=1}^{n} p_k x_k\right\|_X^2\right]. \tag{1.8}$$

Corollary 1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces, C an open convex subset of X and $F : C \to Y$ a twice-differentiable mapping on C. If $x_k \in C$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^n p_k = 1$, then*

$$\left\|\sum_{j=1}^{n} p_j F(x_j) - F\left(\sum_{k=1}^{n} p_k x_k\right)\right\|_Y \leq \frac{1}{2} \sup_{z \in C} \|F''(z)\|_{\mathcal{L}(X^2;Y)} \sum_{j=1}^{n} p_j \left\|x_j - \sum_{k=1}^{n} p_k x_k\right\|_X^2. \tag{1.9}$$

Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. Following Definition 1, we say that the function $F : D \subset \mathbb{C} \to \mathbb{C}$ is called K-*bounded modulus convex*, for the given $K > 0$, if it satisfies the condition

$$|(1-\lambda) F(x) + \lambda F(y) - F((1-\lambda) x + \lambda y)| \leq \frac{1}{2} K \lambda (1-\lambda) |x - y|^2 \tag{1.10}$$

for any $x, y \in D$ and $\lambda \in [0,1]$. For simplicity, we denote this by $F \in \mathcal{BM}_K(D)$.

All the above results can be translated for complex functions defined on convex subsets $D \subset \mathbb{C}$.

In the following, in order to obtain several inequalities for the complex integral, we need the following facts.

Suppose γ is a smooth path from $\mathbb{C}$ parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz$$

where $v := zz$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in D, and open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.11}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz| \le \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{1.12}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\,|dz|\right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\,|dz|.$$

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q} \|f\|_{\gamma,p}.$$

In the recent paper [**5**] we obtained the following results:

THEOREM 4. *Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. Assume that f is holomorphic on D and $f \in \mathcal{BM}_K(D)$. If $\gamma \subset D$ parametrized by $z(t)$, $t \in [a,b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v \in D$, then*

$$\left| \int_\gamma f(z)\,dz - \left[f(v) + f'(v)\left(\frac{w+u}{2} - v\right)\right](w-u) \right| \le \frac{1}{2} K \int_\gamma |z-v|^2 |dz| \tag{1.13}$$

and

$$\left| \frac{1}{2}[f(w)(w-v) + f(u)(v-u) + f(v)(w-u)] - \int_\gamma f(z)\,dz \right| \le \frac{1}{4} K \int_\gamma |z-v|^2 |dz|. \tag{1.14}$$

Motivated by the above results, in this paper we establish some new Jensen's type inequalities for the complex integral on γ, a smooth path from $\mathbb{C}$ and K-bounded modulus convex functions. Some examples for the complex exponential and complex logarithm are also given.

2. General Integral Inequalities

We have:

THEOREM 5. *Let $G \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$ and that F is holomorphic on G with $F \in \mathcal{BM}_K(G)$. Assume also that $f: D \to G$ is continuous on D, $\gamma \subset D$ parametrized by $z(t)$, $t \in [a,b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ with $w \ne u$ and $\frac{1}{w-u}\int_\gamma f(z)\,dz \in G$, then*

$$\left| \frac{1}{w-u} \int_\gamma (F \circ f)(v)\,dv - F\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right) \right| \le \frac{1}{2} K \frac{1}{|w-u|} \int_\gamma \left| f(v) - \frac{1}{w-u}\int_\gamma f(z)\,dz \right|^2 |dv|. \tag{2.1}$$

PROOF. Let x, $y \in G$. Since $F \in \mathcal{BM}_K(G)$, then we have

$$|F((1-\lambda)x + \lambda y) - F(x) + \lambda[F(x) - F(y)]| \le \frac{1}{2} K \lambda (1-\lambda)|x-y|^2$$

that implies that

$$\left| \frac{F(x + \lambda(y-x)) - F(x)}{\lambda} + F(x) - F(y) \right| \le \frac{1}{2} K (1-\lambda)|x-y|^2$$

for $\lambda \in (0,1)$.

Since F is holomorphic on G, then by letting $\lambda \to 0+$, we get

$$|F'(x)(y-x) + F(x) - F(y)| \le \frac{1}{2} K |x-y|^2$$

that is equivalent to

$$|F(y)-F(x)-F'(x)(y-x)| \leq \frac{1}{2}K|y-x|^{2} \tag{2.2}$$

for all $x,\, y\in G$.

If we take in (2.2) $x=\frac{1}{w-u}\int_{\gamma}f(z)\,dz$, then we get

$$\begin{aligned}\Bigg|F(y)-F\left(\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right)\\ -F'\left(\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right)\left(y-\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right)\Bigg|\\ \leq\frac{1}{2}K\left|y-\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right|^{2}\end{aligned} \tag{2.3}$$

for all $y\in G$.

If we take in this inequality $y=f(v)$, $v\in\gamma$, then we get

$$\begin{aligned}\Bigg|(F\circ f)(v)-F\left(\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right)\\ -F'\left(\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right)\left(f(v)-\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right)\Bigg|\\ \leq\frac{1}{2}K\left|f(v)-\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right|^{2}\end{aligned} \tag{2.4}$$

for all $v\in\gamma$.

We have

$$\begin{aligned}\frac{1}{w-u}\int_{\gamma}\Bigg[(F\circ f)(v)-F\left(\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right)\\ -F'\left(\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right)\left(f(v)-\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right)\Bigg]dv\\ =\frac{1}{w-u}\int_{\gamma}(F\circ f)(v)\,dv-F\left(\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right)\\ -F'\left(\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right)\left(\frac{1}{w-u}\int_{\gamma}f(v)\,dv-\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right)\\ =\frac{1}{w-u}\int_{\gamma}(F\circ f)(v)\,dv-F\left(\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right).\end{aligned} \tag{2.5}$$

By using (2.4) and (2.5) we get

$$\left|\frac{1}{w-u}\int_\gamma (F\circ f)(v)\,dv - F\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)\right|$$
$$\leq \frac{1}{|w-u|}\int_\gamma \left|(F\circ f)(v) - F\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)\right.$$
$$\left. -F'\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)\left(f(v)-\frac{1}{w-u}\int_\gamma f(z)\,dz\right)\right||dv|$$
$$\leq \frac{1}{2}K\frac{1}{|w-u|}\int_\gamma \left|f(v)-\frac{1}{w-u}\int_\gamma f(z)\,dz\right|^2|dv|,$$

which proves the inequality (2.1). □

Corollary 2. *With the assumptions of Theorem 5 and if*

$$\|F''\|_{G,\infty} := \sup_{z\in G}|F''(z)| < \infty,$$

then

$$\left|\frac{1}{w-u}\int_\gamma (F\circ f)(v)\,dv - F\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)\right| \leq \frac{1}{2}\|F''\|_{G,\infty}\frac{1}{|w-u|}\int_\gamma \left|f(v)-\frac{1}{w-u}\int_\gamma f(z)\,dz\right|^2|dv|. \tag{2.6}$$

Remark 1. *If we take $D = G$, $\gamma \subset G$ and $f(z) = z$, then by (2.6) we get the Hermite-Hadamard type inequality (see also* [**5**]*)*

$$\left|\frac{1}{w-u}\int_\gamma F(v)\,dv - F\left(\frac{w+u}{2}\right)\right| \leq \frac{1}{2}\|F''\|_{G,\infty}\frac{1}{|w-u|}\int_\gamma \left|v-\frac{w+u}{2}\right|^2|dv|, \tag{2.7}$$

provided F is holomorphic on G and $\|F''\|_{G,\infty} := \sup_{z\in G}|F''(z)| < \infty$.

We also have:

Theorem 6. *Let $G \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$ and that F is holomorphic on G with $F \in \mathcal{BM}_K(G)$. Assume also that $f : D \to G$ is continuous on D, $\gamma \subset D$ parametrized by $z(t)$, $t \in [a,b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ with $w \neq u$,*

$$\int_\gamma (F'\circ f)(v)\,dv \neq 0 \text{ and } \frac{\int_\gamma (F'\circ f)(v)f(v)\,dv}{\int_\gamma (F'\circ f)(v)\,dv} \in G, \tag{2.8}$$

then

$$\left|F\left(\frac{\int_\gamma (F'\circ f)(v) f(v)\,dv}{\int_\gamma (F'\circ f)(v)\,dv}\right)-\frac{1}{w-u}\int_\gamma (F\circ f)(z)\,dz\right| \le \frac{1}{2}K\frac{1}{|w-u|}\int_\gamma \left|\frac{\int_\gamma (F'\circ f)(v) f(v)\,dv}{\int_\gamma (F'\circ f)(v)\,dv}-f(z)\right|^2 |dz|. \tag{2.9}$$

Proof. From (2.2) we get

$$|F(y)-F(f(v))-F'(f(v))(y-f(v))| \le \frac{1}{2}K|y-f(v)|^2 \tag{2.10}$$

for any $y\in G$ and for $v\in D$.

Taking the integral in (2.10) we get

$$\frac{1}{|w-u|}\int_\gamma |F(y)-F(f(v))-F'(f(v))(y-f(v))|\,|dv| \le \frac{1}{2}K\frac{1}{|w-u|}\int_\gamma |y-f(v)|^2\,|dv| \tag{2.11}$$

for $y\in G$.

Using the properties of integral and modulus, we also have

$$\left|\frac{1}{w-u}\int_\gamma [F(y)-F(f(w))-F'(f(w))(y-f(w))]\,dw\right| \le \frac{1}{|w-u|}\int_\gamma |F(y)-F(f(w))-F'(f(w))(y-f(w))|\,|dw| \tag{2.12}$$

for $y\in G$.

Now, observe that

$$\frac{1}{w-u}\int_\gamma [F(y)-F(f(v))-F'(f(v))(y-f(v))]\,dv = F(y)-\frac{1}{w-u}\int_\gamma (F\circ f)(v)\,dv - y\frac{1}{w-u}\int_\gamma (F'\circ f)(v)\,dv+\frac{1}{w-u}\int_\gamma (F'\circ f)(v) f(v)\,dv$$

and by (2.11) and (2.12) we get the following inequality of interest

$$\left|F(y)-\frac{1}{w-u}\int_\gamma (F\circ f)(v)\,dv - y\frac{1}{w-u}\int_\gamma (F'\circ f)(v)\,dv+\frac{1}{w-u}\int_\gamma (F'\circ f)(v) f(v)\,dv\right| \le \frac{1}{2}K\frac{1}{|w-u|}\int_\gamma |y-f(z)|^2\,|dz| \tag{2.13}$$

for $y\in G$.

If we take in (2.13)

$$y = \frac{\int_\gamma (F' \circ f)(v) f(v) dv}{\int_\gamma (F' \circ f)(v) dv} \in G,$$

then we get the desired result (2.9). □

Corollary 3. *With the assumptions of Corollary 2 and Theorem 6 we have*

$$\left| F\left(\frac{\int_\gamma (F' \circ f)(v) f(v) dv}{\int_\gamma (F' \circ f)(v) dv} \right) - \frac{1}{w-u} \int_\gamma (F \circ f)(z) dz \right| \leq \frac{1}{2} \|F''\|_{G,\infty} \frac{1}{|w-u|} \int_\gamma \left| \frac{\int_\gamma (F' \circ f)(v) f(v) dv}{\int_\gamma (F' \circ f)(v) dv} - f(z) \right|^2 |dz| . \tag{2.14}$$

We have by the integration by parts formula (1.11) that

$$\int_\gamma F'(v) v dv = F(w) w - F(u) u - \int_\gamma F(v) dv$$

and

$$\int_\gamma F'(v) dv = F(w) - F(u) .$$

Therefore we can state the following result as well:

Remark 2. *Let $G \subset \mathbb{C}$ be a convex domain of complex numbers and that F is holomorphic on G with $\|F''\|_{G,\infty} := \sup_{z\in G} |F''(z)| < \infty$. Assume also that $\gamma \subset D$ parametrized by $z(t)$, $t \in [a,b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ with $w \neq u$, $F(w) \neq F(u)$ and*

$$\frac{F(w) w - F(u) u - \int_\gamma F(v) dv}{F(w) - F(u)} \in G, \tag{2.15}$$

then by (2.14) we get

$$\left| F\left(\frac{F(w) w - F(u) u - \int_\gamma F(v) dv}{F(w) - F(u)} \right) - \frac{1}{w-u} \int_\gamma F(z) dz \right| \leq \frac{1}{2} \|F''\|_{G,\infty} \frac{1}{|w-u|} \int_\gamma \left| \frac{F(w) w - F(u) u - \int_\gamma F(v) dv}{F(w) - F(u)} - z \right|^2 |dz| . \tag{2.16}$$

3. Some Examples

If we consider the function $F(z) = \exp z$, $z \in \mathbb{C}$ and $\gamma \subset \mathbb{C}$ parametrized by $z(t)$, $t \in [a,b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ with $w \neq u$, then by

(2.6) we have for continuous function $f:\gamma\to\mathbb{C}$

$$\left|\frac{1}{w-u}\int_{\gamma}(\exp\circ f)(v)\,dv-\exp\left(\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right)\right| \le\frac{1}{2}\|\exp\|_{G,\infty}\frac{1}{|w-u|}\int_{\gamma}\left|f(v)-\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right|^{2}|dv|, \tag{3.1}$$

while from (2.6) we obtain

$$\left|\frac{\exp w-\exp u}{w-u}-\exp\left(\frac{w+u}{2}\right)\right| \le\frac{1}{2}\|\exp\|_{G,\infty}\frac{1}{|w-u|}\int_{\gamma}\left|v-\frac{w+u}{2}\right|^{2}|dv|. \tag{3.2}$$

From (2.14) we get

$$\left|\exp\left(\frac{\int_{\gamma}(\exp\circ f)(v)f(v)\,dv}{\int_{\gamma}(\exp\circ f)(v)\,dv}\right)-\frac{1}{w-u}\int_{\gamma}(\exp\circ f)(z)\,dz\right| \le\frac{1}{2}\|\exp\|_{G,\infty}\frac{1}{|w-u|}\int_{\gamma}\left|\frac{\int_{\gamma}(\exp\circ f)(v)f(v)\,dv}{\int_{\gamma}(\exp\circ f)(v)\,dv}-f(z)\right|^{2}|dz|, \tag{3.3}$$

while from (2.15) we get

$$\left|\exp\left(\frac{(w-1)\exp w-(u-1)\exp u}{\exp w-\exp u}\right)-\frac{\exp w-\exp u}{w-u}\right| \le\frac{1}{2}\|\exp\|_{G,\infty}\frac{1}{|w-u|}\int_{\gamma}\left|\frac{(w-1)\exp w-(u-1)\exp u}{\exp w-\exp u}-z\right|^{2}|dz|. \tag{3.4}$$

Consider the function $F(z)=Log(z)$ where $Log(z)=\ln|z|+iArg(z)$ and $Arg(z)$ is such that $-\pi<Arg(z)\le\pi$. Log is called the *"principal branch"* of the complex logarithmic function. F is analytic on all of $\mathbb{L}:=\mathbb{C}\setminus\{x+iy:x\le 0,\ y=0\}$ and $F'(z)=\frac{1}{z}$ on this set.

If we consider $g:D\to\mathbb{C}$, $g(z)=\frac{1}{z}$ where $D\subset\mathbb{L}$, then F is a primitive of g on D and if $\gamma\subset D$ parametrized by $z(t)$, $t\in[a,b]$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$, then

$$\int_{\gamma}\frac{dz}{z}=Log(w)-Log(u).$$

Also, the function $G:\mathbb{L}\to\mathbb{C}$, $G(z)=zLog(z)-z$ is analytic on $\mathbb{L}$ and $G'(z)=Log(z)$, $z\in\mathbb{L}$.

Assume also that $f:D\to\mathbb{L}$ is continuous on D, $\gamma\subset D$ parametrized by $z(t)$, $t\in[a,b]$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ with $w\ne u$ and

$\frac{1}{w-u}\int_\gamma f(z)\,dz \in \mathbb{L}$, then from (2.1) for $F(z) = Logz$, we get

$$\left|\frac{1}{w-u}\int_\gamma (Log\circ f)(v)\,dv - Log\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)\right| \le \frac{1}{2}\frac{1}{d_\gamma^2|w-u|}\int_\gamma \left|f(v)-\frac{1}{w-u}\int_\gamma f(z)\,dz\right|^2|dv|, \tag{3.5}$$

where $d_\gamma := \inf_{z\in\gamma}|z|$ is assumed to be positive and finite.

For $\gamma \subset \mathbb{L}$ and $f(z) = z$, we get from (3.5) that

$$\left|\frac{wLog(w)-uLog(u)}{w-u} - Log\left(\frac{w+u}{2}\right) - 1\right| \le \frac{1}{2}\frac{1}{d_\gamma^2|w-u|}\int_\gamma \left|v-\frac{w+u}{2}\right|^2|dv|, \tag{3.6}$$

where $d_\gamma := \inf_{z\in\gamma}|z|$ is assumed to be positive and finite.

Further, for $F(z) = Logz$ we have

$$\begin{aligned}&\frac{wLogw - uLogu - \int_\gamma Logzdz}{Logw - Logu}\\ &= \frac{wLogw - uLogu - wLog(w) + w + uLog(u) - u}{Logw - Logu}\\ &= \frac{w-u}{Logw - Logu}.\end{aligned}$$

So, if $Logw \neq Logu$ and

$$\frac{w-u}{Logw - Logu} \in \mathbb{L},$$

then by (2.16) we get

$$\left|Log\left(\frac{w-u}{Logw - Logu}\right) - \frac{wLog(w)-uLog(u)}{w-u} + 1\right| \le \frac{1}{2}\frac{1}{d_\gamma^2|w-u|}\int_\gamma \left|\frac{w-u}{Logw-Logu} - z\right|^2|dz|. \tag{3.7}$$

Assume also that $f : D \to \mathbb{L}$ is continuous on D, $\gamma \subset D$ parametrized by $z(t)$, $t \in [a,b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ with $w \neq u$ and $\frac{1}{w-u}\int_\gamma f(z)\,dz \in \mathbb{L}$, then from (2.1) for $F(z) = z^{-1}$, we get

$$\left|\frac{1}{w-u}\int_\gamma [f(v)]^{-1}\,dv - \left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)^{-1}\right| \le \frac{1}{d_\gamma^3|w-u|}\int_\gamma \left|f(v)-\frac{1}{w-u}\int_\gamma f(z)\,dz\right|^2|dv|, \tag{3.8}$$

where $d_\gamma := \inf_{z\in\gamma}|z|$ is assumed to be positive and finite.

For $\gamma \subset \mathbb{L}$ and $f(z) = z$, we get from (3.8) that

$$\left| \frac{Log(w) - Log(u)}{w-u} - \left(\frac{w+u}{2}\right)^{-1} \right| \le \frac{1}{d_\gamma^3 |w-u|} \int_\gamma \left| v - \frac{w+u}{2} \right|^2 |dv| . \tag{3.9}$$

Further, for $F(z) = z^{-1}$ we have

$$\frac{F(w)w - F(u)u - \int_\gamma F(v)\,dv}{F(w) - F(u)} = \frac{-Log(w) + Log(u)}{\frac{1}{w} - \frac{1}{u}} = \frac{Log(w) - Log(u)}{w-u} wu$$

for $w \neq u$ and $u, w \in \mathbb{L}$.

If $w \neq u$ and $u, w \in \mathbb{L}$ with

$$\frac{Log(w) - Log(u)}{w-u} wu \in \mathbb{L},$$

then by (2.16) we get

$$\left| \left(\frac{Log(w) - Log(u)}{w-u} wu\right)^{-1} - \frac{Log(w) - Log(u)}{w-u} \right| \le \frac{1}{d_\gamma^3 |w-u|} \int_\gamma \left| \frac{Log(w) - Log(u)}{w-u} wu - z \right|^2 |dz| . \tag{3.10}$$

References

[1] R. Coleman, *Calculus on Normed Vector Spaces*, Springer New York Heidelberg Dordrecht London, 2012.

[2] S. S. Dragomir, Integral inequalities of Hermite-Hadamard type for K-bounded norm convex mappings, Preprint *RGMIA Res. Rep. Coll.* **17** (2014), Art. 149, 23 pp. [Online https://rgmia.org/papers/v17/v17a149.pdf].

[3] S. S. Dragomir, Integral inequalities of Hermite-Hadamard type for K-bounded norm convex mappings, *Ukrainian Mathematical Journal*, Vol. **68**, No. 10, March, 2017, 1530-1551.

[4] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results, *Australian J. Math. Anal. Appl.*, Volume **14**, Issue 1, Article 1, pp. 1-287, 2017. [Online http://ajmaa.org/cgi-bin/paper.pl?string=v14n1/V14I1P1.tex].

[5] S. S. Dragomir, Inequalities of Hermite-Hadamard type for K-bounded modulus convex complex functions, Preprint *RGMIA Res. Rep. Coll.* **21** (2018), Art. 130, 12 pp. [Online https://rgmia.org/papers/v21/v21a130.pdf].

[6] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000. [Online http://rgmia.org/monographs/hermite_hadamard.html].

Received 30 10 2019, revised 16 05 2020

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa