SCIENTIA
Series A: *Mathematical Sciences*, Vol. 30 (2020), 27–41
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

Novel reductions of Kampé de Fériet function

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Abstract. The aim of this article is to provide in all thirty two novel and general reductions of two and three variables generalized hypergeometric functions. This is achieved by the applications of the so-called Beta integral transform method developed by Krattenthaler and Srinivasa Rao, and Gamma integral method to the well known identities involving products of generalized hypergeometric functions. As special cases, we mention a few known as well as unknown interesting results.

1. Introduction and results required

The generalized hypergeometric function ${}_pF_q[x]$ is defined for complex parameters and argument by the series [**1, 2, 4, 45, 58, 59**]

$$ {}_pF_q\left[\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix} \,\middle|\, x \right] = \sum_{n \geq 0} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!} \,. \tag{1.1} $$

Here the Pochhammer symbol or ascending factorial $(a)_n, a \neq 0$ is given for nonnegative integer n by

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & n = 0 \\ a(a+1)\cdots(a+n-1) & n \in \mathbb{N} \end{cases}, $$

where Γ stands for the familiar Gamma functions, while $(0)_0 = 1$ is conventionally used. When $p \leq q$ this series converges for $x \in \mathbb{C}$, but when $q = p-1$ convergence occurs for $|x| < 1$ unless the series terminates.

The vast popularity and immense usefulness of the hypergeomtric function ${}_2F_1[x]$ and the generalized hypergeometric function ${}_pF_q[x]$ in one variable have inspired and stimulated a number of researchers in mathematics to study hypergeometric functions in two or more variables.

2010 *Mathematics Subject Classification.* Primary: 33C65, 33C20; Secondary: 33B15, 33C05, 33C15.
Key words and phrases. Generalized hypergeometric function, Kampé de Fériet function, Srivastava-Daoust double hypergeometric function, Triple series, Lauricella hypergeometric function.
The research of Tibor K. Pogány was supported in part of the University of Rijeka, Croatia under projects labeled `uniri-pr-prirod-19-16` and `uniri-tehnic-18-66`.
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In this sequel the *Srivastava-Daoust generalization of the Lauricella hypergeometric function* F_D in n variables is defined by multiple power series [**55**, p. 454]

$$\mathscr{S}^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}\left(\begin{array}{c}[(a):\theta',\cdots,\theta^{(n)}]:[(b'):\varphi'];\cdots;[(b^{(n)}):\varphi^{(n)}]\\ [(c):\psi';\cdots;\psi^{(n)}]:[(d'):\delta'];\cdots;[(d^{(n)}):\delta^{(n)}]\end{array}\middle|\,\boldsymbol{x}\right)$$
$$= \sum_{m_1,\cdots,m_n\ge 0}\Omega(m_1,\cdots,m_n)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}, \tag{1.2}$$

with the summation coefficient kernel

$$\Omega(m_1,\cdots,m_n) = \frac{\prod\limits_{j=1}^{A}(a_j)_{m_1\theta'_j+\cdots+m_n\theta^{(n)}_j}\prod\limits_{j=1}^{B'}(b'_j)_{m_1\varphi'_j}\cdots\prod\limits_{j=1}^{B^{(n)}}(b^{(n)}_j)_{m_n\varphi^{(n)}_j}}{\prod\limits_{j=1}^{C}(c_j)_{m_1\psi'_j+\cdots+m_n\psi^{(n)}_j}\prod\limits_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\cdots\prod\limits_{j=1}^{D^{(n)}}(d^{(n)}_j)_{m_n\delta^{(n)}_j}},$$

where the parameters satisfy

$$\theta'_1,\cdots,\theta'_A,\cdots,\delta^{(n)}_1,\cdots,\delta^{(n)}_{D^{(n)}} > 0,$$

and the shorthands $\boldsymbol{x} := (x_1,\cdots,x_n)^T$ and $[X:1]=X$ are used in the symbolic description (1.2). For convenience, we write (a) to denote the sequence of A parameters $a_1,\cdots,a_A$, with similar interpretations for $(b'),\cdots,(d^{(n)})$. Empty products should be interpreted as unity. Srivastava and Daoust [**56**, pp. 157–158] reported that the series in (1.2) converges absolutely

(i) for all $\boldsymbol{x}\in\mathbb{C}^n$ when

$$\Delta_\ell = 1+\sum_{j=1}^{C}\psi^{(\ell)}_j+\sum_{j=1}^{D^{(\ell)}}\delta^{(\ell)}_j-\sum_{j=1}^{A}\theta^{(\ell)}_j-\sum_{j=1}^{B^{(\ell)}}\varphi^{(\ell)}_j > 0, \qquad \ell=\overline{1,n};$$

(ii) for $|x_\ell|<\eta_\ell$ when $\Delta_\ell=0$, $\ell=\overline{1,n}$, where

$$\eta_\ell := \min_{\mu_1,\cdots,\mu_n>0}\left\{\mu_\ell^{1+\sum\limits_{j=1}^{D^{(\ell)}}\delta^{(\ell)}_j-\sum\limits_{j=1}^{B^{(\ell)}}\varphi^{(\ell)}_j}\;\frac{\prod\limits_{j=1}^{C}\left(\sum\limits_{\ell=1}^{n}\mu_\ell\psi^{(\ell)}_j\right)^{\psi^{(\ell)}_j}\prod\limits_{j=1}^{D^{(\ell)}}\left(\delta^{(\ell)}_j\right)^{\delta^{(\ell)}_j}}{\prod\limits_{j=1}^{A}\left(\sum\limits_{\ell=1}^{n}\mu_\ell\theta^{(\ell)}_j\right)^{\theta^{(\ell)}_j}\prod\limits_{j=1}^{B^{(\ell)}}\left(\varphi^{(\ell)}_j\right)^{\varphi^{(\ell)}_j}}\right\}.$$

When all $\Delta_\ell<0$, $\mathscr{S}^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}(x_1,\cdots,x_n)$ diverges except at the origin, that is, this series is formal.

The further set of conditions for convergence of the series $\mathscr{S}^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}$ is given in [**56**]. We remark at this point that the Srivastava–Daoust $\mathscr{S}$ generalized Lauricella hypergeometric function for $n=2$ reduces to $\mathscr{S}^{A:B;B'}_{C:D;D'}$, the Srivastava–Daoust generalized Kampé de Fériet hypergeometric function of two variables initially introduced in [**54, 55**]. Detailed account of the above function can be found in [**56**] and in the monograph [**58**].

In turn, specifying the parameter–array $(\theta'_1, \cdots, \theta'_A, \cdots, \delta_1^{(n)}, \cdots, \delta_{D^{(n)}}^{(n)}) = (1, \cdots, 1)$, we call the resulting $\mathscr{S}$ function a *Kampé de Fériet generalized hypergeometric function* signifying it by F (instead of $\mathscr{S}$). The *Kampé de Fériet hypergeometric function of two variables* (abbreviated also as KdF function in the sequel) defined by the double power series [**3**, p. 150] in a notation given e.g. by Srivastava and Panda [**60**, p. 423, Eq. (26)]

$$F^{p:q;k}_{l:m;n}\left[\begin{matrix} (a_p) : (b_q); (c_k) \\ (\alpha_l) : (\beta_m); (\gamma_n) \end{matrix} \,\middle|\, \begin{matrix} x \\ y \end{matrix} \right] = \sum_{r,t\geq 0} \frac{\prod_{j=1}^{p} (a_j)_{r+t} \prod_{j=1}^{q} (b_j)_r \prod_{j=1}^{k} (c_j)_t}{\prod_{j=1}^{l} (\alpha_j)_{r+t} \prod_{j=1}^{m} (\beta_j)_r \prod_{j=1}^{n} (\gamma_j)_t} \frac{x^r}{r!} \frac{y^t}{t!}, \tag{1.3}$$

which converges when [**56**]

(iii) $p + q < l + m + 1,\, p + k < l + n + 1,\, \max\{|x|, |y|\} < \infty,$ or

(iv) $p + q = l + m + 1,\, p + k = l + n + 1$ and

$$\begin{cases} |x|^{\frac{1}{p-l}} + |y|^{\frac{1}{p-l}} < 1, & l < p \\ \max\{|x|, |y|\} < 1, & l > p \end{cases}.$$

We point out that these convergence conditions immediately follow from **(i)** and **(ii)**.

Special attention is given to the *triple hypergeometric series* $F^{(3)}[x, y, z]$ of Srivastava and its special cases [**53**, p. 428]. The exact power series definition of this higher transcendental function reads [**58**, p. 44, Eqs. (14-15)]

$$F^{(3)}\left[\begin{matrix} (a) :: (b); (b'); (b'') : (c); (c'), (c'') \\ (e) :: (g); (g'); (g'') : (h); (h'); (h'') \end{matrix} \,\middle|\, x, y, z \right] = \sum_{m,n,p\geq 0} \Lambda(m, n, p) \frac{x^m}{m!} \frac{x^n}{n!} \frac{x^p}{p!},$$

for which the shorthand notation $F^{(3)}[z, y, z]$ is used, and the kernel expression

$$\Lambda(m, n, p) = \frac{\prod_{j=1}^{A} (a_j)_{m+n+p} \prod_{j=1}^{B} (b_j)_{m+n} \prod_{j=1}^{B'} (b'_j)_{m+p} \prod_{j=1}^{B''} (b''_j)_{n+p}}{\prod_{j=1}^{E} (e_j)_{m+n+p} \prod_{j=1}^{G} (g_j)_{m+n} \prod_{j=1}^{G'} (g'_j)_{m+p} \prod_{j=1}^{G''} (g''_j)_{n+p}} \frac{\prod_{j=1}^{C} (c_j)_m \prod_{j=1}^{C'} (c'_j)_n \prod_{j=1}^{C''} (c''_j)_p}{\prod_{j=1}^{H} (h_j)_m \prod_{j=1}^{H'} (h'_j)_n \prod_{j=1}^{H''} (h''_j)_p}.$$

For the Kampé de Fériet function, Srivastava–Daoust double hypergeometric function and Srivastava triple series, there are lots of reductions formulae available in the literature. Early ones can be found in the research articles like [**6**], [**14–17**], [**19**], [**21–23**], [**29, 30**], [**32, 33**], [**34, 35**], [**39–44**], [**47–56**] and [**61–66**] and for many more results, we refer to the standard monographs [**20, 58**]. For recent ones, we refer the research articles [**5**], [**7–13**], [**24, 25**], [**28**], [**36**], [**46**], [**57**].

Krattenthaler and Srinivasa Rao [**26**] showed how identities for hypergeometric series for some fixed value of the argument (usually 1) can be derived from known hypergeometric identities with a similar number of parameters involving arguments $x, 1 - x$ or a combination of their powers. The basic idea of the method isto multiply the known hypergeometric indetity by the factor $x^{d-1}(1 - x)^{e-d-1}$ with suitable parameters d, e, integrate term by term making use of the Beta integral representation

$$\int_0^1 x^{d-1}(1-x)^{e-d-1}\,\mathrm{d}t = \mathrm{B}(d,e-d) = \frac{\Gamma(d)\,\Gamma(e-d)}{\Gamma(e)}, \qquad \Re(e) > \Re(d) > 0\,,$$

where $\mathrm{B}(\cdot,\cdot)$ stands for the Euler Beta function. For the hypergeometric function and finally re-write the result in terms of a new hypergeometric series. This so–called Beta integral method has been automated using computer algebra and the software package HYP to generate both old and new results. We may, by this technique, apply Gamma integral method as well.

In our present investigation we shall require the following known identities see e.g. Prudnikov *et al.* [**38**, p. 488, Eq. 7.11.1.7]

$${}_1F_1\left[\begin{array}{c}\alpha\\ 2\alpha+n\end{array}\Big| x\right] = \frac{\Gamma(\alpha-\frac12)}{\left(\frac14 x\right)^{\alpha-\frac12}}\,\mathrm{e}^{\frac12 x}\sum_{k=0}^n \frac{(-n)_k(2\alpha-1)_k}{(2\alpha+n)_k\,k!}(\alpha+k-\tfrac12)\,I_{\alpha+k-\frac12}\left(\tfrac12 x\right)\,,$$

valid for all $n\in\mathbb{N}_0$ and where I_ν stands for the modified Bessel function of the first kind. By virtue of the well-known transformation formula [**1**, p. 377, Eq. 9.6.47]

$$I_\nu(z) = \frac{\left(\frac12 z\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left[\begin{array}{c}-\\ \nu+1\end{array}\Big|\tfrac14 z^2\right], \tag{1.4}$$

we re–write the previous display in terms of the confluent hypergeometric function into the equivalent form:

$${}_1F_1\left[\begin{array}{c}\alpha\\ 2\alpha+n\end{array}\Big| x\right] = \mathrm{e}^{\frac12 x}\sum_{k=0}^n \frac{(-n)_k(2\alpha-1)_k}{(2\alpha+n)_k\,(\alpha-\frac12)_k\,k!}\left(\tfrac14 x\right)^k {}_0F_1\left[\begin{array}{c}-\\ \alpha+k+\frac12\end{array}\Big|\tfrac1{16}x^2\right], \tag{1.5}$$

in turn [**38**, p. 487, Eq. 7.11.1.6]

$$\begin{aligned}{}_1F_1\left[\begin{array}{c}\alpha\\ 2\alpha-n\end{array}\Big| x\right] &= \Gamma(\alpha-n-\tfrac12)\left(\tfrac14 x\right)^{n-\alpha+\frac12}\,\mathrm{e}^{\frac12 x}\\ &\times\sum_{k=0}^n \frac{(-1)^k(-n)_k(2\alpha-2n-1)_k}{(2\alpha-n)_k\,k!}(\alpha+k-n-\tfrac12)\,I_{\alpha+k-n-\frac12}\left(\tfrac12 x\right)\,,\end{aligned} \tag{1.6}$$

holds for all $n\in\mathbb{N}_0$. Applying the same transformation formula (1.4) by which we re–write the modified Bessel function of the first kind into a confluent hypergeometric function the previous display becomes

$$\begin{aligned}\mathrm{e}^{-\frac12 x}\,{}_1F_1\left[\begin{array}{c}\alpha\\ 2\alpha-n\end{array}\Big| x\right] &= \sum_{k=0}^n \frac{(-1)^k(-n)_k(2\alpha-2n-1)_k\left(\frac14 x\right)^k}{(2\alpha-n)_k\,(\alpha-n-\frac12)_k\,k!}\\ &\qquad\times{}_0F_1\left[\begin{array}{c}-\\ \alpha+k-n+\frac12\end{array}\Big|\tfrac1{16}x^2\right],\end{aligned} \tag{1.7}$$

where $n\in\mathbb{N}_0$.

Using the same method as Rathie in [**48**], Choi and Rathie [**10**] expressed the square of the confluent (or Kummer's) hypergeometric function having contiguous parameters built three results in the form of a double finite sum involving the associated generalized hypergeometric ${}_2F_3$ function terms [**10**, p. 1597–98, Eqs. (2.1), (2.2) and (2.3)]

$$\Big\{{}_1F_1\Big[\begin{matrix}\alpha\\ 2\alpha+n\end{matrix}\Big|x\Big]\Big\}^2 = \mathrm{e}^x\sum_{j,k=0}^{n}\frac{(-n)_j(-n)_k(2\alpha-1)_j(2\alpha-1)_k\,x^{j+k}}{4^{j+k}(\alpha-\frac12)_j(\alpha-\frac12)_k(2\alpha+n)_j(2\alpha+n)_k j!k!}$$
$$\times\,{}_2F_3\Big[\begin{matrix}\alpha+\frac{j+k}{2}),\ \alpha+\frac{j+k+1}{2}\\ \alpha+j+\frac12,\ \alpha+k+\frac12,\ 2\alpha+j+k\end{matrix}\Big|\tfrac14 x^2\Big] \tag{1.8}$$

$$\Big\{{}_1F_1\Big[\begin{matrix}\alpha\\ 2\alpha-n\end{matrix}\Big|x\Big]\Big\}^2 = \mathrm{e}^x\sum_{j,k=0}^{n}\frac{(-1)^{j+k}(-n)_j(-n)_k(2\alpha-2n-1)_j(2\alpha-2n-1)_k\,x^{j+k}}{4^{j+k}(\alpha-n-\frac12)_j(\alpha-n-\frac12)_k(2\alpha-n)_j(2\alpha-n)_k j!k!}$$
$$\times\,{}_2F_3\Big[\begin{matrix}\alpha-n+\frac{j+k}{2},\ \alpha-n+\frac{j+k+1}{2}\\ \alpha-n+j+\frac12,\ \alpha-n+k+\frac12,\ 2(\alpha-n)+j+k\end{matrix}\Big|\tfrac14 x^2\Big]. \tag{1.9}$$

$${}_1F_1\Big[\begin{matrix}\alpha\\ 2\alpha+n\end{matrix}\Big|x\Big]{}_1F_1\Big[\begin{matrix}\alpha\\ 2\alpha-n\end{matrix}\Big|x\Big]$$
$$= \mathrm{e}^x\sum_{j,k=0}^{n}\frac{(-1)^{k}(-n)_j(-n)_k(2\alpha-1)_j(2\alpha-2n-1)_k\,x^{j+k}}{4^{j+k}(\alpha-\frac12)_j(\alpha-n-\frac12)_k(2\alpha+n)_j(2\alpha-n)_k j!k!}$$
$$\times\,{}_2F_3\Big[\begin{matrix}\alpha+\frac{j+k-n}{2},\ \alpha+\frac{j+k-n+1}{2}\\ \alpha+j+\frac12,\ \alpha-n+k+\frac12,\ 2\alpha-n+j+k\end{matrix}\Big|\frac{x^2}{4}\Big]. \tag{1.10}$$

For $n=0$ the results (1.5) and (1.7) reduce to the following Kummer second theorem [**27**] *viz.*

$$\mathrm{e}^{-\frac12 x}{}_1F_1\Big[\begin{matrix}\alpha\\ 2\alpha\end{matrix}\Big|x\Big] = {}_0F_1\Big[\begin{matrix}-\\ \alpha+\frac12\end{matrix}\Big|\frac{x^2}{16}\Big],$$

whereas for $n=0$ the formulae (1.8), (1.9) and (1.10) reduce to the following well–known Preece's indetity [**37**]

$$\Big\{{}_1F_1\Big[\begin{matrix}\alpha\\ 2\alpha\end{matrix}\Big|x\Big]\Big\}^2 = \mathrm{e}^x{}_1F_2\Big[\begin{matrix}\alpha\\ \alpha+\frac12,\ 2\alpha\end{matrix}\Big|\frac{x^2}{4}\Big].$$

The organization of the article is following. In sections 2 and 3, we shall establsh reduction formulae between generalized hypergeometric functions, Kampé de Fériet function and Srivastava–Daoust $\mathscr{S}$–function of two variables by the application of the Beta and Gamma integral methods to the known identities (1.5) and (1.7) respectively. Sections 4, 5, and 6 deal with the reduction formulae between generalized hypergeometric function, Kampé de Fériet function and Srivastava–Daoust $\mathscr{S}$–function of two variables and Srivastava triple series by the application of the Beta and Gamma integration methods to the identities (1.8), (1.9) and (1.10), respectively.

2. The first set of reduction formulae

Proposition 2.1. *For all $n \in \mathbb{N}_0$ and $\Re(e) > \Re(d) > 0$ there hold true:*

$$F^{\,1:0;1}_{\,1:0;1}\left[\begin{array}{c} d : \; - \; ; \; \alpha \\ e : - \; ; 2\alpha + n \end{array} \middle| \begin{array}{c} -\frac{1}{2}x \\ x \end{array} \right] = \sum_{k=0}^{n} \frac{(-n)_k (2\alpha - 1)_k \, (d)_k}{(2\alpha + n)_k \, (\alpha - \frac{1}{2})_k \, (e)_k \, k!} \big(\tfrac{1}{4}x\big)^k$$

(2.1)
$$\times \, {}_2F_3\left[\begin{array}{c} \frac{d+k}{2}, \; \frac{d+k+1}{2} \\ \alpha + k + \frac{1}{2}, \; \frac{e+k}{2}, \; \frac{e+k+1}{2} \end{array} \middle| \frac{1}{16}x^2 \right],$$

$${}_2F_2\left[\begin{array}{c} \alpha, \, d \\ 2\alpha + n, \, e \end{array} \middle| x \right] = \sum_{k=0}^{n} \frac{(-n)_k (2\alpha - n)_k \, (d)_k}{(2\alpha + n)_k \, (\alpha - \frac{1}{2})_k \, (e)_k \, k!} \Big(\frac{x}{4}\Big)^k$$

(2.2)
$$\times \, \mathscr{S}^{1:0;0}_{1:0;1}\left(\begin{array}{c} [d : 1, 2] : \; - \; ; \; - \\ [e : 1, 2] : -; [\alpha + k + \frac{1}{2}, 1] \end{array} \middle| \begin{array}{c} \frac{1}{2}x \\ \frac{1}{16}x^2 \end{array} \right),$$

(2.3)
$${}_2F_1\left[\begin{array}{c} \alpha, \, d \\ 2\alpha + n \end{array} \middle| x \right] = \sum_{k=0}^{n} \frac{(-n)_k (2\alpha - 1)_k (d)_k \, x^k}{4^k (2\alpha + n)_k \, (\alpha - \frac{1}{2})_k \, k!} \mathscr{S}^{1:0;0}_{0:0;1}\left(\begin{array}{c} [d + k : 1; 2] : -; - \\ - : -; [\alpha + k + \frac{1}{2}, 1] \end{array} \middle| \begin{array}{c} \frac{1}{2}x \\ \frac{1}{16}x^2 \end{array} \right),$$

(2.4)
$$F^{1:0;1}_{0:0;1}\left[\begin{array}{c} d : \; - \; ; \alpha \\ - : - ; 2\alpha + n \end{array} \middle| \begin{array}{c} -\frac{1}{2}x \\ x \end{array} \right] = \sum_{k=0}^{n} \frac{(-n)_k (2\alpha - 1)_k \, (d)_k \, x^k}{4^k \, (2\alpha + n)_k \, (\alpha - \frac{1}{2})_k \, k!} \, {}_2F_1\left[\begin{array}{c} \frac{d+k}{2}, \; \frac{d+k+1}{2} \\ \alpha + k + \frac{1}{2} \end{array} \middle| \frac{1}{4}x^2 \right].$$

Proof. The proofs of all these results are quiet straightforward. In order to establish the display (2.1) use proceed as follows. Re-write (1.5) into:

(2.5)
$$\mathrm{e}^{-\frac{1}{2}x} \, {}_1F_1\left[\begin{array}{c} \alpha \\ 2\alpha + n \end{array} \middle| x \right] = \sum_{k=0}^{n} \frac{(-n)_k (2\alpha - 1)_k}{(2\alpha + n)_k \, (\alpha - \frac{1}{2})_k \, k!} \big(\tfrac{1}{4}x\big)^k \, {}_0F_1\left[\begin{array}{c} - \\ \alpha + k + \frac{1}{2} \end{array} \middle| \frac{x^2}{16} \right].$$

Firstly, express

$$\mathrm{e}^{-\frac{1}{2}x} = {}_0F_0\left[\begin{array}{c} - \\ - \end{array} \middle| -\frac{1}{2}x \right],$$

then replace x with xt, multiply both sides by $t^{d-1}(1-t)^{e-d-1}$ and integrate with respect to t in the unit interval, getting

$$L_1 := \int_0^1 t^{d-1}(1-t)^{e-d-1} \, {}_0F_0\left[\begin{array}{c} - \\ - \end{array} \middle| -\frac{1}{2}xt \right] \, {}_1F_1\left[\begin{array}{c} \alpha \\ 2\alpha + n \end{array} \middle| xt \right] \mathrm{d}t$$

(2.6)
$$= \sum_{k=0}^{n} \frac{(-n)_k (2\alpha - 1)_k \, \big(\frac{1}{4}x\big)^k}{(2\alpha + n)_k (\alpha - \frac{1}{2})_k \, k!} \int_0^1 t^{d+k-1}(1-t)^{e-d-1} \, {}_0F_1\left[\begin{array}{c} - \\ \alpha + k + \frac{1}{2} \end{array} \middle| \frac{x^2t^2}{16} \right] \mathrm{d}t =: R_1.$$

Representing both hypergeometric terms as series, after some simplification we get

$$\begin{aligned} L_1 &= \sum_{k,m\ge 0} \frac{(\alpha)_m}{(2\alpha+n)_m} \frac{(-\frac12 x)^k\, x^m}{k!m!} \int_0^1 t^{d+k+m-1}(1-t)^{e-d-1}\,\mathrm{d}t \\ &= \mathrm{B}(d,e-d) \sum_{k,m\ge 0} \frac{(d)_{k+m}}{(e)_{k+m}} \frac{(\alpha)_m}{(2\alpha+n)_m} \frac{(-\frac12 x)^k\, x^m}{k!\; m!}\,. \end{aligned}$$

Finally, by virtue of the definition (1.3) of the KdF function we conclude that

$$L_1 = \mathrm{B}(d,e-d)\, F^{1:0;1}_{1:0;1}\left[\begin{array}{c} d: \; - \;\; ;\; \alpha \\ e: - \;; 2\alpha+n \end{array} \middle| \begin{array}{c} -\frac12 x \\ x \end{array} \right].$$

On the other hand, as to the right–hand–side of (2.6) R_1, say, express ${}_0F_1$ as a series, change the order if integration and summation (which is a legitimate operation), evaluate the Beta integral, using the Pochhammer symbol's definition and after some reduction of the matter, summing up with the generalized hypergeometric series (1.1) we deduce

$$\begin{aligned} R_1 &= \frac{\Gamma(d)\Gamma(e-d)}{\Gamma(e)} \sum_{k=0}^n \frac{(-n)_k(2\alpha-1)_k\,(d)_k}{(2\alpha+n)_k\,(\alpha-\frac12)_k\,(e)_k\,k!}\left(\tfrac14 x\right)^k \\ &\quad\times {}_2F_3\left[\begin{array}{c} \frac{d+k}{2},\; \frac{d+k+1}{2} \\ \alpha+k+\frac12,\; \frac{e+k}{2},\; \frac{e+k+1}{2} \end{array} \middle| \tfrac{1}{16}x^2 \right]. \end{aligned}$$

Finally, equating L_1 and R_1, we arrive at the statement (2.1).

For establishing the second statement (2.2) apply the result (1.5) and repeat similar lines as that given in the previous proof.

In obtaining the third formula (2.3) consider (1.5) in which substitute xt instead of x and write the hypergeometric term on the left-hand-side $L_2(t)$, say, as the power series. Multiplying this expression with the Gamma integral kernel $\mathrm{e}^{-t}\, t^{d-1}, \Re(d) > 0$ and integrating with respect to t on the positive real half–axis, after routine transformations getting

$$\int_0^\infty \mathrm{e}^{-t}\, t^{d-1}\, L_2(t)\,\mathrm{d}t = \Gamma(d) \sum_{j\ge 0} \frac{(\alpha)_j (d)_j}{(2\alpha+n)_j} \frac{x^j}{j!} = \Gamma(d)\; {}_2F_1\left[\begin{array}{c} \alpha,\, d \\ 2\alpha+n \end{array} \middle| x \right].$$

Now, replacing x with xt, expanding the right-hand-side's $R_2(t)$, say, both exponential and hypergeometric terms into Maclaurin series. This results in a triple series

$$R_2(t) = \sum_{k=0}^n \sum_{j,m\ge 0} \frac{(-n)_k(2\alpha-1)_k\, \left(\frac14 x\right)^k \left(\frac12 x\right)^j \left(\frac1{16}x^2\right)^m}{(2\alpha+n)_k\,(\alpha-\frac12)_k\,(\alpha+k+\frac12)_m\; k!\; j!\; m!}\, t^{k+j+2m}\,;$$

Than we repeat the previous Gamma integral procedure in (1.5) by which the subsequent integral becomes

$$\int_0^\infty \mathrm{e}^{-t}\, t^{d-1}\, R_2(t)\,\mathrm{d}t = \Gamma(d) \sum_{k=0}^n \frac{(-n)_k(2\alpha-1)_k \left(\frac14 x\right)^k}{(2\alpha+n)_k(\alpha-\frac12)_k\; k!} \sum_{j,m\ge 0} \frac{(d+k)_{j+2m}}{(\alpha+k+1)_m} \frac{\left(\frac{x}{2}\right)^j}{j!} \frac{\left(\frac{x^2}{16}\right)^m}{m!}\,,$$

such that coincides with the right–hand–side expression in (2.3). The rest is obvious.

Finally, as to the result (2.4) we start with (2.5) using the same method in obtaining the asserted result. □

2.1. Special cases. If we set $n = 0$ in the formulae (2.1)–(2.4) we immediately deduce the following corollaries of the Proposition 2.1., respectively:

$$F^{1:0;1}_{1:0;1}\left[\begin{matrix} d: & - & ; & \alpha \\ e: & - & ; & 2\alpha \end{matrix}\,\middle|\,\begin{matrix} -\frac12 x \\ x \end{matrix}\right] = {}_2F_3\left[\begin{matrix} \frac{d}{2},\ \frac{d+1}{2} \\ \alpha+\frac12,\ \frac{e}{2},\ \frac{e+1}{2}\end{matrix}\,\middle|\,\frac{1}{16}x^2\right], \tag{2.7}$$

$$\mathscr{S}^{1:0;0}_{1:0;1}\left(\begin{matrix} [d:1,2]: & - & ; & - \\ [e:1,2]: & -; & [\alpha+\frac12,1] \end{matrix}\,\middle|\,\begin{matrix} \frac12 x \\ \frac{1}{16}x^2 \end{matrix}\right) = {}_2F_2\left[\begin{matrix} \alpha,\ d \\ 2\alpha,\ e \end{matrix}\,\middle|\,x\right], \tag{2.8}$$

$$\mathscr{S}^{1:0;0}_{0:0;1}\left(\begin{matrix} [d:1;2]:-;- \\ -:-;[\alpha+\frac12,1] \end{matrix}\,\middle|\,\begin{matrix} \frac12 x \\ \frac{1}{16}x^2 \end{matrix}\right) = {}_2F_1\left[\begin{matrix} \alpha,\ d \\ 2\alpha \end{matrix}\,\middle|\,x\right], \tag{2.9}$$

$$F^{1:0;1}_{0:0;1}\left[\begin{matrix} d: & - & ;\alpha \\ -: & - & ;2\alpha \end{matrix}\,\middle|\,\begin{matrix} -\frac12 x \\ x \end{matrix}\right] = {}_2F_1\left[\begin{matrix} \frac{d}{2},\ \frac{d+1}{2} \\ \alpha+\frac12 \end{matrix}\,\middle|\,\frac14 x^2\right]. \tag{2.10}$$

A plethora of further particular results can be inferred by appropriately fixing n in Proposition 2.1. We also point out that these four formulae hold for all $\Re(e) > \Re(d) > 0$, according to the ancestor proposition's preambula.

We close this section remarking that another reduction formulae given in subsequent sections can be proven now in a straightforward way mimicking both proving procedures of the Proposition 2.1. Accordingly, the subsequent results are given without proofs which are left as exercises for the interested readers.

3. The second set of reduction formulae

Here we establish reduction formulae which originate back to (1.6), that is (1.7).

PROPOSITION 3.1. *For all $n \in \mathbb{N}_0$ and $\Re(e) > \Re(d) > 0$ we have*

$$\begin{aligned} F^{1:0;1}_{1:0;1}\left[\begin{matrix} d: & - & ; & \alpha \\ e: & - & ; & 2\alpha-n \end{matrix}\,\middle|\,\begin{matrix} -\frac12 x \\ x \end{matrix}\right] &= \sum_{k=0}^n \frac{(-1)^k(-n)_k(2\alpha-2n-1)_k\,(d)_k}{(2\alpha-n)_k\,(\alpha-n-\frac12)_k\,(e)_k\,k!}\left(\tfrac14 x\right)^k \\ &\quad\times {}_2F_3\left[\begin{matrix} \frac{d+k}{2},\ \frac{d+k+1}{2} \\ \alpha+k-n+\frac12,\ \frac{e+k}{2},\ \frac{e+k+1}{2}\end{matrix}\,\middle|\,\frac{1}{16}x^2\right], \end{aligned} \tag{3.1}$$

$$\begin{aligned} {}_2F_2\left[\begin{matrix} \alpha,\ d \\ 2\alpha-n,\ e \end{matrix}\,\middle|\,x\right] &= \sum_{k=0}^n \frac{(-1)^k(-n)_k(2\alpha-2n-1)_k\,(d)_k}{(2\alpha-n)_k\,(\alpha-n-\frac12)_k\,(e)_k\,k!}\left(\frac{x}{4}\right)^k \\ &\quad\times \mathscr{S}^{1:0;0}_{1:0;1}\left(\begin{matrix} [d:1,2]: & - & ; & - \\ [e:1,2]: & -; & [\alpha+k-n+\frac12,1] \end{matrix}\,\middle|\,\begin{matrix} \frac12 x \\ \frac{1}{16}x^2 \end{matrix}\right), \end{aligned} \tag{3.2}$$

$$\begin{aligned} {}_2F_1\left[\begin{matrix} \alpha,\ d \\ 2\alpha-n \end{matrix}\,\middle|\,x\right] &= \sum_{k=0}^n \frac{(-1)^k(-n)_k(2\alpha-2n-1)_k(d)_k\,x^k}{4^k(2\alpha-n)_k\,(\alpha-n-\frac12)_k\,k!} \\ &\quad\times \mathscr{S}^{1:0;0}_{0:0;1}\left(\begin{matrix} [d+k:1;2]:-;- \\ -:-;[\alpha+k-n-\frac12,1] \end{matrix}\,\middle|\,\begin{matrix} \frac12 x \\ \frac{1}{16}x^2 \end{matrix}\right), \end{aligned} \tag{3.3}$$

$$F^{1:0;1}_{0:0;1}\left[\begin{matrix} d: & - & ;\alpha \\ -: & -; & 2\alpha-n\end{matrix}\,\middle|\,\begin{matrix}-\frac{1}{2}x\\ x\end{matrix}\right] = \sum_{k=0}^{n}\frac{(-1)^k(-n)_k(2\alpha-2n-1)_k\,(d)_k\,x^k}{4^k\,(2\alpha-n)_k\,(\alpha-n-\frac{1}{2})_k\,k!}$$

$$\times\,{}_2F_1\left[\begin{matrix}\frac{d+k}{2},\ \frac{d+k+1}{2}\\ \alpha+k-n+\frac{1}{2}\end{matrix}\,\middle|\,\frac{1}{4}x^2\right]. \tag{3.4}$$

It is interesting to mention here that if in (3.1)–(3.4) we set $n = 0$ we arrive immediately at the corollaries (2.7)–(2.9) of the previous proposition.

4. The third set of reduction formulae

In this section we establish further reduction formulae which hold for all nonnegative integer n and complex input parameters α. To infer the results exposed below, the starting equality is (1.8). Here we evidently show the connecting relations between Kampé de Fériet functions $F^{1:1;1}_{1:1;1}[x,-x]$, $F^{1:1;1}_{0:1;1}[x,-x]$ and the appropriate triple Lauricella functions $F^{(3)}[-x,x,x]$ *via* finite linear combinations of contiguous generalized hypergeometric ${}_4F_5[x^2/4]$ and ${}_4F_3[x^2/4]$, respectively.

Proposition 4.1. *For all $n\in\mathbb{N}_0$, $d,e;\Re(e)>\Re(d)>0$ and $\alpha\in\mathbb{C}$, we have*

$$F^{1:1;1}_{1:1;1}\left[\begin{matrix} d: & \alpha & ; & \alpha\\ e: & 2\alpha+n; & & 2\alpha+n\end{matrix}\,\middle|\,\begin{matrix}x\\ x\end{matrix}\right] = \sum_{j,\ k=0}^{n}\frac{(d)_{j+k}(-n)_j(-n)_k(2\alpha-1)_j(2\alpha-1)_k\left(\frac{1}{4}x\right)^{j+k}}{(e)_{j+k}(\alpha-\frac{1}{2})_j(\alpha-\frac{1}{2})_k\,(2\alpha+n)_j(2\alpha+n)_k\ j!k!}$$

$$\times\,\mathscr{S}^{1:0;2}_{1:0;3}\left(\begin{matrix}[d+j+k:1,2]:-\,;[\alpha+\frac{j+k}{2},1];[\alpha+\frac{j+k+1}{2},1]\\ [e+j+k:1,2]:-;[\alpha+j+\frac{1}{2},1];[\alpha+k+\frac{1}{2},1];[2\alpha+j+k,1]\end{matrix}\,\middle|\,\begin{matrix}x\\ \frac{1}{4}x^2\end{matrix}\right), \tag{4.1}$$

$$F^{1:1;1}_{1:1;1}\left[\begin{matrix} d: & \alpha & ; & \alpha+n\\ e: & 2\alpha+n; & & 2\alpha+n\end{matrix}\,\middle|\,\begin{matrix}x\\ -x\end{matrix}\right] = F^{(3)}\left[\begin{matrix} d:: & -;-;-;-;\alpha;\alpha\\ e:: & -;-;-;-;2\alpha+n;2\alpha+n\end{matrix}\,\middle|\,-x,x,x\right]$$

$$= \sum_{j,k=0}^{n}\frac{(-n)_j(-n)_k(2\alpha-1)_j(2\alpha-1)_k\,(d)_{j+k}x^{j+k}}{4^{j+k}(\alpha-\frac{1}{2})_j(\alpha-\frac{1}{2})_k\,(2\alpha+n)_j(2\alpha+n)_k\,(e)_{j+k}j!\,k!}$$

$$\times\,{}_4F_5\left[\begin{matrix}\alpha+\frac{j+k}{2},\ \alpha+\frac{j+k+1}{2},\ \frac{d+j+k}{2},\ \frac{d+j+k+1}{2}\\ \alpha+j+\frac{1}{2},\ \alpha+k+\frac{1}{2},\ 2\alpha+j+k,\ \frac{e+j+k}{2},\ \frac{e+j+k+1}{2}\end{matrix}\,\middle|\,\frac{1}{4}x^2\right], \tag{4.2}$$

Moreover, we have

$$F^{1:1;1}_{0:1;1}\left[\begin{matrix} d: & \alpha & ; & \alpha\\ -: & 2\alpha+n; & & 2\alpha+n\end{matrix}\,\middle|\,\begin{matrix}x\\ x\end{matrix}\right] = \sum_{j,\ k=0}^{n}\frac{(-n)_j(-n)_k(2\alpha-1)_j(2\alpha-1)_k\,(d)_{j+k}\,x^{j+k}}{4^{j+k}(\alpha-\frac{1}{2})_j(\alpha-\frac{1}{2})_k\,(2\alpha+n)_j(2\alpha+n)_k\,j!\,k!}$$

$$\times\,\mathscr{S}^{1:0;2}_{0:0;3}\left(\begin{matrix}[d+j+k:1,2]:-\,;[\alpha+\frac{j+k}{2},1];[\alpha+\frac{j+k+1}{2},1]\\ -:-;[\alpha+j+\frac{1}{2},1];[\alpha+k+\frac{1}{2},1];[2\alpha+j+k,1]\end{matrix}\,\middle|\,\begin{matrix}x\\ \frac{1}{4}x^2\end{matrix}\right), \tag{4.3}$$

$$
\begin{aligned}
F^{1:1;1}_{0:1;1}\left[\begin{matrix} d:\ \alpha\ ;\ \alpha+n \\ -:2\alpha+n;2\alpha+n\end{matrix}\,\middle|\,\begin{matrix} x\\ -x\end{matrix}\right] &= F^{(3)}\left[\begin{matrix} d::\ -;\ -;\ -;\ -;\ \alpha;\ \alpha \\ -::-;-;-;-;2\alpha+n;2\alpha+n\end{matrix}\,\middle|\, -x,x,x\right] \\
&= \sum_{j,k=0}^{n} \frac{(-n)_j(-n)_k(2\alpha-1)_j(2\alpha-1)_k\,(d)_{j+k}x^{j+k}}{4^{j+k}(\alpha-\frac{1}{2})_j(\alpha-\frac{1}{2})_k\,(2\alpha+n)_j(2\alpha+n)_k\,j!\,k!} \\
&\quad\times {}_4F_3\left[\begin{matrix}\alpha+\frac{j+k}{2},\ \alpha+\frac{j+k+1}{2},\ \frac{d+j+k}{2},\ \frac{d+j+k+1}{2} \\ \alpha+j+\frac{1}{2},\ \alpha+k+\frac{1}{2},\ 2\alpha+j+k\end{matrix}\,\middle|\,\frac{1}{4}x^2\right]. \qquad (4.4)
\end{aligned}
$$

4.1. Special cases. Setting $n=0$, keeping valid the parameters' range $d,e;\Re(e)>\Re(d)>0$ and $\alpha\in\mathbb{C}$ in (4.1)–(4.4), we get the following results.

$$
F^{1:1;1}_{1:1;1}\left[\begin{matrix} d:\alpha;\alpha \\ e:2\alpha;2\alpha\end{matrix}\,\middle|\,\begin{matrix}x\\x\end{matrix}\right] = \mathscr{S}^{1:0;1}_{1:0;2}\left(\begin{matrix}[d:1,2]:-\ ;[\alpha,1] \\ [e:1,2]:-\ ;[\alpha+\frac{1}{2},1];[2\alpha,1]\end{matrix}\,\middle|\,\begin{matrix}x\\ \frac{1}{4}x^2\end{matrix}\right),
$$

$$
{}_3F_4\left[\begin{matrix}\alpha,\ \frac{d}{2},\ \frac{d+1}{2} \\ \alpha+\frac{1}{2},\ 2\alpha,\ \frac{e}{2},\ \frac{e+1}{2}\end{matrix}\,\middle|\,\frac{1}{4}x^2\right] = \begin{cases} F^{1:1;1}_{1:1;1}\left[\begin{matrix}d:\ \alpha;\ \alpha \\ e:2\alpha;2\alpha\end{matrix}\,\middle|\,\begin{matrix}x\\-x\end{matrix}\right] \\ F^{(3)}\left[\begin{matrix}d::-;-;-;-;\ \alpha\ ;\ \alpha \\ e::-;-;-;-;2\alpha;2\alpha\end{matrix}\,\middle|\,-x,x,x\right]\end{cases}; \qquad (4.5)
$$

$$
F^{1:1;1}_{0:1;1}\left[\begin{matrix} d:\alpha;\alpha \\ -:2\alpha;2\alpha\end{matrix}\,\middle|\,\begin{matrix}x\\x\end{matrix}\right] = \mathscr{S}^{1:0;1}_{0:0;2}\left(\begin{matrix}[d:1,2]:-\ ;[\alpha,1] \\ -:-\ ;[\alpha+\frac{1}{2},1];[2\alpha,1]\end{matrix}\,\middle|\,\begin{matrix}x\\ \frac{1}{4}x^2\end{matrix}\right),
$$

$$
{}_3F_2\left[\begin{matrix}\alpha,\ \frac{d}{2},\ \frac{d+1}{2} \\ \alpha+\frac{1}{2},\ 2\alpha\end{matrix}\,\middle|\,\frac{1}{4}x^2\right] = \begin{cases} F^{1:1;1}_{0:1;1}\left[\begin{matrix}d:\ \alpha;\alpha \\ -:2\alpha;2\alpha\end{matrix}\,\middle|\,\begin{matrix}x\\-x\end{matrix}\right] \\ F^{(3)}\left[\begin{matrix}d::-;-;-;-;\ \alpha\ ;\ \alpha \\ -::-;-;-;-;2\alpha;2\alpha\end{matrix}\,\middle|\,-x,x,x\right]\end{cases}. \qquad (4.6)
$$

The double relations (4.5) and (4.6) give very interesting interconnections between the generalized hypergeometric functions, KdF functions and Lauricella triple hypergeometric functions.

Similarly, by fixing the values of parameters involved another summation and reduction results can be obtained.

5. Fourth set of reduction formulae

Next, another reduction of the KdF function can be earned by treating (1.9) by the Beta and/or Gamma integral techniques.

PROPOSITION 5.1. *Let $n \in \mathbb{N}_0$, the parameters $d, e, \Re(e) > \Re(d) > 0$ and $\alpha \in \mathbb{C}$. Introducing the shorthand $\alpha_n^\eta := \alpha + \eta - n, \eta \in \{j,k\}$, we have*

$$F^{1:1;1}_{1:1;1}\left[\begin{matrix} d : \alpha; \alpha \\ e : 2\alpha - n; 2\alpha - n \end{matrix} \middle| \begin{matrix} x \\ x \end{matrix}\right]$$
$$= \sum_{j,k=0}^{n} \frac{(-1)^{j+k}(-n)_j(-n)_k(2\alpha-2n-1)_j(2\alpha-2n-1)_k}{(\alpha-n-\frac12)_j(\alpha-n-\frac12)_k\,(2\alpha-n)_j(2\alpha-n)_k}\,\frac{(d)_{j+k}}{(e)_{j+k}}\,\frac{\left(\frac14 x\right)^{j+k}}{j!k!}$$
$$\times \mathscr{S}^{1:0;2}_{1:0;3}\left(\begin{matrix} [d+j+k:1,2]:-;[\alpha-n+\frac{j+k}{2},1];[\alpha-n+\frac{j+k+1}{2},1] \\ [e+j+k:1,2]:-;[\alpha_n^j+\frac12,1];[\alpha_n^k+\frac12,1];[\alpha_n^j+\alpha_n^k,1] \end{matrix} \middle| \begin{matrix} x \\ \frac14 x^2 \end{matrix}\right),$$

$$F^{1:1;1}_{1:1;1}\left[\begin{matrix} d : \alpha \;;\; \alpha - n \\ e : 2\alpha - n; 2\alpha - n \end{matrix} \middle| \begin{matrix} x \\ -x \end{matrix}\right] = F^{(3)}\left[\begin{matrix} d :: -;-;-;-;\alpha;\alpha \\ e :: -;-;-;-;2\alpha-n;2\alpha-n \end{matrix} \middle| -x, x, x\right]$$
$$= \sum_{j,k=0}^{n} \frac{(-1)^{j+k}(-n)_j(-n)_k(2\alpha-2n-1)_j(2\alpha-2n-1)_k}{(\alpha-n-\frac12)_j(\alpha-n-\frac12)_k\,(2\alpha-n)_j(2\alpha-n)_k}\,\frac{(d)_{j+k}}{(e)_{j+k}\,j!\,k!}\,\left(\frac{x}{4}\right)^{j+k}$$
$$\times {}_4F_5\left[\begin{matrix} \alpha-n+\frac{j+k}{2},\ \alpha-n+\frac{j+k+1}{2},\ \frac{d+j+k}{2},\ \frac{d+j+k+1}{2} \\ \alpha_n^j+\frac12,\ \alpha_n^k+\frac12,\ \alpha_n^j+\alpha_n^k,\ \frac{e+j+k}{2},\ \frac{e+j+k+1}{2} \end{matrix} \middle| \frac14 x^2\right].$$

Moreover

$$F^{1:1;1}_{0:1;1}\left[\begin{matrix} d : \alpha \;;\; \alpha \\ - : 2\alpha - n; 2\alpha - n \end{matrix} \middle| \begin{matrix} x \\ x \end{matrix}\right]$$
$$= \sum_{j,k=0}^{n} \frac{(-1)^{j+k}(-n)_j(-n)_k(2\alpha-2n-1)_j(2\alpha-2n-1)_k\,(d)_{j+k}\,x^{j+k}}{4^{j+k}(\alpha-n-\frac12)_j(\alpha-n-\frac12)_k\,(2\alpha-n)_j(2\alpha-n)_k\,j!\,k!}$$
$$\times \mathscr{S}^{1:0;2}_{0:0;3}\left(\begin{matrix} [d+j+k:1,2]:-;[\alpha+\frac{j+k}{2},1];[\alpha+\frac{j+k+1}{2},1] \\ -:-;[\alpha_n^j+\frac12,1];[\alpha_n^k+\frac12,1];[\alpha_n^j+\alpha_n^k,1] \end{matrix} \middle| \begin{matrix} x \\ \frac14 x^2 \end{matrix}\right),$$

and

$$F^{1:1;1}_{0:1;1}\left[\begin{matrix} d : \alpha; \alpha - n \\ - : 2\alpha - n; 2\alpha - n \end{matrix} \middle| \begin{matrix} x \\ -x \end{matrix}\right] = F^{(3)}\left[\begin{matrix} d :: -;-;-;-;\alpha;\alpha \\ - :: -;-;-;-;2\alpha-n;2\alpha-n \end{matrix} \middle| -x, x, x\right]$$
$$= \sum_{j,k=0}^{n} \frac{(-1)^{j+k}(d)_{j+k}(-n)_j(-n)_k(2\alpha-1)_j(2\alpha-1)_k}{(\alpha-n-\frac12)_j(\alpha-n-\frac12)_k\,(2\alpha-n)_j(2\alpha-n)_k\,j!\,k!}\,\frac{\left(\frac14 x\right)^{j+k}}{j!k!}$$
$$\times {}_4F_3\left[\begin{matrix} \alpha-n+\frac{j+k}{2},\ \alpha-n+\frac{j+k+1}{2},\ \frac{d+j+k}{2},\ \frac{d+j+k+1}{2} \\ \alpha_n^j+\frac12,\ \alpha_n^k+\frac12,\ \alpha_n^j+\alpha_n^k \end{matrix} \middle| \frac14 x^2\right].$$

6. Fifth set of summation results

The results presented in this section are obtained by virtue of the Beta integral transform (two auxiliary parameters d, e) and the Gamma integral transformation method with one additional parameter d, which ones are applied to the starting formula (1.10).

PROPOSITION 6.1. *Let $n \in \mathbb{N}_0$, the parameters $d, e, \Re(e) > \Re(d) > 0$ and $\alpha \in \mathbb{C}$. Let $\alpha_n^k := \alpha + k - n$ be a shorthand abbreviation throughout. Then we have*

$$F^{1:1;1}_{1:1;1}\left[\begin{array}{c} d: \quad \alpha \quad ; \quad \alpha \\ e: 2\alpha+n; 2\alpha-n \end{array}\middle|\begin{array}{c} x \\ x \end{array}\right]$$
$$= \sum_{j,k=0}^{n} \frac{(-1)^k(-n)_j(-n)_k(2\alpha-1)_j(2\alpha-2n-1)_k}{(\alpha-\frac12)_j(\alpha-n-\frac12)_k\,(2\alpha+n)_j(2\alpha-n)_k}\,\frac{(d)_{j+k}}{(e)_{j+k}\,j!\,k!}\,\left(\frac{x}{4}\right)^{j+k}$$
$$\times \mathscr{S}^{1:0;2}_{1:0;3}\left(\begin{array}{c} [d+j+k:1,2]:-\,;[\frac{\alpha+j}{2}+\frac{\alpha_n^k}{2},1];[\frac{\alpha+j+1}{2}+\frac{\alpha_n^k}{2},1] \\ [e+j+k:1,2]:-;[\alpha+j+\frac12,1];[\alpha_n^k+\frac12,1];[\alpha+\alpha_n^k+j,1] \end{array}\middle|\begin{array}{c} x \\ \frac14 x^2 \end{array}\right);$$

$$F^{1:1;1}_{1:1;1}\left[\begin{array}{c} d: \ \alpha \ ; \ \alpha-n \\ e: 2\alpha+n; 2\alpha-n \end{array}\middle|\begin{array}{c} x \\ -x \end{array}\right] = F^{(3)}\left[\begin{array}{c} d::\ -;\,-;\,-;\,-;\,\alpha;\,\alpha \\ e::-;-;-;-;2\alpha+n;2\alpha-n \end{array}\middle| -x,x,x\right]$$
$$= \sum_{j,k=0}^{n} \frac{(-1)^k(-n)_j(-n)_k(2\alpha-1)_j(2\alpha-2n-1)_k}{(\alpha-\frac12)_j(\alpha-n-\frac12)_k\,(2\alpha+n)_j(2\alpha-n)_k}\,\frac{(d)_{j+k}}{(e)_{j+k}\,j!\,k!}\,\left(\frac{x}{4}\right)^{j+k}$$
$$\times\,{}_4F_5\left[\begin{array}{c} \frac{\alpha+j}{2}+\frac{\alpha_n^k}{2},\ \frac{\alpha+j+1}{2}+\frac{\alpha_n^k}{2},\ \frac{d+j+k}{2},\ \frac{d+j+k+1}{2} \\ \alpha+j+\frac12,\ \alpha_n^k+\frac12,\ \alpha+j+\alpha_n^k,\ \frac{e+j+k}{2},\ \frac{e+j+k+1}{2} \end{array}\middle|\frac14 x^2\right].$$

Moreover,

$$F^{1:1;1}_{0:1;1}\left[\begin{array}{c} d:\ \alpha;\,\alpha \\ -: 2\alpha+n; 2\alpha-n \end{array}\middle|\begin{array}{c} x \\ x \end{array}\right]$$
$$= \sum_{j,k=0}^{n} \frac{(-1)^k(-n)_j(-n)_k(2\alpha-1)_j(2\alpha-2n-1)_k\,(d)_{j+k}\,x^{j+k}}{4^{j+k}(\alpha-\frac12)_j(\alpha-n-\frac12)_k\,(2\alpha+n)_j(2\alpha-n)_k\,j!\,k!}$$
$$\times \mathscr{S}^{1:0;2}_{0:0;3}\left(\begin{array}{c} [d+j+k:1,2]:-\,;[\frac{\alpha+j}{2}+\frac{\alpha_n^k}{2},1];[\frac{\alpha+j+1}{2}+\frac{\alpha_n^k}{2},1] \\ -:-;[\alpha+j+\frac12,1];[\alpha_n^k+\frac12,1];[\alpha+j+\alpha_n^k,1] \end{array}\middle|\begin{array}{c} x \\ \frac14 x^2 \end{array}\right),$$

$$F^{1:1;1}_{0:1;1}\left[\begin{array}{c} d:\ \alpha;\,\alpha-n \\ -: 2\alpha+n; 2\alpha-n \end{array}\middle|\begin{array}{c} x \\ -x \end{array}\right] = F^{(3)}\left[\begin{array}{c} d::\ -;\,-;\,-;\,-;\,\alpha;\,\alpha \\ -::-;-;-;-;2\alpha+n;2\alpha-n \end{array}\middle| -x,x,x\right]$$
$$= \sum_{j,k=0}^{n} \frac{(-1)^k(-n)_j(-n)_k(2\alpha-1)_j(2\alpha-2n-1)_k\,(d)_{j+k}\,x^{j+k}}{4^{j+k}(\alpha-\frac12)_j(\alpha-n-\frac12)_k\,(2\alpha+n)_j(2\alpha-n)_k\,j!\,k!}$$
$$\times\,{}_4F_3\left[\begin{array}{c} \frac{\alpha+j}{2}+\frac{\alpha_n^k}{2},\ \frac{\alpha+j+1}{2}+\frac{\alpha_n^k}{2},\ \frac{d+j+k}{2},\ \frac{d+j+k+1}{2} \\ \alpha+j+\frac12,\ \alpha_n^k+\frac12,\ \alpha+j+\alpha_n^k \end{array}\middle|\frac14 x^2\right].$$

REMARK 1. *The main difference between the starting ancestor formula* (1.10) *in deriving the results of* Proposition 6.1. *from one side and in formulae* (1.8) *and* (1.9) *on the other side occur in their contiguous parameter signs which in present case are opposite in the confluent Kummer functions ${}_1F_1[x]$.*

REMARK 2. *Putting $n = 0$ eveywhere in* Proposition 6.1. *we recognize that the* 2.1. Specific cases *and also* 4.1. Specific cases *are exactly covered in this way too.*

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Received 11 01 2020, revised 16 05 2020

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