SCIENTIA
Series A: *Mathematical Sciences*, Vol. 30 (2020), 43–53
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

A special collection of definite integrals

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Abstract. A compiled list of definite integrals related to special constants is presented. These include Riemann zeta function and the Dirichlet beta function.

1. Introduction

The goal of the current work is to present in a unified manner a collection of definite integrals involving some classical function, such as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1.1}$$

$$\lambda(s) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} = (1-2^{-s})\zeta(s) \tag{1.2}$$

and their alternating counterparts,

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s), \tag{1.3}$$

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}. \tag{1.4}$$

These functions are named after Riemann and Dirichlet in view of their studies in relation to the distribution of prime numbers. The reader will find in [**5**] information on these functions.

These functions have corresponding finite counterparts. The notation used here is meant to be suggestive. For instance,

$$\zeta_N(s) = \sum_{n=1}^{N} \frac{1}{n^s}, \tag{1.5}$$

2000 *Mathematics Subject Classification.* Primary 33C99, Secondary 11M06, 30B50.
Key words and phrases. Definite integrals, zeta functions, Dirichlet beta function, Catalan constant.

and similar for the other functions. A common notation for the harmonic numbers, $\zeta_N(1)$, is H_N. Analogously, we have

$$\lambda_N(s) = \sum_{n=0}^{N} \frac{1}{(2n+1)^s}, \tag{1.6}$$

$$\eta_N(s) = \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^s}, \tag{1.7}$$

and

$$\beta_N(s) = \sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)^s}. \tag{1.8}$$

There is a large variety of relations among these functions, for instance,

$$2\lambda_N(1) = 2\zeta_{2N}(1) - \zeta_N(1). \tag{1.9}$$

The goal of the paper is to present the evaluation of definite integrals and express them in terms of these functions.

2. Classical forms and β sums

The first evaluation generalizes an identity appearing in [**5**, p. 55] we find the double sum identity

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\lambda_{n-1}(1)}{n^2} = \pi G - \frac{7\zeta(3)}{4},$$

where G is the Catalan's constant defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}. \tag{2.1}$$

A well known classical result is:

Theorem 2.1. *The evaluation*

$$\int_0^1 \frac{(\arctan x)^2}{x} dx = \frac{\pi G}{2} - \frac{7\zeta(3)}{8},$$

holds.

Proof. This is classical and is obtained by expanding the integrand in power series. Details appear in [**2**]. ∎

The next result presents a generalization.

Theorem 2.2. *Let $\alpha \geqslant 1$. Then*

$$\int_0^1 \frac{(\arctan x^{1/\alpha})^2}{x} dx = \alpha \int_0^1 \frac{(\arctan x)^2}{x} dx.$$

PROOF. Here we write $\lambda_n(1) = \lambda_n$. It is known that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\lambda_{n-1}x^{2n}}{n} = (\arctan x)^2.$$

Replace x by $x^{1/\alpha}$ to produce

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\lambda_{n-1}(x^{1/\alpha})^{2n}x^{-1}}{n} = \frac{(\arctan x^{1/\alpha})^2}{x}. \tag{2.2}$$

Integrate both sides, and evaluate the left side in terms of

$$\frac{(-1)^{n-1}\lambda_{n-1}}{n}\int_0^1 (x^{1/\alpha})^{2n}x^{-1}dx = \frac{\lambda_{n-1}}{n}\int_0^1 x^{(2n-\alpha)/\alpha}dx. \tag{2.3}$$

Now use

$$\int x^{(2n-\alpha)/\alpha}dx = \frac{\alpha(x^{1/\alpha})^{2n}}{2n}, \tag{2.4}$$

to obtain

$$\frac{\alpha}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\lambda_{n-1}(1)(x^{1/\alpha})^{2n}}{n^2}dx = \int_0^x \frac{(\arctan t^{1/\alpha})^2}{t}dt. \tag{2.5}$$

Therefore

$$\frac{\alpha}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\lambda_{n-1}}{n^2}dx = \alpha\int_0^1 \frac{(\arctan x)^2}{x}dx, \tag{2.6}$$

and this becomes

$$\int_0^1 \frac{(\arctan x^{1/\alpha})^2}{x}dx = \alpha\int_0^1 \frac{(\arctan x)^2}{x}dx. \tag{2.7}$$

The proof is complete. □

Example 2.1. An attractive special case is

$$\int_0^1 \frac{(\arctan \sqrt[8]{x})^2}{x}dx = 4\pi G - 7\zeta(3). \tag{2.8}$$

Example 2.2. The previous example is now compared with the evaluation

$$\int_0^1 \frac{(\arcsin \sqrt[8]{x})^2}{x}dx = 2\pi^2\log 2 - 7\zeta(3), \tag{2.9}$$

appearing in [**2**, p. 122], written in the equivalent form

$$\int_0^1 \frac{(\arcsin x)^2}{x}dx = \frac{\pi^2\log 2}{4} - \frac{7\zeta(3)}{8}.$$

There is also an indeterminate form of these evaluations:

$$\int_0^x \frac{(\arcsin t)^2}{t}dt = \sum_{n=1}^{\infty} \frac{4^{n-1}}{n^3\binom{2n}{n}}x^{2n}. \tag{2.10}$$

The previous examples are now generalized.

PROPOSITION 2.1. *If $\alpha \geqslant 1$, then*

$$\int_0^1 \frac{(\arcsin x^{1/\alpha})^2}{x} dx = \alpha \int_0^1 \frac{(\arcsin x)^2}{x} dx.$$

PROOF. Use the series (2.10) and follow the proof of Proposition 2.2 to obtain the result. □

NOTE 2.1. The evaluation in Example 2.2 is connected to the integral

$$2\int_0^{\pi/2} x \ln \sin x \, dx = -\Big(\frac{\pi^2 \log 2}{4} - \frac{7\zeta(3)}{8} \Big) \tag{2.11}$$

appearing in [**7**, p. 234].

NOTE 2.2. A direct application of the identity

$$\arctan x = \arcsin \frac{x}{\sqrt{x^2+1}} \tag{2.12}$$

combined with (2.10) gives

$$4\int_0^1 \frac{(\arctan x)^2}{x} dx = \sum_{n=1}^{\infty} \frac{2^n}{n^3 \binom{2n}{n}} = 2\pi G - \frac{7\zeta(3)}{2}. \tag{2.13}$$

The next results follow closely from the discussion presented in [**2**, p. 129-133]. Start with the classical expansion of the cotangent function (see [**10**])

$$x \cot x = 1 - 2\sum_{n=1}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} x^{2n}. \tag{2.14}$$

and use

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \Big[1 - \Big(\frac{x}{\pi n} \Big)^2 \Big], \tag{2.15}$$

and

$$\frac{d(\ln \sin x)}{dx} = \cot x$$

to obtain the series expansion (2.14) involving $\zeta(2n)$. The following expansions are established next:

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n} B_{2n}}{(2n)!} x^{2n-1} \tag{2.16}$$

and

$$\sec x = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!} x^{2n}. \tag{2.17}$$

and this yields

$$\zeta(2n) = (-1)^{n-1} \frac{4^{n-1} B_{2n}}{(2n)!} \pi^{2n} \quad \text{and} \quad \beta(2n+1) = (-1)^n \frac{E_{2n}}{4^{n+1}(2n)!} \pi^{2n+1}. \tag{2.18}$$

PROPOSITION 2.2. *The expansion of the secant function is given by*

$$\sec x = \sum_{n=0}^{\infty} \frac{4^{n+1}\beta(2n+1)}{\pi^{2n+1}} x^{2n}. \tag{2.19}$$

PROOF. Start from the right-hand side of (2.18) to produce

$$E_{2n} = (-1)^n \frac{2^{2(n+1)}(2n)!\beta(2n+1)}{\pi^{2n+1}}. \tag{2.20}$$

Now recall the expansion of the inverse Gudermannian (*viz* series no. 754 in [**6**]) to obtain

$$gd^{-1}(x) = \log(\sec x + \tan x) = x + \sum_{n=1}^{\infty} \frac{E_{2n}}{(2n+1)!} x^{2n+1} \tag{2.21}$$

and by substitution of (2.20) produces

$$\log(\sec x + \tan x) = x + \sum_{n=1}^{\infty} (-1)^n \frac{4^{n+1}\beta(2n+1)}{\pi^{2n+1}(2n+1)} x^{2n+1}. \tag{2.22}$$

A simple calculation now gives the result. □

Proposition 2.2 yields the identity

$$\frac{\pi}{2} \sec \frac{\pi x}{2} = 2 \sum_{n=0}^{\infty} \beta(2n+1) x^{2n}, \tag{2.23}$$

that is the even-series analogue of

$$\pi \cot \pi x = \sum_{n=1}^{\infty} \zeta(2n) x^{2n-1}. \tag{2.24}$$

NOTE 2.3. In connection to the inverse Gudermannian used in the argument above, note two interesting integrals that appear in the literature:

$$\int_0^{\pi/2} \log(\sec x + \tan x) dx = 2G \tag{2.25}$$

and

$$\int_0^{\pi/2} x \log(\sec x + \tan x) dx = \frac{7\zeta(3)}{4}. \tag{2.26}$$

These are connected to the present context by the identity

$$\int_0^{\pi/2} \left(\frac{\pi}{2} - x\right) \text{gd}^{-1} x \, dx = \int_0^1 \frac{(\arctan x)^2}{x} dx. \tag{2.27}$$

The next result appears in [**11**]. The ideas there are related to the argument in [**4**].

THEOREM 2.3. *(Yue and Williams) The Apéry constant $\zeta(3)$ is given by*

$$\zeta(3) = -\pi^2 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2^{2n-1}(2n+2)(2n+3)}.$$

PROOF. This is just a reproduction of the proof given in the sources mentioned above. It is given for completeness purpose.

Recall that

$$\int_0^t \frac{(\arcsin x)^2}{x} dx = \sum_{n=1}^{\infty} \frac{4^{n-1}}{n^3 \binom{2n}{n}} t^{2n},$$

and therefore,

$$\int_0^{\sin t} \frac{(\arcsin x)^2}{x} dx = \sum_{n=1}^{\infty} \frac{4^{n-1}}{n^3 \binom{2n}{n}} \sin^{2n} t.$$

The substitution $x = \sin u$ gives

$$\int_0^t \frac{u^2}{\sin u} du = \int_0^t u^2 \cot u \, du = \sum_{n=1}^{\infty} \frac{4^{n-1}}{n^3 \binom{2n}{n}} \sin^{2n} t.$$

Then (2.14) produces

$$u^2 \cot u = -2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} u^{2n+1}$$

and integrating the left side yields

$$-\int_0^t \left(2 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{\pi^{2n+1}} u^{2n+1} \right) du = \sum_{n=1}^{\infty} \frac{4^{n-1}}{n^3 \binom{2n}{n}} \sin^{2n} t. \tag{2.28}$$

The series inside the integral converges uniformly, therefore

$$-2 \sum_{n=0}^{\infty} \left(\frac{\zeta(2n)}{\pi^{2n+1}} \int_0^t u^{2n+1} \, du \right) = -\sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n+1}(n+1)} t^{2n+2}. \tag{2.29}$$

Integrate a second time from 0 to t, to produce

$$-\sum_{n=1}^{\infty} \frac{\zeta(2n)}{\pi^{2n+1}(n+1)(2n+3)} t^{2n+3}.$$

and with $t = \pi/2$ this becomes

$$\int_0^{\pi/2} \int_0^t u^2 \cot u \, du \, dt = -\frac{\pi^2}{8} \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(n+1)(2n+3)2^{2n}} \tag{2.30}$$

Next use the notation

$$\int_0^{\pi/2} \sin^{2n+1} x \, dx = W_n. \tag{2.31}$$

and use Wallis' formula in the form

$$\frac{1}{W_n} = \frac{4^n}{(2n+1)\binom{2n}{n}} \tag{2.32}$$

in the identity

$$\int_0^t \sum_{n=1}^{\infty} \frac{4^{n-1}}{n^3 \binom{2n}{n}} \sin^{2n} t \, dt = \sum_{n=1}^{\infty} \frac{4^{n-1}}{n^3 \binom{2n}{n}(2n+1)} \sin^{2n+1} t. \tag{2.33}$$

to produce

$$\frac{1}{4}\int_0^{\pi/2}\sum_{n=1}^{\infty}\frac{\sin^{2n+1}t}{n^3W_n}\,dt = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3}. \tag{2.34}$$

Combine (2.34) and (2.30) completes the proof of the theorem. □

A very similar procedure gives a companion result.

PROPOSITION 2.3. *The evaluation*

$$\frac{\pi^2}{4}\sum_{n=0}^{\infty}\frac{\beta(2n+1)}{(2n+1)(2n+2)(2n+3)} = \frac{\pi G}{2} - \frac{7\zeta(3)}{8}. \tag{2.35}$$

holds.

PROOF. Start with

$$\int_0^{\tan t}\frac{(\arctan x)^2}{x}dx = 2\int_0^t u^2\csc 2u\,du, \tag{2.36}$$

and use the elementary identity $\csc x = \sec(x-\pi/2)$ to write

$$2\int_0^t u^2\csc 2u\,du = 2\int_0^t u^2\sec\left(2u-\frac{\pi}{2}\right)du. \tag{2.37}$$

Now use Proposition (2.2) to write

$$2\int_0^t u^2\sec\left(2u-\frac{\pi}{2}\right)du = \int_0^t\sum_{n=0}^{\infty}\frac{4^{n+1}\beta(2n+1)}{\pi^{2n+1}}\left(2u-\frac{\pi}{2}\right)^{2n}u^2\,du. \tag{2.38}$$

Change the order of integration to derive

$$\int_0^t\left(2u-\frac{\pi}{2}\right)^{2n}u^2du = \frac{\left(2t-\frac{\pi}{2}\right)^{2n}(\pi-4t)[A_nt^2+B_nt\pi+\pi^2]}{32(2n+1)(2n+2)(2n+3)}\Bigg|_0^t, \tag{2.39}$$

where $A_n = (2n+1)(2n+2)$, $B_n = 2n+1$. Thus, we have shown

$$\int_0^{\tan t}\frac{(\arctan x)^2}{x}dx = \frac{1}{8}\sum_{n=0}^{\infty}\frac{4^n\beta(2n+1)\left[\left(2t-\frac{\pi}{2}\right)^{2n}(\pi-4t)(A_nt^2+B_nt\pi+\pi^2)\Big|_0^t\right]}{\pi^{2n+1}(2n+1)(2n+2)(2n+3)} \tag{2.40}$$

The special value $t=\frac{\pi}{4}$ now gives

$$\int_0^1\frac{(\arctan x)^2}{x}dx = \frac{\pi^2}{4}\sum_{n=0}^{\infty}\frac{\beta(2n+1)}{(2n+1)(2n+2)(2n+3)} = \frac{\pi G}{2} - \frac{7\zeta(3)}{8} \tag{2.41}$$

and the evaluation is complete. □

NOTE 2.4. An outline of a general approach presented here can be found in [**3**]. In particular, it is shown that

$$\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\beta(2k+1)}{(2k+1)(2k+2)} = G. \tag{2.42}$$

However, the relationship with ζ-values is not presented by these authors.

3. Juxtaposing the logarithm and sine functions

This section begins with some standard notations for a group of special functions. For what follows we remind the reader of some standard definitions.

DEFINITION 3.1. The `cosine integral` for $-\pi < \text{Arg}(x) < \pi$ is given by

$$\text{Ci}(x) = -\int_x^\infty \frac{\cos t}{t} dt = \gamma + \log x - \int_0^x \frac{1-\cos t}{t} dt.$$

Here γ is the Euler-Mascheroni constant. The notation

$$\text{cin}(x) = \int_0^x \frac{1-\cos t}{t} dt \tag{3.1}$$

is used throughout.

The first result uses the notation $\overline{x} = 1 - x$.

PROPOSITION 3.1. *The evaluation*

$$\int_0^1 \log x \sin x = \int_0^1 \log \overline{x} \sin \overline{x} \, dx = \text{Ci}(1) - \gamma.$$

holds.

PROOF. Start with the uniformly convergent series

$$\sin x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \tag{3.2}$$

so that

$$\log x \sin x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1} \log x}{(2n+1)!}, \tag{3.3}$$

and the original integral is written as

$$\sum_{n=0}^\infty (-1)^n \int_0^1 \frac{\log x}{(2n+1)!} x^{2n+1} = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} \int_0^1 x^{2n+1} \log x \, dx. \tag{3.4}$$

Integration by parts gives

$$\int_0^1 x^{2n+1} \log x \, dx = -\frac{1}{4(n+1)^2}, \tag{3.5}$$

and then

$$\int_0^1 \log x \sin x = \sum_{n=0}^\infty \frac{(-1)^{n+1}}{2(n+1)(2(n+1))!} = \sum_{n=1}^\infty \frac{(-1)^n}{2n(2n)!}. \tag{3.6}$$

The classical expansion

$$\text{cin}(x) = \sum_{n=1}^\infty \frac{(-1)^{n-1} x^{2n}}{2n(2n)!}. \tag{3.7}$$

produces

$$\int_0^1 \log x \sin x \, dx = \mathrm{cin}(1). \tag{3.8}$$

The proof is complete. □

NOTE 3.1. Proceeding as above and using

$$\log t \sin t = \int_0^1 \int_0^1 \frac{\cos(tx)}{1+\bar{t}y} dxdy, \tag{3.9}$$

gives

$$\int_0^1 \log x \sin x = \int_0^1 \int_0^1 \int_0^1 \frac{\cos(xy)}{1+\overline{x}z}\, dxdydz. \tag{3.10}$$

This integral is similar to well-known integrals for other special constants such as $\zeta(3)$ and γ. For example,

$$\zeta(3) = \int_0^1 \int_0^1 \int_0^1 \frac{1}{1-xyz}\, dxdydz$$

as shown in [**1**]. Other examples include

$$\zeta(2) = \int_0^1 \int_0^1 \frac{1}{1-xy}\, dxdy,$$

$$\zeta(3) = -\int_0^1 \int_0^1 \frac{\ln xy}{1-xy}\, dxdy,$$

and

$$\gamma = -\int_0^1 \int_0^1 \frac{\overline{x}}{(1-xy)\ln xy}\, dxdy;$$

for details see [**9**].

The arithmetic character of the value obtained above is discussed next.

PROPOSITION 3.2. *The number cin*(1) *is irrational.*

PROOF. It has been established that

$$\mathrm{cin}(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n(2n)!}.$$

The same argument used to show $1/e$ is irrational *mutatis mutandis* will work; see [**8**, p.2-3]. □

The next result is a reformulation of Proposition 3.2. The authors remain skeptical of its value toward showing γ is an irrational number.

COROLLARY 3.1. *One of the numbers ci*(1) *or* γ *is irrational.*

PROOF. Since

$$\mathrm{cin}(1) = \gamma + \mathrm{ci}(1)$$

and since cin(1) is irrational, the proof is complete. □

The next result is a reciprocity theorem for the function Ci, found to be of interest. An immediate application to the study of γ.

PROPOSITION 3.3. *Let m, $n \in \mathbb{N}$ be such that $m < n$. Then*

$$\mathrm{Ci}(1) - \int_0^{m/n} \ln \overline{x} \sin \overline{x}\, dx = \mathrm{Ci}\Big(\frac{n-m}{n}\Big) + \ln\Big(\frac{n}{n-m}\Big)\cos\Big(\frac{n-m}{n}\Big)$$

PROOF. A similar argument as the one presented in the proof Proposition 3.1 gives

$$\int_0^b \log \overline{x} \sin \overline{x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^b \overline{x}^{2n+1} \log \overline{x}\, dx. \tag{3.11}$$

An elementary evaluation gives, for $b \leqslant 1$,

$$\int_0^b \overline{x}^{2n+1} \log \overline{x}\, dx = \frac{\overline{b}^{2(n+1)}(1 - 2(n+1)\log \overline{b}) - 1}{4(n+1)^2}, \tag{3.12}$$

and the right-hand side is

$$\frac{\overline{b}^{2(n+1)}}{4(n+1)^2} - \frac{\overline{b}\log \overline{b}}{2(n+1)} - \frac{1}{4(n+1)^2}. \tag{3.13}$$

This yields

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\overline{b}^{2n}}{2n(2n)!} + \log \overline{b} \sum_{n=1}^{\infty} \frac{(-1)^n \overline{b}^{2n}}{(2n)!} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n(2n)!}. \tag{3.14}$$

Now let $b = m/n$ to complete the proof. □

The next result follows from Proposition 3.3.

PROPOSITION 3.4. *The limit*

$$\lim_{x\to\infty} \Big[\mathrm{Ci}\Big(\frac{1}{x}\Big) + \ln(x)\cos\Big(\frac{1}{x}\Big)\Big] = \gamma,$$

holds.

PROOF. Start with

$$\mathrm{Ci}(x) = \gamma + \log x - \int_0^x \frac{1-\cos t}{t} dt,$$

and restate the result as

$$\lim_{x\to\infty} \Big[\gamma + \log x \cos x^{-1} + \log x^{-1} - \int_0^{x^{-1}} \frac{1-\cos t}{t} dt\Big] = \gamma.$$

The expression in brackets is now simplified as

$$\Big[\gamma - \log x(1-\cos x^{-1}) - \int_0^{x^{-1}} \frac{1-\cos t}{t} dt\Big].$$

The limiting values $\log x(1-\cos x^{-1}) \to 0$ and $\displaystyle\int_0^{x^{-1}} \frac{1-\cos t}{t} dt \to 0$ as $x \to \infty$ complete the proof. □

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Received 10 05 2020 revised 25 06 2020