

A congruence for a double harmonic sum

Tewodros Amdeberhan and Roberto Tauraso

ABSTRACT. In this short note, our primary purpose is to prove the congruence

$$\sum_{k=1}^{\frac{p-1}{2}} \frac{(-1)^k}{k} \sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k \frac{1}{2j-1} \equiv 0 \pmod{p}.$$

Along the way, a number of auxiliary results of independent interest are found.

1. Introduction

The main target and motivation for this work is the present authors' intent to respond to a certain challenge proposed at the public forum called *Mathoverflow* in which the proposer asks a proof for the congruence

$$\sum_{k=1}^{\frac{p-1}{2}} \frac{(-1)^k}{k} \sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k \frac{1}{2j-1} \equiv 0 \pmod{p}.$$

After some effort, we succeed in doing so.

The following notations and conventions will be adhered to throughout the discussion.

Let $p \geq 5$ be a prime. Denote $p' = \frac{p-1}{2}$, $p'' = \lfloor \frac{p-1}{4} \rfloor$ and the Fermat's quotients by $q_2 = \frac{2^{p-1}-1}{p}$ while $\left(\frac{a}{p}\right)$ stands for the Legendre's symbol. For brevity, \equiv_p designates congruence modulo p . The Euler numbers E_n are defined by the exponential generating function

$$\frac{2e^t}{e^{2t} + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

The generalized harmonic numbers are given by $H_n(a) = \sum_{j=1}^n \frac{1}{j^a}$ so that the classical harmonic numbers become $H_n = H_n(1)$.

The organization of the paper is as follows. In Section 2, we list some relevant congruences for harmonic numbers which appear in the existing literature. In Section 3,

2000 *Mathematics Subject Classification.* 11A07, 11A41, 65B10.

Key words and phrases. harmonic numbers, Euler numbers, congruences.

we prove a few preparatory statements to conclude with our main result as advertised at the beginning of this section and the Abstract. In Section 4, we show an evaluation of a related definite sum

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k \frac{1}{2j-1} = -\frac{5\pi^2}{48}.$$

2. Background results

In this section, we shall recall certain well-known congruences which play a direct roll in the sequel. The most basic result in this direction is that of Wolstenholme's $H_{p-1} \equiv_p 0$, which has been strengthened since.

LEMMA 2.1. (*Wolstenholme*)

$$(2.1) \quad H_{p-1} \equiv_{p^2} 0.$$

LEMMA 2.2. (*Eisenstein*)

$$(2.2) \quad H_{p'} \equiv_p -2q_2.$$

LEMMA 2.3. *We have the elementary congruences*

$$(2.3) \quad \sum_{k=1}^{p-1} \frac{1}{k^2} \equiv_p 0, \quad \sum_{k=1}^{p'} \frac{1}{k^2} \equiv_p 0 \quad \text{and} \quad \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \equiv_p 0.$$

PROOF. The first congruence is clear from $\sum_{k=1}^{p-1} \frac{1}{k^2} \equiv_p \sum_{k=1}^{p-1} k^2 = \frac{p(p-1)(2p-1)}{6} \equiv_p 0$. With the change of variables $k \rightarrow p' - k + 1$, we obtain

$$\sum_{k=1}^{p-1} \frac{1}{(2k-1)^2} = \sum_{k=1}^{p'} \frac{1}{(2(p'-k+1)-1)^2} \equiv_p \frac{1}{4} \sum_{k=1}^{p'} \frac{1}{k^2}$$

which implies that

$$0 \equiv_p \sum_{k=1}^{p-1} \frac{1}{k^2} = \frac{1}{4} \sum_{k=1}^{p'} \frac{1}{k^2} + \sum_{k=1}^{p-1} \frac{1}{(2k-1)^2} \equiv_p \frac{1}{2} \sum_{k=1}^{p'} \frac{1}{k^2}.$$

and also

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} = \frac{1}{4} \sum_{k=1}^{p'} \frac{1}{k^2} - \sum_{k=1}^{p'} \frac{1}{(2k-1)^2} \equiv_p \frac{1}{4} \sum_{k=1}^{p'} \frac{1}{k^2} - \frac{1}{4} \sum_{k=1}^{p'} \frac{1}{k^2} = 0.$$

The proof is complete. \square

LEMMA 2.4. (*Glaisher*) [2, (43)]

$$(2.4) \quad H_{p''} \equiv_p -3q_2.$$

LEMMA 2.5. [2, (19) and (20)]

$$(2.5) \quad H_{p'}(2) \equiv_p 0 \quad \text{and} \quad H_{p''}(2) \equiv_p 4(-1)^{p'} E_{p-3}.$$

Hence,

LEMMA 2.6.

$$(2.6) \quad \sum_{k=1}^{p-1} \frac{H_k}{k} = \frac{1}{2} (H_{p-1}^2 - H_{p-1}(2)) \equiv_p 0,$$

$$(2.7) \quad \sum_{k=1}^{p'} \frac{H_k}{k} = \frac{1}{2} (H_{p'}^2 - H_{p'}(2)) \equiv_p 2q_2^2,$$

$$(2.8) \quad \sum_{k=1}^{p''} \frac{H_k}{k} = \frac{1}{2} (H_{p''}^2 + H_{p''}(2)) \equiv_p \frac{9q_2^2}{2} + 2(-1)^{p'} E_{p-3}.$$

LEMMA 2.7. Define the function $\mathcal{L}_d(x) = \sum_{j=1}^{p-1} \frac{x^j}{j^d}$. Then,

$$(2.9) \quad \sum_{j=1}^{p-1} \frac{x^j H_j}{j} \equiv_p \mathcal{L}_2(x) - \mathcal{L}_2(1-x).$$

PROOF. We have

$$\sum_{j=1}^{p-1} \frac{(1-x)^j - 1}{j} = \sum_{j=1}^{p-1} \binom{p-1}{j} \frac{(-x)^j}{j} \equiv_{p^2} \sum_{j=1}^{p-1} \frac{x^j}{j} (1 - pH_j)$$

which implies that

$$\sum_{j=1}^{p-1} \frac{x^j H_j}{j} \equiv_p \frac{\mathcal{L}_1(x) - \mathcal{L}_1(1-x) + H_{p-1}}{p} \equiv_p \mathcal{L}_2(x) - \mathcal{L}_2(1-x).$$

where we used [1, (6)] and (2.1), as well as

$$-\mathcal{L}_2(x) \equiv_p \frac{1}{p} \left(\frac{x^p + (1-x)^p - 1}{p} + \mathcal{L}_1(1-x) \right).$$

□

LEMMA 2.8. We have

$$(2.10) \quad \sum_{k=1}^{p-1} \frac{(-1)^k}{k} H_k \equiv_p q_2^2.$$

PROOF. By Lemma 2.7, the last congruence in (2.3) and [1, (2)], we gather that

$$(2.11) \quad \sum_{k=1}^{p-1} \frac{(-1)^k H_k}{k} \equiv_p \mathcal{L}_2(-1) - \mathcal{L}_2(2) \equiv_p q_2^2.$$

□

LEMMA 2.9. We have

$$(2.12) \quad \sum_{k=1}^{p'} \frac{(-1)^k H_k}{k} \equiv_p \frac{q_2^2}{2} + (-1)^{p'} E_{p-3}.$$

PROOF. We proceed as follows:

$$\begin{aligned}
\sum_{k=1}^{p-1} \frac{(-1)^k H_k}{k} &= \sum_{k=1}^{p'} \frac{(-1)^k H_k}{k} + \sum_{k=p'+1}^{p-1} \frac{(-1)^k H_k}{k} = \sum_{k=1}^{p'} \frac{(-1)^k H_k}{k} + \sum_{k=1}^{p'} \frac{(-1)^{p-k} H_{p-k}}{p-k} \\
&\equiv_p \sum_{k=1}^{p'} \frac{(-1)^k H_k}{k} + \sum_{k=1}^{p'} \frac{(-1)^k H_{p-k}}{k} \equiv_p \sum_{k=1}^{p'} \frac{(-1)^k H_k}{k} + \sum_{k=1}^{p'} \frac{(-1)^k H_{k-1}}{k} \\
&= 2 \sum_{k=1}^{p'} \frac{(-1)^k H_k}{k} - \sum_{k=1}^{p'} \frac{(-1)^k}{k^2}
\end{aligned}$$

where we used the fact that $H_{p-k} \equiv_p H_{k-1}$. Hence, by (2.5) and (2.11),

$$\begin{aligned}
\sum_{k=1}^{p'} \frac{(-1)^k H_k}{k} &\equiv_p \frac{1}{2} \sum_{k=1}^{p-1} \frac{(-1)^k H_k}{k} + \frac{1}{2} \sum_{k=1}^{p'} \frac{(-1)^k}{k^2} \\
&= \frac{1}{2} \sum_{k=1}^{p-1} \frac{(-1)^k H_k}{k} + \frac{1}{2} \left[-H_{p'}(2) + \frac{1}{2} H_{p''}(2) \right] \\
&\equiv_p \frac{q_2^2}{2} + (-1)^{p'} E_{p-3}.
\end{aligned}$$

□

3. Main Results

In order to reach the main goal of this paper, we first establish a series of crucial preparatory statements. From (2.6) and (2.11), it is immediate that

$$(3.1) \quad \sum_{k=1}^{p'} \frac{H_{2k}}{k} = \sum_{k=1}^{p-1} \frac{H_k}{k} + \sum_{k=1}^{p-1} \frac{(-1)^k H_k}{k} \equiv_p q_2^2.$$

On the other hand, (2.7) and (2.12) lead to

$$(3.2) \quad \sum_{k=1}^{p''} \frac{H_{2k}}{k} = \sum_{k=1}^{p'} \frac{H_k}{k} + \sum_{k=1}^{p'} \frac{(-1)^k H_k}{k} \equiv_p \frac{5q_2^2}{2} + (-1)^{p'} E_{p-3}.$$

LEMMA 3.1. *We have*

$$(3.3) \quad \sum_{k=1}^{p'} \frac{(-1)^k H_{2k}}{k} \equiv_p \frac{q_2^2}{4}.$$

PROOF. By [3, Section 4]

$$\begin{aligned}
\sum_{k=1}^{p'} \frac{(-1)^k H_{2k}}{k} &= 2 \sum_{k=1}^{p'} \frac{(i^2)^k H_{2k}}{2k} = 2 \operatorname{Re} \left(\sum_{k=1}^{p-1} \frac{i^k H_k}{k} \right) \\
&\equiv_p 2 \operatorname{Re} (\mathcal{L}_2(i) - \mathcal{L}_2(1-i)) \\
&\equiv_p 2 \operatorname{Re} \left(\frac{((-1)^{p'} + i) E_{p-3}}{2} - \frac{-q_2^2(1-i(-1)^{p'}) + 4(-1)^{p'} E_{p-3}}{8} \right) \\
&\equiv_p \frac{q_2^2}{4}.
\end{aligned}$$

□

LEMMA 3.2. *We have*

$$(3.4) \quad \sum_{k=1}^{p'} \frac{(-1)^k}{k} H_{2\lfloor \frac{k}{2} \rfloor} \equiv_p \frac{q_2^2}{2}.$$

PROOF. Since $H_{2k} \equiv_p H_{2(p'-k)}$,

$$\begin{aligned}
\sum_{k=1}^{p'} \frac{(-1)^k}{k} H_{2\lfloor \frac{k}{2} \rfloor} &= \sum_{k=1}^{p''} \frac{H_{2k}}{2k} - \sum_{k=1}^{\lfloor \frac{p'}{2} \rfloor} \frac{H_{2(k-1)}}{2k-1} \\
&= \sum_{k=1}^{p''} \frac{H_{2k}}{2k} - \sum_{k=1}^{p'} \frac{H_{2(p'-k)}}{p-2k} + \sum_{k=1}^{p''} \frac{H_{2(p'-k)}}{p-2k} \\
&\equiv_p \sum_{k=1}^{p''} \frac{H_{2k}}{2k} + \sum_{k=1}^{p'} \frac{H_{2k}}{2k} - \sum_{k=1}^{p''} \frac{H_{2k}}{2k} = \frac{1}{2} \sum_{k=1}^{p'} \frac{H_{2k}}{k} \equiv_p \frac{q_2^2}{2}
\end{aligned}$$

where in the last step we used (3.1). □

LEMMA 3.3. *We have*

$$(3.5) \quad \sum_{k=1}^{p'} \frac{(-1)^k}{k} H_{\lfloor \frac{k}{2} \rfloor} \equiv_p q_2^2 + (-1)^{\frac{p-1}{2}} E_{p-3}.$$

PROOF. The argument goes as follows:

$$\begin{aligned}
\sum_{k=1}^{p'} \frac{(-1)^k}{k} H_{\lfloor \frac{k}{2} \rfloor} &= \sum_{k=1}^{p''} \frac{H_k}{2k} - \sum_{k=1}^{\lfloor \frac{p'}{2} \rfloor} \frac{H_{k-1}}{2k-1} \\
&= \sum_{k=1}^{p''} \frac{H_k}{2k} - \sum_{k=1}^{p'} \frac{H_{p'-k}}{p-2k} + \sum_{k=1}^{p''} \frac{H_{p'-k}}{p-2k} \\
&\equiv_p \sum_{k=1}^{p''} \frac{H_k}{2k} + \sum_{k=1}^{p'} \frac{H_{p'-k}}{2k} - \sum_{k=1}^{p''} \frac{H_{p'-k}}{2k}.
\end{aligned}$$

By using

$$H_{p'-k} = H_{p'} - \sum_{j=0}^{k-1} \frac{1}{p'-j} \equiv_p H_{p'} + 2 \sum_{j=0}^{k-1} \frac{1}{2j+1} = H_{p'} + 2H_{2k} - H_k$$

we get

$$\sum_{k=1}^{p'} \frac{(-1)^k}{k} H_{\lfloor \frac{k}{2} \rfloor} \equiv_p -\frac{1}{2} \sum_{k=1}^{p'} \frac{H_k}{k} + \sum_{k=1}^{p''} \frac{H_k}{k} + \sum_{k=1}^{p'} \frac{H_{2k}}{k} - \sum_{k=1}^{p''} \frac{H_{2k}}{k} - \frac{H_{p'}^2}{2} + \frac{H_{p'} H_{p''}}{2}.$$

Invoke (2.2), (2.4), (2.7), (2.8), (3.2), and (3.3) to complete the proof. \square

Finally, we are ready to state and prove our main result.

THEOREM 3.1. *We have*

$$(3.6) \quad \sum_{k=1}^{p'} \frac{(-1)^k}{k} \sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k \frac{1}{2j-1} \equiv_p 0.$$

PROOF. The inner sum can be rewritten as:

$$\sum_{\lfloor \frac{k}{2} \rfloor + 1}^k \frac{1}{2j-1} = \sum_{j=1}^k \frac{1}{2j-1} - \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{2j-1} = \left[H_{2k} - \frac{1}{2} H_k \right] - \left[H_{2\lfloor \frac{k}{2} \rfloor} - \frac{1}{2} H_{\lfloor \frac{k}{2} \rfloor} \right].$$

In view of this, the theorem can be reformulated as

$$\sum_{k=1}^{p'} \frac{(-1)^k}{k} \left[H_{2k} - \frac{1}{2} H_k - H_{2\lfloor \frac{k}{2} \rfloor} + \frac{1}{2} H_{\lfloor \frac{k}{2} \rfloor} \right] \equiv_p 0.$$

The proof follows from (2.12), (3.3), (3.4), and (3.5). \square

4. An infinite series evaluation

In the present section, we consider an infinite series counterpart to the harmonic sum that has been the subject of much of this paper.

THEOREM 4.1. *We have*

$$S := \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k \frac{1}{2j-1} = -\frac{5}{8} \zeta(2).$$

PROOF. Split the sum S according to parity to express in terms of harmonic sums,

$$\begin{aligned} S &= \sum_{k=1}^{\infty} \frac{H_{4k} - \frac{1}{2} H_{2k} - H_{2k} + \frac{1}{2} H_k}{2k} - \sum_{k=1}^{\infty} \frac{H_{4k-2} - \frac{1}{2} H_{2k-1} - H_{2k-2} + \frac{1}{2} H_{k-1}}{2k-1} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k}{k} H_{2k} - \frac{3}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} H_k - \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} H_{\lfloor \frac{k}{2} \rfloor}. \end{aligned}$$

Next, we compute each series one-by-one, the easiest of which being $\sum_{k \geq 1} \frac{1}{(2k-1)^2} = \frac{3}{4}\zeta(2)$. For the rest, the representation $H_k = \int_0^1 \frac{1-t^k}{1-t} dt$ will be employed repeatedly.

$$\sum_{k \geq 1} \frac{(-1)^k H_k}{k} = \int_0^1 \frac{dt}{1-t} \sum_{k \geq 1} \frac{(-1)^k - (-t)^k}{k} = \int_0^1 \frac{\log(1+t) - \log 2}{1-t} dt = \frac{\log^2 2 - \zeta(2)}{2},$$

$$\begin{aligned} \sum_{k \geq 1} \frac{(-1)^k H_{2k}}{k} &= \int_0^1 \frac{\log(1+t^2) - \log 2}{1-t} dt = \int_0^1 \left[\frac{\log(1+t^2) - \log 2}{1-t^2} \right] (1+t) dt \\ &= \int_0^1 \frac{\log(1+t^2) - \log 2}{1-t^2} dt + \frac{1}{2} \int_0^1 \frac{\log(1+t) - \log 2}{1-t} dt \\ &= -\frac{3}{8}\zeta(2) + \frac{\log^2 2 - \zeta(2)}{4}, \end{aligned}$$

$$\sum_{k \geq 1} \frac{(-1)^k}{k} H_{\lfloor \frac{k}{2} \rfloor} = \frac{1}{2} \int_0^1 \frac{\frac{1}{\sqrt{t}} \log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right) + \log(1-t) - 2 \log 2}{1-t} dt = S_1 + S_2 + S_3;$$

where

$$\begin{aligned} S_1 &:= \frac{1}{2} \int_0^1 \frac{\log(1+\sqrt{t}) - \log 2}{1-t} dt = \int_0^1 \left(\frac{\log(1+t) - \log 2}{1-t^2} \right) t dt \\ &= \frac{1}{2} \int_0^1 \frac{\log(1+t) - \log 2}{1-t} dt - \frac{1}{2} \int_0^1 \frac{\log(1+t) - \log 2}{1+t} dt \\ &= \frac{\log^2 2 - \zeta(2)}{4} + \frac{1}{4} \log^2 2, \end{aligned}$$

$$S_2 := \frac{1}{2} \int_0^1 \log(1-\sqrt{t}) \left[\frac{1 - \frac{1}{\sqrt{t}}}{1-t} \right] dt = - \int_0^1 \frac{\log(1-t)}{1+t} dt = \frac{\zeta(2) - \log^2 2}{2},$$

$$\begin{aligned} S_3 &:= \frac{1}{2} \int_0^1 \frac{\frac{1}{\sqrt{t}} \log(1+\sqrt{t}) - \log 2}{1-t} dt = \int_0^1 \frac{\log(1+t) - t \log 2}{1-t^2} dt \\ &= \frac{1}{2} \int_0^1 \frac{\log(1+t) - \log 2}{1-t} dt + \frac{1}{2} \int_0^1 \frac{\log(1+t) - \log 2}{1+t} dt + \log 2 \int_0^1 \frac{dt}{1+t} \\ &= \frac{\log^2 2 - \zeta(2)}{4} - \frac{1}{4} \log^2 2 + \log^2 2. \end{aligned}$$

Combining all the above calculations yields $S = -\frac{5}{8}\zeta(2)$, as required. \square

References

- [1] A. Granville, *The square of the Fermat quotient*, Integers **4** (2004), #A22.
- [2] E. Lehmer, *On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson*, Ann. of Math. **39** (1938), 350–360.
- [3] S. Mattarei and R. Tauraso, *Congruences for central binomial sums and finite polylogarithms*, J. Number Theory **133** (2013), 131–157.

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118
E-mail address: `tamdeber@tulane.edu`

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA “TOR VERGATA”, 00133 ROMA, ITALY
E-mail address: `tauraso@mat.uniroma2.it`

Received 29 05 2019 revised 15 07 2019