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Generalized Bessel recursion relations

M.L. Glasser

Abstract. This paper presents the equality of finite index sums of Bessel functions containing arbitrary numbers of terms. These reduce to the familiar three term recursion formulas in simple cases.

1. Introduction

The motivation for this note was the observation that the basic recursion relation for the modified Bessel function K[1],

$$K_0(z) + \left(\frac{2}{z}\right) K_1(z) = K_2(z)$$

can be expressed as the symmetry with respect to $m = 0$ and $n = 1$ of the sum

$$\sum_{k=0}^{n} K_{k-m-1}(z) \left(\frac{z}{2}\right)^{k+m}. \tag{1}$$

The attempt to generalize this to arbitrary m and n led to our principal result

Theorem 1

For positive integers m and n the expression

$$(n+1)! \sum_{k=0}^{n} \frac{1}{k!} \binom{m+k+1}{m} K_{k-m-1}(z) \left(\frac{z}{2}\right)^{k+m} \tag{2}$$

is symmetric with respect to m and n.

This will be proven in the following section and some similar results presented in the concluding paragraph.

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2. Calculation

Consider the sum

$$F(n,p,q) = \frac{(n+q+1)!}{q!(q+1)!}\sum_{k=0}^{p}\frac{(q+k+1)!}{(k+1)!}\frac{(n+k)!}{k!} \tag{3}$$

for $p, q, n \in \mathcal{Z}^+$. One finds that, e.g.

$$F(1,p,q) = \frac{(p+q+2)!}{p!q!}$$

$$F(2,p,q) = \frac{(p+q+2)!}{p!q!}[6+2(p+q)+pq]$$

and by induction on n one obtains

Lemma 1

$$\frac{p!q!}{(p+q+2)!}F(n,p,q)$$

is a polynomial $P(p,q) = P(q,p)$ of degree $n-1$ in p and q.

Next, by interchanging the order of summation and invoking lemma 1, one has

Lemma 2

$$G(p,q,z) = \sum_{n=0}^{\infty}\frac{1}{(n!)^2}F(n,p,q)z^n = \sum_{k=0}^{p}\binom{q+k+1}{q}\,{}_2F_1(k+1,q+2;1;z)$$

is analytic for $|z| < 1$ and symmetric with respect to p and q.

Finally, noting that[2]

$$\int_0^{\infty} J_0(z\sqrt{x})\,{}_2F_1(k+1,q+2;1;-x)dx = \frac{2^{-k-q}z^{k+q+1}}{k!(q+1)!}K_{k-q-1}(z) \tag{4}$$

(changing q to m and p to n) we have Theorem 1.

For example, with $m = 0$ we get the possibly new summation

$$\sum_{k=0}^{n}\frac{1}{k!}K_{k-1}(z)(z/2)^k = \frac{1}{n!}K_{n+1}(z)(z/2)^n. \tag{5}$$

Setting $z = -ix$ in the relation

$$K_\nu(z) = \frac{\pi}{2}i^{\nu+1}[J_\nu(iz) + iY_\nu(iz)] \tag{6}$$

after a small manipulation one obtains

Theorem 2

$$(-1)^m(n+1)!\sum_{k=0}^{n}\frac{1}{k!}\binom{m+k+1}{m}J_{k-m-1}(x)(x/2)^{k+m} \tag{7}$$

$$(-1)^m(n+1)!\sum_{k=0}^{n}\frac{1}{k!}\binom{m+k+1}{m}Y_{k-m-1}(x)(x/2)^{k+m} \tag{8}$$

are both symmetric with respect to m and n.

Corollary

$$\sum_{k=0}^{n}\frac{1}{k!}\,\mathcal{C}_{k-1}(x)(x/2)^k=-\frac{1}{n!}\,\mathcal{C}_{n+1}(x)(x/2)^n \tag{9}$$

where $\mathcal{C}=aJ+bY$.

3. Discussion

Since many integrals of the Gauss hypergeometric function are known, one of the most extensive tabulations being[2], Lemma 2 is the gateway to a myriad of unexpected finite sum identities involving various classes of special functions. We conclude by listing a small selection..

From[2]

$$\int_0^\infty (1-e^{-t})^{\lambda-1}e^{-xt}\;{}_2F_1(k+1,m+2;1;ze^{-t})dt$$
$$=B(x,\lambda)\;{}_3F_2(k+1,m+2,x;1,x+\lambda;z) \tag{10}$$

and one has the symmetry of

$$\sum_{k=0}^{n}\binom{m+k+1}{m}\;{}_3F_2(k+1,m+2,x;1,x+\lambda;z) \tag{11}$$

For example for $m=0$

$$\sum_{k=0}^{n}\;{}_3F_2(k+1,2,x;1,x+\lambda;z)=(n+1)\;{}_2F_1(n+2,x;x+\lambda;z). \tag{12}$$

Similarly,

$$\frac{n!(n+1)!}{\Gamma(n+2-a)}\sum_{k=0}^{n}\frac{(m+k+1)!\Gamma(k+1-a)}{k!(k+1)!}.$$
$$=\frac{m!(m+1)!}{\Gamma(m+2-a)}\sum_{k=0}^{m}\frac{(n+k+1)!\Gamma(k+1-a)}{k!(k+1)!} \tag{13}$$

$$\sum_{k=0}^{n}\binom{m+k+1}{m}=\sum_{k=0}^{m}\binom{n+k+1}{n}. \tag{14}$$

$$\sum_{k=0}^{n}\binom{m+k+1}{m}\,{}_3F_2(k+1,m+2,a;1,a+b;z)$$
$$=\sum_{k=0}^{m}\binom{n+k+1}{n}\,{}_3F_2(k+1,n+2,a;1,a+b;z). \tag{15}$$

$$\sum_{k=0}^{n}\,{}_3F_2(k+1,2,a;1,a+b;1)=\frac{(n+1)\Gamma(b-n-2)\Gamma(a+b)}{\Gamma(a+b-n-2)\Gamma(b)}. \tag{16}$$

$$\sum_{k=0}^{n}\frac{(p+k)!}{k!}=\frac{(n+p+1)!}{(p+1)n!},\quad p=0,1,2,\cdots \tag{17}$$

$$\sum_{k=0}^{n}\binom{m+k+1}{m}z^{(k+m)/2}S_{-k-m-2,k-m-1}(z)$$
$$=\sum_{k=0}^{m}\binom{n+k+1}{n}z^{(k+n)/2}S_{-k-n-2,k-n-1}(z). \tag{18}$$

$$\sum_{k=0}^{n}\binom{m+k+1}{m}z^{(k+m)/2}W_{-k-m-2,k-m-1}(z)$$
$$=\sum_{k=0}^{m}\binom{n+k+1}{n}z^{(k+n)/2}W_{-k-n-2,k-n-1}(z). \tag{19}$$

4. References

[1] G.E. Andrews, R. Askey and R. Roy, *Special Functions* [Cambridge University Press, 1999]

[2] A.P. Prudnikov, Yu. A.Brychkov and O.I. Marichev,*Integrals and Series, Vol. 3* [Gordon and Breach, NY 1986] Section 2.21.1.

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Department of Physics,
Clarkson University, Potsdam, NY (USA).
E-mail address: M Glasser <lglasser@clarkson.edu>