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Integrals involving Bernoulli and Euler polynomials

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Abstract. Integrals containing the Bernoulli and Euler polynomials appear in different parts of the literature. The goal of this work is to present them in a unified manner.

1. Introduction

The Bernoulli polynomials are defined by the generating function

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n = e^{xz} \frac{z}{e^z - 1}, \; |z| < 2\pi. \tag{1.1}$$

The constant term yields the value $B_0(x) = 1$ and differentiation with respect to x produces the relation

$$B_n'(x) = nB_{n-1}(x), \text{ for } n \geqslant 1, \tag{1.2}$$

showing that $B_n(x)$ is indeed a polynomial in x (of degree n). The constant terms $B_n(0) \equiv B_n$ are the Bernoulli numbers with generating function

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = \frac{z}{e^z - 1}. \tag{1.3}$$

Many of the identities for Bernoulli polynomials are easily obtained from the generating function. For instance, replacing x by $1-x$ and z by $-z$ yields

$$B_n(1-x) = (-1)^n B_n(x), \quad \text{for } n \geqslant 0, \tag{1.4}$$

and for $x = 0$ it gives

$$B_n(1) = (-1)^n B_n(0). \tag{1.5}$$

The identity

$$\frac{z}{e^z - 1} - 1 + \frac{z}{2} = \sum_{n=2}^{\infty} \frac{B_n}{n!} z^n \tag{1.6}$$

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and the fact that the left hand-side is an even function of z, shows that $B_{2n+1} = 0$ for $n \geqslant 1$. The only non-zero Bernoulli number of odd index is $B_1 = -\frac{1}{2}$. A direct multiplication of the two factors appearing in (1.1) gives the expression

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \tag{1.7}$$

for the Bernoulli polynomials in terms of the Bernoulli numbers.

The identity

$$B_n = -\frac{1}{n+1} \sum_{j=0}^{n-1} \binom{n+1}{j} B_j \tag{1.8}$$

shows that B_n is a rational number and Mordell [**20**] used it to prove

$$B_{2n} = -\sum_{r=0}^{n-1} \frac{2^{2r}-1}{2^{2n}-1} \binom{2n}{2r} B_{2r} B_{2n-2r} \tag{1.9}$$

and to establish that $(-1)^{n-1} B_{2n} > 0$. Therefore the signs of B_{2n} alternate; see also (3.10). Details appear in [**19**]. The Bernoulli numbers have curious arithmetic properties. Writing the fraction $B_{2n} = (-1)^{n-1} N_{2n}/D_{2n}$ in reduced form, the von Staudt-Clausen theorem states that D_{2n} is the product of all primes p such that $p-1$ divides $2n$. For example, $D_{100} = 33330 = 2 \cdot 3 \cdot 5 \cdot 11 \cdot 101$. On the other hand, the numerators N_{2n} are more mysterious. For example, N_{100} is an 83 digits number that factors as 263×379 times three other large primes. This is a pity since they are connected to Fermat's last theorem: Kummer introduced the concept of regular prime and proved that the equation $x^n + y^n = z^n$ has no solutions for these primes. It turns out that p is regular if it does not divide the numerator N_{2k} for $k = 1, 2, \cdots, (p-3)/2$. See [**24, 25**] for more details.

The Bernoulli polynomials also appear in the famous Faulhaber identity

$$\sum_{k=0}^{r-1} k^n = \frac{B_{n+1}(r) - B_{n+1}(0)}{n+1}, \quad r \geqslant 1,\ n \geqslant 1 \tag{1.10}$$

that comes directly from the difference equation

$$B_n(x+1) - B_n(x) = n x^{n-1}, \quad n \geqslant 1. \tag{1.11}$$

In particular $B_n(1) = B_n(0)$ for $n \geqslant 2$. The reader is invited to provide a proof of (1.11) using the generating function.

The Bernoulli polynomials also appear in the Euler-MacLaurin summation formula [**3**], linking series to integrals: for a positive integer l and any function f $l-$times continuously differentiable on $[a, \infty)$,

$$\sum_{n=a}^{\infty} f(n) = \int_a^{\infty} f(x)\,dx + \frac{1}{2} f(a) - \sum_{\ell=2}^{k} \frac{(-1)^{\ell}}{\ell!} f^{(\ell-1)}(a) B_{\ell} - \frac{(-1)^k}{k!} \int_a^{\infty} f^{(k)}(x) B_k(\{x\}) dx, \tag{1.12}$$

with $\{x\}$ being the fractional part of x.

The goal of this note is to survey integrals containing the Bernoulli polynomials as part of the integrand. It is natural to include examples containing the companion family of Euler polynomials $E_n(x)$ defined by the generating function

$$\sum_{n=0}^{\infty} \frac{E_n(x)}{n!} z^n = e^{xz} \frac{2}{e^z+1}. \tag{1.13}$$

2. Elementary examples

This section contains elementary integrals.

Example 2.1. The relation (1.2) gives

$$\int B_n(x)\,dx = \frac{B_{n+1}(x)}{n+1} \tag{2.1}$$

In particular

$$\int_a^{a+1} B_n(x)\,dx = \frac{B_{n+1}(a+1)-B_{n+1}(a)}{n+1} = a^n, \quad n \geqslant 0, \tag{2.2}$$

where the last equality comes from the difference equation (1.11). The special case $a=0$ gives

$$\int_0^1 B_n(x)\,dx = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n \neq 0. \end{cases} \tag{2.3}$$

The extension of the first example to an integral over an interval of length $1/2$ is presented next. It involves the Euler polynomials defined by the generating function (10.1) below. The proof begins with a preliminary result. As a corollary, it follows that $E_n(x)$ is also a polynomial in x.

Lemma 2.2. The Bernoulli and Euler polynomials are related by

$$B_n\left(x+\tfrac{1}{2}\right) - B_n(x) = \frac{nE_{n-1}(2x)}{2^n}. \tag{2.4}$$

Proof. Compute the generating function

$$\sum_{n=0}^{\infty} \frac{B_n(x+\frac{1}{2})-B_n(x)}{n!} z^n = \left(e^{z(x+1/2)} - e^{xz}\right) \frac{z}{e^z-1} = ze^{xz}\frac{1}{e^{z/2}+1}.$$

The result now follows from (1.13). □

Example 2.3. The Bernoulli polynomials satsify

$$\int_a^{a+1/2} B_n(x)\,dx = \frac{1}{2^{n+1}} E_n(2a), \quad n \geqslant 0. \tag{2.5}$$

In the special case $a=0$, the identity becomes

$$\int_0^{1/2} B_n(x)\,dx = \frac{E_n(0)}{2^{n+1}} = \frac{1-2^{n+1}}{2^n}\frac{B_{n+1}}{n+1}. \tag{2.6}$$

The proof of identity (2.5) starts with (1.2) to obtain

$$\int_a^{a+1/2} B_n(x)\,dx = \frac{1}{n+1}\left(B_{n+1}(a+\tfrac{1}{2}) - B_{n+1}(a)\right). \tag{2.7}$$

The result now follows from (2.4). The last step in (2.6) is satted below as Lemma 10.1.

3. Fourier coefficients

The periodized Bernoulli polynomials, defined by

$$\widetilde{B}_n(x) = B_n(\{x\}), \tag{3.1}$$

being a periodic function of period 1, can be expanded in a Fourier series

$$\widetilde{B}_n(x) = \sum_{k=-\infty}^{\infty} c_{k,n} e^{2\pi i k x}. \tag{3.2}$$

The coefficients $c_{k,n}$ are given by the integral

$$c_{k,n} = \int_0^1 \widetilde{B}_n(t) e^{-2\pi i k t}\,dt = \int_0^1 B_n(t) e^{-2\pi i k t}\,dt, \tag{3.3}$$

since $\widetilde{B}_n(t) = B_n(t)$ for $0 \leqslant t \leqslant 1$. The next statement gives explicit expressions for these coefficients.

Theorem 3.1. The Fourier coefficients $c_{k,n}$ are given by

$$c_{k,0} = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0; \end{cases} \tag{3.4}$$

and for $n \geqslant 1$,

$$c_{k,n} = \begin{cases} -n!/(2\pi i k)^n & \text{if } k \neq 0, \\ 0 & \text{if } k = 0. \end{cases} \tag{3.5}$$

Proof. The values of $c_{k,0}$ come directly from $B_0(t) = 1$. For $n \geqslant 1$ and $k = 0$:

$$c_{0,n} = \int_0^1 B_n(t)\,dt = \frac{B_{n+1}(1) - B_{n+1}(0)}{n+1} \tag{3.6}$$

from (2.2). The relation (1.11), with $x = 0$, shows that $c_{0,n} = 0$. For $k \neq 0$, integration by parts gives

$$c_{k,n} = \frac{1}{2\pi i k} \int_0^1 n B_{n-1}(t) e^{-2\pi i k t}\,dt, \tag{3.7}$$

since the boundary terms vanish. This gives the recurrence

$$c_{k,n} = \frac{1}{2\pi i k} n c_{k,n-1}, \tag{3.8}$$

which can be iterated to produce

$$c_{k,n} = \frac{n!}{(2\pi i k)^{n-1}} c_{k,1}. \tag{3.9}$$

The initial condition $c_{k,1} = -1/(2\pi ik)$ is checked by direct integration using $B_1(t) = t - \frac{1}{2}$. The computation is complete. □

The formulas for the Fourier coefficients given above yield the expansion stated below.

Corollary 3.2. The identity

$$B_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{e^{2\pi ikx}}{k^n} \tag{3.10}$$

holds for $x \in [0,1]$ if $n \geqslant 2$ and $x \in (0,1)$ if $n = 1$.

Remark 3.3. A variety of integral evaluations involving Bernoulli polynomials were obtained in [**12, 13**], as special cases of integrals of the Hurwitz zeta function

$$\zeta(z,q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^z}. \tag{3.11}$$

The relation

$$B_n(q) = -n\zeta(1-n,q) \quad \text{for } n \in \mathbb{N} \tag{3.12}$$

is responsible for them. These more general integral evaluations have been established starting with the Fourier expansion

$$\zeta(z,q) = \frac{2\Gamma(1-z)}{(2\pi)^{1-z}} \times \left(\sin\left(\frac{\pi z}{2}\right) \sum_{n=1}^{\infty} \frac{\cos(2\pi qn)}{n^{1-z}} + \cos\left(\frac{\pi z}{2}\right) \sum_{n=1}^{\infty} \frac{\sin(2\pi qn)}{n^{1-z}} \right). \tag{3.13}$$

The reader will find in [**5**] and [**26**] proofs of this expansion.

4. Product of two Bernoulli polynomials

This section discusses the evaluation of integrals involving the product of two Bernoulli polynomials. The family $\{B_n(x)\}_{n\geqslant 0}$ consists on one polynomial per degree. Therefore, since $B_{n_1}(x)B_{n_2}(x)$ is a polynomial of degree n_1+n_2, there are linearization constants $\alpha_j(n_1,n_2)$ such that

$$B_{n_1}(x)B_{n_2}(x) = \sum_{j=0}^{n_1+n_2} \alpha_j(n_1,n_2)B_j(x). \tag{4.1}$$

Then (2.3) gives

$$\int_0^1 B_{n_1}(x)B_{n_2}(x)\,dx = \alpha_0(n_1,n_2). \tag{4.2}$$

The evaluation of the integral has been reduced to the computation of the 0^{th}-coefficient in (4.1). In order to produce such a computation, by a naive approach, we start with

$$
\begin{aligned}
(4.3)\quad B_{n_1}(x)B_{n_2}(x) &= \left(\sum_{k_1=0}^{n_1}\binom{n_1}{k_1}B_{n_1-k_1}x^{k_1}\right)\left(\sum_{k_2=0}^{n_2}\binom{n_2}{k_2}B_{n_2-k_2}x^{k_2}\right) \\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\binom{n_1}{k_1}\binom{n_2}{k_2}\widehat{B}_{n_1-k_1}\widehat{B}_{n_2-k_2}x^{k_1+k_2} \\
&= \sum_{m=0}^{\infty}\left(\sum_{k_1=0}^{m}\binom{n_1}{k_1}\binom{n_2}{m-k_1}\widehat{B}_{n_1-k_1}\widehat{B}_{n_2-m+k_1}\right)x^m, \\
&= \sum_{m=0}^{n_1+n_2}\left(\sum_{k_1=0}^{m}\binom{n_1}{k_1}\binom{n_2}{m-k_1}\widehat{B}_{n_1-k_1}\widehat{B}_{n_2-m+k_1}\right)x^m,
\end{aligned}
$$

where the limits of summation have been extended to $+\infty$ and

$$
(4.4)\qquad \widehat{B}_r = \begin{cases} B_r & \text{if } r \geqslant 0, \\ 0 & \text{if } r < 0. \end{cases}
$$

Since the added terms vanish, because of the binomial coefficients, this is justified. The sum has been stopped at $m = n_1 + n_2$, since $B_{n_1}(x)B_{n_2}(x)$ has degree $n_1 + n_2$.

The expression of x^m in terms of Bernoulli polynomials is obtained by inverting (1.7) to obtain

$$
(4.5)\qquad x^m = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}B_j(x).
$$

Replacing in the expression above yields
(4.6)

$$
B_{n_1}(x)B_{n_2}(x) = \sum_{m=0}^{n_1+n_2}\left(\sum_{k_1=0}^{m}\binom{n_1}{k_1}\binom{n_2}{m-k_1}\widehat{B}_{n_1-k_1}\widehat{B}_{n_2-m+k_1}\right)\left[\frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}B_j(x)\right].
$$

Integrating this expression over $[0,1]$, only the terms for $j = 0$ survive, yielding
(4.7)

$$
\int_0^1 B_{n_1}(x)B_{n_2}(x)\,dx = \sum_{m=0}^{n_1+n_2}\frac{1}{m+1}\left(\sum_{k_1=0}^{m}\binom{n_1}{k_1}\binom{n_2}{m-k_1}\widehat{B}_{n_1-k_1}\widehat{B}_{n_2-m+k_1}\right).
$$

A simplified version of this integral comes from the identity
(4.8)

$$
B_{n_1}(x)B_{n_2}(x) = \sum_{0\leqslant r<\frac{n_1+n_2}{2}}^{\infty}\left[n_2\binom{n_1}{2r}+n_1\binom{n_2}{2r}\right]\frac{B_{2r}B_{n_1+n_2-2r}(x)}{n_1+n_2-2r}+(-1)^{n_1+1}\frac{B_{n_1+n_2}}{\binom{n_1+n_2}{n_1}}
$$

for $n_1 + n_2 \geqslant 2$. This was established by Carlitz [**7, 8**] and also appears in Nielsen [**21**, page 75].

Integrating over $[0,1]$ gives

$$\int_0^1 B_{n_1}(x)B_{n_2}(x)\,dx = (-1)^{n_1+1}\frac{B_{n_1+n_2}}{\binom{n_1+n_2}{n_1}}. \tag{4.9}$$

The special case $n_1 = n_2 = n$ gives

$$\int_0^1 B_n^2(x)\,dx = \frac{|B_{2n}|}{\binom{2n}{n}}, \tag{4.10}$$

that may be written as

$$\int_0^1 \frac{B_n(x)}{n!}\cdot\frac{B_n(x)}{n!}\,dx = \frac{|B_{2n}|}{(2n)!}. \tag{4.11}$$

Two elementary proofs of (4.12) have been given by D. Zagier in the book by Arakawa et al. [**4**]. The first proof is based on manipulations of the (exponential) generating function of the integral. The second one, reproduced below, is based on the Fourier expansion of the Bernoulli numbers.

Example 4.1. For $n_1,\, n_2 \geqslant 0$,

$$\int_0^1 B_{n_1}(x)B_{n_2}(x)\,dx = (-1)^{n_1+1}\frac{n_1!\,n_2!}{(n_1+n_2)!}B_{n_1+n_2}. \tag{4.12}$$

This is the same as (4.9). The proof starts by replacing the Bernoulli polynomials by their Fourier expansion given in (3.10) to obtain

$$\begin{aligned}\int_0^1 B_{n_1}(x)B_{n_2}(x)\,dx &= \frac{n_1!\,n_2!}{(2\pi i)^{n_1+n_2}}\int_0^1 \sum_{k,\,\ell\neq 0}\frac{e^{2\pi i(k+\ell)t}}{k^{n_1}\,\ell^{n_2}}\,dt \\ &= \frac{n_1!n_2!}{(2\pi i)^{n_1+n_2}}\sum_{\ell\neq 0}\frac{1}{(-\ell)^{n_1}\ell^{n_2}}.\end{aligned} \tag{4.13}$$

The result now follows from the relation between values of the Riemann zeta function and Bernoulli numbers:

$$\zeta(2n) = \sum_{r=1}^{\infty}\frac{1}{r^{2n}} = (-1)^{n+1}\frac{B_{2n}(2\pi)^{2n}}{2(2n)!}. \tag{4.14}$$

See [**26**] for a proof of this classical identity.

The indefinite version of (4.12) appears in [**2**]. Their result is stated below.

Theorem 4.2. For $n_1,\, n_2 \geqslant 0$,

$$\int_0^x B_{n_1}(t)B_{n_2}(t)\,dt = \tag{4.15}$$

$$\frac{n_1!\,n_2!}{(n_1+n_2+1)!}\sum_{j=0}^{n_1}(-1)^j\binom{n_1+n_2+1}{n_1-j}\left[B_{n_1-j}(x)B_{n_2+j+1}(x) - B_{n_1-j}B_{n_2+j+1}\right].$$

PROOF. The proof follows the argument in [**2**]. Introduce the notation

$$I_{a,b} = \int_0^x B_a(t)B_b(t)\,dt \quad \text{and} \quad C_{a,b} = B_a(x)B_b(x) - B_aB_b. \tag{4.16}$$

Integration by parts yields the recursion

$$I_{n_1,n_2} = \frac{C_{n_1,n_2+1}}{n_2+1} - \frac{n_1}{n_2+1} I_{n_1-1,n_2+1}, \tag{4.17}$$

with the initial condition $I_{0,n_1+n_2} = C_{0,n_1+n_2+1}/(n_1+n_2+1)$. This is now solved explicitly to obtain the result. □

Remark 4.3. Integrating Carlitz identity (4.8) gives the alternative expression

$$\begin{aligned}\int_0^x B_{n_1}(t)B_{n_2}(t)\,dt = \\ \sum_{0 \leqslant \ell < \frac{n_1+n_2}{2}} \left[n_2\binom{n_1}{2\ell} + n_1\binom{n_2}{2\ell}\right] B_{2\ell} \frac{B_{n_1+n_2-2\ell+1}(x) - B_{n_1+n_2-2\ell+1}}{(n_1+n_2-2\ell)(n_1+n_2-2\ell+1)} \\ + (-1)^{n_1-1}\frac{n_1!\,n_2!}{(n_1+n_2)!} B_{n_1+n_2} x.\end{aligned} \tag{4.18}$$

5. Product of more than two Bernoulli polynomials

This section discusses integrals where the integrand contains products of Bernoulli polynomials with more than two factors. The first result is due to Carliz [**8**].

Theorem 5.1 (Carlitz)**.** Let $n, m_1, \cdots, m_n$ be integers and $x_1, \cdots, x_n$ be real numbers. Then

$$\int_0^1 \prod_{j=1}^n B_{m_j}(x_j+y)\,dy = \sum_{r_1=0}^{m_1} \cdots \sum_{r_n=0}^{m_n} \frac{1}{r_1+\cdots+r_n+1} \prod_{j=1}^n \binom{m_j}{r_j} B_{m_j-r_j}(x_j). \tag{5.1}$$

In particular,

$$\int_0^1 \prod_{j=1}^n B_{m_j}(y)\,dy = \sum_{r_1=0}^{m_1} \cdots \sum_{r_n=0}^{m_n} \frac{1}{r_1+\cdots+r_n+1} \prod_{j=1}^n \binom{m_j}{r_j} B_{m_j-r_j}. \tag{5.2}$$

PROOF. The identity

$$e^{z(x+y)}\frac{z}{e^z-1} = e^{zx}\frac{ze^{zy}}{e^z-1} \tag{5.3}$$

is translated via the generating functions (1.1) and (1.3) into

$$B_m(x+y) = \sum_{r=0}^m \binom{n}{r} x^r B_{m-r}(y). \tag{5.4}$$

Integrating this relation yields the result. □

The next result also appears in Carlitz.

Theorem 5.2 (Carlitz)**.** Assume $m, n, p, q \geqslant 1$. Then

$$\int_0^1 B_m(x)B_n(x)B_p(x)dx = (-1)^{p+1}\, p! \sum_{r \geqslant 0} \left[n\binom{m}{2r} + m\binom{n}{2r} \right] \frac{(m+n-2r-1)!}{(m+n+p-2r)!} B_{2r}B_{m+n+p-2r} \tag{5.5}$$

and

$$\begin{aligned} \int_0^1 B_m(x)B_n(x)B_p(x)B_q(x)dx = (-1)^{m+n+p+q} & \sum_{r,s \geqslant 0} \left[n\binom{m}{2r} + m\binom{n}{2r} \right] \left[q\binom{p}{2s} + p\binom{q}{2s} \right] \\ & \times \frac{(m+n-2r-1)!\,(p+q-2s-1)!}{(m+n+p+q-2r-2s)!} B_{2r}B_{2s}B_{m+n+p+q-2r-2s} \\ & + (-1)^{m+p} \frac{m!n!p!q!}{(m+n)!\,(p+q)!} B_{m+n}B_{p+q}. \end{aligned} \tag{5.6}$$

Proof. Use (4.12) and (4.8). □

6. Franel integrals

In 1924, the Swiss mathematician J. Franel published an article [**14**] in which he provided an equivalent condition for the Riemann hypothesis to hold. The result is based on the asymptotic behavior of a certain specific function. On his way to this result, he needed to compute an integral involving the periodic version of a product of two Bernoulli polynomials. Integral of this type are called `Franel integrals` and their `order` refers to the number of Bernoulli factors.

6.1. Second order Franel integrals. The original problem amounts to the evaluation of

$$\int_0^1 B_1(\{ax\})B_1(\{bx\})dx,$$

for positive integers a and b. Franel writes: *One easily makes sure that*

$$\int_0^1 B_1(\{ax\})B_1(\{bx\})dx = \frac{(a,b)^2}{12ab} \tag{6.1}$$

adding in a footnote *using the trigonometric series expansion of* $f(x)$. Here (a, b) denotes the greatest common divisor of a and b.

This result is indeed easily obtained using the expansion given in Theorem 3.1 in the case $n = 1$:

$$B_1(\{x\}) = -\frac{1}{2\pi i} \sum_{\substack{k \neq 0 \\ k=-\infty}}^{\infty} \frac{e^{2\pi i k x}}{k},$$

so that

$$B_1(\{ax\})B_1(\{bx\}) = -\frac{1}{4\pi^2} \sum_{k,l \neq 0} \frac{e^{2\pi i (kax + lbx)}}{kl}.$$

Integrating produces

$$\int_0^1 B_1(\{ax\})B_1(\{bx\})dx = -\frac{1}{4\pi^2}\sum_{k,l\neq 0}\int_0^1 \frac{e^{2\pi i x(ka+lb)}}{kl}dx.$$

The integral above vanishes if $ka+lb$ is non zero, and is equal to 1 if k and l are such

$$ka + lb = 0. \tag{6.2}$$

It follows that

$$\int_0^1 B_1(\{ax\})B_1(\{bx\})dx = -\frac{1}{4\pi^2}\sum_{\substack{ak+bl=0\\ k,l\neq 0}} \frac{1}{kl}.$$

A direct change of variables permits to reduce to the case in which a and b are coprime. Then $(a,b) = ab$ and the solutions k,l to the equation (6.2) are

$$k = nb,\ \ l = -na,\ \ n \in \mathbb{Z}, n \neq 0.$$

It follows that

$$\sum_{k,l\neq 0}\int_0^1 \frac{e^{2\pi i x(ka+lb)}}{kl}dx = \sum_{n\neq 0}\frac{1}{(nb)\,(-na)} = -\frac{1}{ab}\sum_{n\neq 0}\frac{1}{n^2},$$

and since the last sum is twice

$$\sum_{n\geqslant 1}\frac{1}{n^2} = \frac{\pi^2}{6}$$

the evaluation is complete.

Remark 6.1. An extended version of this result appears in an article by Kluyver [**17**], twenty-one years earlier, under the form

$$\int_0^1 \widetilde{B}_m(bx)\widetilde{B}_n(ax)dx = (-1)^{n-1}\frac{m!n!}{(m+n)!}B_{m+n}\frac{(a,b)^{m+n}}{a^m b^n},\ \ m,n\in\mathbb{N}.$$

The proof uses the Fourier expansion of the integrand as previously.

6.2. Higher-order Franel integrals. Order-three Franel integrals can be found in Wilson [**27**]. The integral

$$I_{l,m,n}\,(a,b,c) = \int_0^1 \widetilde{B}_l(ax)\widetilde{B}_m(bx)\widetilde{B}_n(cx)dx$$

is evaluated in terms of the generalized Dedekind sums

$$S_{m,n}\,(h,k) = \sum_{r=0}^{k-1}\widetilde{B}_m\left(\frac{r}{k}\right)\widetilde{B}_n\left(\frac{rh}{k}\right).$$

The details are stated next.

Theorem 6.2 (Wilson)**.** Assuming that a, b and c are pairwise coprime and that $l+m+n$ is even (since the integral vanishes otherwise), and with $a\bar{a}+b\bar{b}=1$,

$$I_{l,m,n}(a,b,c) = \frac{(-1)^{l+n}\, l!m!n!a^{l-1}}{c^l} \sum_{s=0}^{m} \binom{l+s-1}{s} \left(\frac{b}{c}\right)^s (-1)^{s+1} \frac{S_{m-s,l+n+s}(c\bar{b},a)}{(m-s)!\,(l+n+s)!} \tag{6.3}$$

$$+ \frac{(-1)^{m+n}\, l!m!n!b^{m-1}}{c^m} \sum_{s=0}^{l} \binom{m+s-1}{s} \left(\frac{a}{c}\right)^s (-1)^{s+1} \frac{S_{l-s,m+n+s}(c\bar{a},b)}{(l-s)!\,(m+n+s)!}. \tag{6.4}$$

The proof involves applying Parseval formula to the functions $\widetilde{B}_l(ax)\widetilde{B}_m(bx)$ and $\widetilde{B}_n(cx)$.

Remark 6.3. The order four Franel integral, for the special case $m=n=1$, is addressed in McIntosh [**18**]. With the notation

$$I(a,b,c,d) = \int_0^1 B_1(\{ax\})B_1(\{bx\})B_1(\{cx\})B_1(\{dx\})\,dx,$$

the author uses the Fourier expansion method above to obtain

$$I(a,b,c,d) = \frac{1}{16\pi^4} \sum_{\substack{ai+bj+ck+dl=0\\ i,j,k,l\neq 0}} \frac{1}{ijkl}.$$

The quadruple sum above is denoted by $L(a,b,c,d)$. Some special values are

$$\begin{aligned} L(1,1,1,1) &= \frac{\pi^4}{5}, \\ L(a,1,1,1) &= \left(\frac{1}{3a} - \frac{2}{15a^3}\right)\pi^4, \\ L(a,a,b,b) &= \left(\frac{1}{9} + \frac{4}{45}\frac{(a,b)^4}{a^2b^2}\right)\pi^4. \end{aligned} \tag{6.5}$$

A more involved result is

$$L(a,a,b,1) = \left(\frac{1}{9b} + \frac{2b}{45a^2} + \frac{2}{45}\frac{(a,b)^4}{a^2b^3} + \frac{16}{3a}S_3(a,b)\right)\pi^4,$$

with the generalized Dedekind sum

$$S_3(h,k) = \sum_{r=1}^{k-1} \frac{r}{k} B_3\left(\left\{\frac{hr}{k}\right\}\right) = -\frac{3}{4\pi^3} \sum_{\substack{k\nmid r\\ r=1}}^{\infty} \frac{1}{r^3}\cot\left(\frac{rh\pi}{k}\right).$$

These results are obtained using an idea by P. Montgomery, namely the fact that each sum $L(a,b,c,d)$ can be expressed as the convolution

$$L(a,b,c,d) = \sum_{M=-\infty}^{\infty} L_M(a,b)L_M(c,d)$$

where

$$L_M(a,b) = \sum_{\substack{ak+bl=M \\ k,l\neq 0}} \frac{1}{kl}.$$

More details can be found in [**18**].

Example 6.4. For the evaluation of the integral of $B_1^4(\{x\})$, it is required to compute

$$L_0(1,1) = \sum_{\substack{k,l\neq 0 \\ k+l=0}} \frac{1}{kl} = -\sum_{k\neq 0} \frac{1}{k^2} = -2\zeta(2) = -\frac{\pi^2}{3},$$

whereas, for $M \neq 0$,

$$L_M(1,1) = \sum_{\substack{k,l\neq 0 \\ k+l=M}} \frac{1}{kl} = \sum_{k=0}^{M-1} \frac{1}{k(M-k)} - 2\sum_{k=1}^{\infty} \frac{1}{k(M+k)} = -\frac{2}{M^2}$$

so that

$$L(1,1,1,1) = 2\sum_{M\geqslant 1} \frac{4}{M^4} + \left(\frac{\pi^2}{3}\right)^2 = 8\zeta(4) + \frac{\pi^4}{9} = \frac{\pi^4}{5}$$

and finally

$$\int_0^1 B_1^4(\{x\})\,dx = \frac{1}{16\pi^4}\frac{\pi^4}{5} = \frac{1}{80}.$$

This is consistent with the direct integral obtained from $B_1(x) = x - \frac{1}{2}$.

7. Some integrals involving hyperbolic secant functions

A few results involving Bernoulli polynomials can be deduced from the integral representation

$$B_n(x) = \frac{\pi}{2}\int_{-\infty}^{+\infty} \left(x - \frac{1}{2} + it\right)^n \operatorname{sech}^2(\pi t)\,dt. \tag{7.1}$$

For example, identities (1.2), (1.4) and (1.5) are simple consequences of (7.1).

This integral representation appears in [**22**, p.75] under the equivalent form

$$B_n(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} (x+z)^n \left(\frac{\pi}{\sin \pi z}\right)^2 dz.$$

A separate proof is presented next: start from entry 3.982.1 in [**15**]

$$\int_{-\infty}^{+\infty} \operatorname{sech}^2(at)\cos(tu)dt = \frac{\pi u}{a^2}\operatorname{csch}\left(\frac{\pi u}{2a}\right)$$

so that

$$\frac{\pi}{2}\int_{-\infty}^{+\infty} \operatorname{sech}^2(\pi t)\cos(tu)dt = \frac{u}{2}\operatorname{csch}\left(\frac{u}{2}\right)$$

and by parity,

$$\frac{\pi}{2}\int_{-\infty}^{+\infty} \operatorname{sech}^2(\pi t)\, e^{itu}dt = \frac{u}{2}\operatorname{csch}\left(\frac{u}{2}\right).$$

Multiplying both sides by $e^{u\left(x-\frac{1}{2}\right)}$, we obtain

$$\frac{\pi}{2}\int_{-\infty}^{+\infty}\operatorname{sech}^2(\pi t)e^{\left(it+x-\frac{1}{2}\right)u}dt=\frac{u}{e^u-1}e^{ux}=\sum_{n\geqslant 0}\frac{B_n\left(x\right)}{n!}u^n.$$

Identifying the coefficients of u on both sides gives the result.

8. Some $\log\left(\tan t\right)$ integrals

This section contains some integrals including the function $\log(\tan t)$. The relation with Bernoulli polynomials is explained in the proof. The first result appears in [**11**]. As a motivating example, start with the Mathematica evaluation of

$$\begin{aligned}
\int_0^{\pi/2} t\log(\tan t) &= \frac{7}{8}\zeta(3) \\
\int_0^{\pi/2} t^3\log(\tan t) &= \frac{7}{16}\pi\zeta(3) \\
\int_0^{\pi/2} t^3\log(\tan t)\,dt &= \frac{21}{64}\pi^2\zeta(3)-\frac{93}{64}\zeta(5) \\
\int_0^{\pi/2} t^4\log(\tan t)\,dt &= \frac{7}{32}\pi^3\zeta(3)-\frac{93}{64}\pi\zeta(5) \\
\int_0^{\pi/2} t^5\log(\tan t)\,dt &= \frac{35}{256}\pi^4\zeta(3)-\frac{465}{256}\pi^2\zeta(5)+\frac{1905}{256}\zeta(7).
\end{aligned}\tag{8.1}$$

These examples suggest that the integral of a polynomial times $\log(\tan t)$ is a combination of zeta values.

The general result is stated below.

Theorem 8.1. Assume $P(t)$ is a polynomial of degree m, then

$$\begin{aligned}
&\int_0^{\pi/2} P\left(t\right)\log\left(\tan t\right)dt \\
&\quad=\sum_{k=1}^{\left\lfloor\frac{m+1}{2}\right\rfloor}\frac{\left(-1\right)^{k-1}}{2^{2k-1}}\left[P^{(2k-1)}\left(\frac{\pi}{2}\right)+P^{(2k-1)}\left(0\right)\right]\left(1-\frac{1}{2^{2k+1}}\right)\zeta\left(2k+1\right).
\end{aligned}\tag{8.2}$$

Proof. Integrate by parts to obtain

$$\int_0^{\pi/2} P(t)\log\left(\tan t\right)dt=-\int_0^{\pi/2} P'(t)\left(\int_0^t\log\left(\tan u\right)du\right)dt.$$

The inner integral has been evaluated by D. Bradley [**6**]:

Lemma 8.2. For $t\in[0,\pi/2]$,

$$\int_0^t\log\left(\tan u\right)du=-\sum_{n=0}^{\infty}\frac{\sin\left(2\left(2n+1\right)t\right)}{\left(2n+1\right)^2}.$$

Proof. This result is obtained by integrating the Fourier expansion of the function $\log(\tan u)$ deduced from the expansions

$$\log(\sin u) = -\sum_{n \geqslant 1} \frac{\cos(2nu)}{n} - \log 2$$

and

$$\log(\cos u) = -\sum_{n \geqslant 1} (-1)^n \frac{\cos(2nu)}{n} - \log 2,$$

so that

$$\log(\tan u) = \sum_{n \geqslant 1} \left[(-1)^n - 1\right] \frac{\cos(2nu)}{n} = -2 \sum_{n \geqslant 0} \frac{\cos(2(2n+1))u}{2n+1}.$$

□

Thus it is required to evaluate integrals of the form

$$\int_0^{\pi/2} P'(t) \sin(2(2n+1)t)\, dt.$$

These are obtained by successive integrations by parts as

$$\int_0^{\pi/2} P'(t) \sin(2(2n+1)t)\, dt = \sum_{k=1}^{\lfloor \frac{m+1}{2} \rfloor} \frac{(-1)^{k-1}}{2^{2k-1}} \left[P^{(2k-1)}\left(\frac{\pi}{2}\right) + P^{(2k-1)}(0)\right] \frac{1}{(2n+1)^{2k-1}}.$$

Exchanging the two sums produces

$$\int_0^{\pi/2} P(t) \log(\tan t)\, dt = \sum_{k=1}^{\lfloor \frac{m+1}{2} \rfloor} \frac{(-1)^{k-1}}{2^{2k-1}} \left[P^{(2k-1)}\left(\frac{\pi}{2}\right) + P^{(2k-1)}(0)\right] \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2k+1}}.$$

The last step is the series in terms of the zeta function. It is an easy exercise to check that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2k+1}} = \left(1 - \frac{1}{2^{2k+1}}\right) \zeta(2k+1),$$

and the result follows. □

In the case of Bernoulli polynomials, Theorem 8.1 produces the following result.

Theorem 8.3. *If $m = 2p+1$ then*

$$\int_0^{\pi/2} B_{2p+1}\left(\frac{2t}{\pi}\right) \log(\tan t)\, dt = 2 \sum_{k=1}^{p+1} \frac{(-1)^{k-1}}{\pi^{2k-1}} \frac{(2p+1)!}{(2p-2k+2)!} B_{2p-2k+2} \left(1 - \frac{1}{2^{2k+1}}\right) \zeta(2k+1). \tag{8.3}$$

If $m = 2p$,

$$\int_0^{\pi/2} B_{2p}\left(\frac{2t}{\pi}\right) \log(\tan t)\, dt = 0.$$

Proof. If $m = 2p+1$, then

$$\int_0^{\pi/2} P(t)\log(\tan t)\,dt = \sum_{k=1}^{p+1} \frac{(-1)^{k-1}}{2^{2k-1}} \left[P^{(2k-1)}\left(\frac{\pi}{2}\right) + P^{(2k-1)}(0)\right]\left(1-\frac{1}{2^{2k+1}}\right)\zeta(2k+1).$$

Substituting

$$P(t) = B_m\left(\frac{2t}{\pi}\right) = B_{2p+1}\left(\frac{2t}{\pi}\right)$$

gives

$$P^{(2k-1)}(t) = \frac{(2p+1)!}{(2p-2k+2)!}\left(\frac{2}{\pi}\right)^{2k-1} B_{2p-2k+2}\left(\frac{2t}{\pi}\right)$$

and

$$\begin{aligned} P^{(2k-1)}\left(\frac{\pi}{2}\right) &= \frac{(2p+1)!}{(2p-2k+2)!}\left(\frac{2}{\pi}\right)^{2k-1} B_{2p-2k+2}(1), \\ P^{(2k-1)}(0) &= \frac{(2p+1)!}{(2p-2k+2)!}\left(\frac{2}{\pi}\right)^{2k-1} B_{2p-2k+2}(0). \end{aligned}$$

This produces

$$P^{(2k-1)}\left(\frac{\pi}{2}\right) + P^{(2k-1)}(0) = 2\frac{(2p+1)!}{(2p-2k+2)!}\left(\frac{2}{\pi}\right)^{2k-1} B_{2p-2k+2}$$

and

$$\int_0^{\pi/2} B_{2p+1}\left(\frac{2t}{\pi}\right)\log(\tan t)\,dt = 2\sum_{k=1}^{p+1}\frac{(-1)^{k-1}}{\pi^{2k-1}}\frac{(2p+1)!}{(2p-2k+2)!}B_{2p-2k+2}\left(1-\frac{1}{2^{2k+1}}\right)\zeta(2k+1). \tag{8.4}$$

If $m = 2p$, then the function

$$f(t) = B_{2p}\left(\frac{2t}{\pi}\right)\log(\tan t)$$

is antisymmetric with respect to $t = \frac{\pi}{4}$; that is, $f(t) = -f(\pi/2 - t)$, so that its integral over $[0, \pi/2]$ vanishes. □

9. Additional integrals from Prudnikov

The examples discussed in this section appear in the table of integrals by A. P. Prudnikov el al. [**23**].

9.1. Moments for Bernoulli polynomials. Entry 2.4.1.3 is

$$\int_0^1 x^m B_n(x)dx = \sum_{k=0}^{n} \binom{n}{k} \frac{B_k}{m+n-k+1}. \tag{9.1}$$

PROOF. Use the Taylor expansion (1.7) to obtain

$$\int_0^1 x^m B_n\left(x\right)dx = \int_0^1 x^m \sum_{k=0}^{n} \binom{n}{k} x^{n-k} B_k dx = \sum_{k=0}^{n} \binom{n}{k} \frac{B_k}{m+n-k+1}. \tag{9.2}$$

□

9.2. Laplace transform over $[0,1]$**.** Entry 2.4.1.4 provides an expression for the integral

$$\int_0^1 e^{ax} B_n\left(x\right)dx \tag{9.3}$$

with a $n!$ factor missing. The correct version of this integral is given next.

Theorem 9.1.

$$\int_0^1 e^{ax} B_n(x)dx = n!\left(-1\right)^{n+1} \frac{e^a-1}{2a^{n+1}} \left(a\coth\frac{a}{2} - a - 2\sum_{k=0}^{n} \frac{a^k}{k!} B_k \right), \; n \geqslant 1. \tag{9.4}$$

PROOF. Let I_n be the left-hand side. Integration by parts, using the derivation rule (1.2), produces

$$I_n = \frac{1}{a}\left[e^a B_n\left(1\right) - B_n\left(0\right)\right] - \frac{n}{a} I_{n-1}.$$

For $n \geqslant 1$, formula (1.5) simplifies the previous expression to

$$I_n = \frac{e^a-1}{a} B_n - \frac{n}{a} I_{n-1}.$$

This recurrence is solved as

$$I_n = \frac{e^a-1}{a} \sum_{k=0}^{n-2} \frac{\left(-1\right)^k}{a^k} \frac{n!}{(n-k)!} B_{n-k} + \frac{n!\left(-1\right)^{n-1}}{a^{n-1}} I_1,$$

and the initial value I_1 is computed as

$$I_1 = \int_0^1 e^{ax}\left(x - \frac{1}{2}\right)dx = \frac{a+2}{2a^2} + e^a \frac{a-2}{2a^2},$$

producing

$$\begin{aligned} I_n &= \frac{e^a-1}{a}\sum_{k=0}^{n-2}\frac{(-1)^k}{a^k}\frac{n!}{(n-k)!}B_{n-k} + (-1)^n n!\frac{e^a-1}{a^{n+1}} + (-1)^{n-1} n!\frac{e^a+1}{2a^n} \\ &= n!\frac{e^a-1}{2a^{n+1}}(-1)^{n+1}\left[2\sum_{k=2}^{n}(-1)^{k+1}a^k\frac{B_k}{k!} - 2 + a\coth\frac{a}{2}\right]. \end{aligned}$$

The Bernoulli numbers satisfy $(-1)^k B_k = B_k,\ k \neq 1$, so that

$$I_n = n!\frac{e^a-1}{2a^{n+1}}(-1)^{n+1}\left[-2\sum_{k=2}^{n}a^k\frac{B_k}{k!} - 2 + a\coth\frac{a}{2}\right]$$

and then the result follows from

$$\sum_{k=2}^{n}a^k\frac{B_k}{k!} = \sum_{k=0}^{n}a^k\frac{B_k}{k!} - 1 + \frac{a}{2}.$$

□

Remark 9.2. Identification of the coefficient of $a^n/n!$ in (9.4) recovers (9.1).

9.3. Entry 2.4.1.8 **from Prudnikov.** This entry is

$$\int_0^1 \cot(\pi t)\, B_{2n+1}(t)\, dt = (-1)^{n+1}\, 2\frac{(2n+1)!}{(2\pi)^{2n+1}}\zeta(2n+1). \tag{9.5}$$

This is evaluated next. Note that in the neighborhood of 0,

$$\cot(\pi t)\, B_m(t) = \frac{B_m}{\pi t} + \frac{mB_{m-1}}{\pi} + O(t),$$

so that convergence of the integral requires $m = 2n+1,\ m \geqslant 1$. In the neighborhood of 1,

$$\cot(\pi t)\, B_m(t) = \frac{B_m(1)}{\pi(t-1)} + \frac{mB_{m-1}(1)}{\pi} + O(t-1).$$

Using (1.5), it follows that convergence in the neighborhood of 1 requires the same condition $m = 2n+1,\ m \geqslant 1$. Hence the integral in (9.5) exists for $n \geqslant 1$.

Remark 9.3. None of the integrals

$$\int_0^1 t^k \cot(\pi t)\, dt$$

are convergent since in the neighborhood of 1,

$$t^k \cot(\pi t) = \frac{1}{\pi(t-1)} + \frac{k}{\pi} + O(t)$$

for all integer k. Thus we can not simply use the Taylor expansion (1.7) and compute each term separately.

This integral appears in [**10**, Prop. 1.4] as Prop. 1.4 referring to [**9**] which in turns refers to [**1**, Eq. 23.2.17]. Cvijovic's derivation of this integral is presented here. Start with the identity

$$\sin(2kx)\cot x = \frac{1}{2}\left[\frac{\sin(2k+1)x}{\sin x} + \frac{\sin(2k-1)x}{\sin x}\right]$$

and then use the Fourier expansion

$$B_{2n+1}(x) = \frac{(-1)^{n+1}\,2\,(2n+1)!}{(2\pi)^{2n+1}}\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{k^{2n+1}} \tag{9.6}$$

deduced from Theorem 3.1 by a parity argument. Therefore the desired integral is expressed as

$$\begin{aligned}\int_0^1 \cot(\pi t)\,B_{2n+1}(t)\,dt &= \frac{(-1)^{n+1}\,2\,(2n+1)!}{(2\pi)^{2n+1}}\sum_{k=1}^{+\infty}\int_0^1 \cot(\pi t)\,\frac{\sin(2\pi kt)}{k^{2n+1}}dt \\ &= \frac{(-1)^{n+1}\,2\,(2n+1)!}{(2\pi)^{2n+1}}\sum_{k=1}^{+\infty}\frac{1}{k^{2n+1}} \\ &\quad\times\int_0^1 \frac{1}{2}\left[\frac{\sin(2k+1)\pi t}{\sin \pi t} + \frac{\sin(2k-1)\pi t}{\sin \pi t}\right]dt.\end{aligned}$$

Now use

$$\int_0^1 \frac{\sin(2k+1)\pi t}{\sin \pi t}dt = 1,\ \ k \in \mathbb{N}$$

to obtain

$$\int_0^1 \cot(\pi t)\,B_{2n+1}(t)\,dt = \frac{(-1)^{n+1}\,2\,(2n+1)!}{(2\pi)^{2n+1}}\zeta(2n+1).$$

This completes the evaluation of (9.5).

Remark 9.4. [Cvijovic] It can be shown using the same approach as above that

$$\int_0^1 B_{2n+1}(t)\tan(\pi t)\,dt = \frac{2\,(2n+1)!}{(2\pi)^{2n+1}}(-1)^{n+1}\,\eta(2n+1),$$

where

$$\eta(s) = \sum_{k\geqslant 1}\frac{(-1)^{k-1}}{k^s}.$$

Remark 9.5. Eie et al. [**10**] used the evaluation (9.5) to study the `extended Euler sums`

$$E_{p,q}^{(r)} = \sum_{k=1}^{\infty}\frac{1}{k^q}\left(\sum_{j=1}^{\lfloor kr\rfloor}\frac{1}{j^p}\right).$$

Their results include the evaluation of more exotic integrals such as

$$\int_0^1 B_{2m}(\{bx\})B_{2n+1}(\{ax\})\cot(\pi x)\,dx = c(2m)c(2n+1) \times \left\{ E^{(r)}_{2m,2n+1} - \frac{1}{2a^{2m}b^{2n+1}}\zeta(2m+2n+1) \right\}$$

with

$$c(n) = \frac{2(-1)^{\lfloor \frac{n}{2} \rfloor + 1} n!}{(2\pi)^n},$$

using the same techniques as above.

10. The Euler numbers and polynomials

The Euler polynomials, defined by the generating function

$$\sum_{n\geqslant 0} \frac{E_n(x)}{n!} z^n = e^{zx}\frac{2}{e^z+1}, \ |z| < \frac{\pi}{2}, \tag{10.1}$$

satisfy properties very similar to those of the Bernoulli polynomials. Nevertheless, there are some conventional differences in the definitions. For instance, the Euler numbers are not defined as the constant terms of $E_n(x)$, but rather as

$$E_n = 2^n E_n\left(\frac{1}{2}\right). \tag{10.2}$$

One of the reasons for this choice of definition is to preserve the integrality of these numbers. In this setting the Euler number have combinatorial interpretations. For instance, the sequence of `Euler zig-zag numbers`

$$A_n = (-1)^{n/2} E_n,$$

with first values

$$A_0 = A_1 = A_2 = 1,\ A_3 = 2,\ A_4 = 5,\ A_5 = 16,$$

counts the number of alternating permutations of the set $1, 2, \ldots, n$.

The Taylor expansion of the Euler polynomials at $x = 0$ is given by

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_k(0)\, x^{n-k} \tag{10.3}$$

and around $x = \frac{1}{2}$ by

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}. \tag{10.4}$$

The Euler polynomials satisfy similarly the Appell property (see (1.2)):

$$\frac{d}{dx}E_n(x) = nE_{n-1}(x) \tag{10.5}$$

or equivalently

$$\int E_n(x)\,dx = \frac{E_{n+1}(x)}{n+1}. \tag{10.6}$$

The symmetry (1.4) around $x = \frac{1}{2}$ satisfied by the Bernoulli polynomials extends to the Euler case as

$$E_n(1-x) = (-1)^n E_n(x); \tag{10.7}$$

and this leads to the particular case

$$E_n(1) = (-1)^n E_n(0). \tag{10.8}$$

Since the generating function

$$\frac{2}{e^z+1} - 1 + \frac{z}{2} = \sum_{n\geqslant 2} \frac{E_n}{n!} z^n$$

is odd, it follows that

$$E_{2n}(0) = 0,\ n \geqslant 1$$

and, from (10.8),

$$E_{2n}(1) = 0,\ n \geqslant 1$$

so that

$$E_n(1) = -E_n(0),\ n \geqslant 1. \tag{10.9}$$

Lemma 10.1. For $n \geqslant 1$, the identity

$$E_n(0) = -E_n(1) = -2\left(2^{n+1}-1\right)\frac{B_{n+1}}{n+1}. \tag{10.10}$$

holds. It follows that

$$\int_0^1 E_n(t)\,dt = -2\frac{E_{n+1}(0)}{n+1} = 4\left(2^{n+2}-1\right)\frac{B_{n+2}}{(n+1)(n+2)} \tag{10.11}$$

and, using (10.7),

$$\int_0^{1/2} E_{2n}(t)\,dt = -\frac{E_{2n+1}(0)}{2n+1} = 2\left(2^{2n+2}-1\right)\frac{B_{2n+2}}{(2n+1)(2n+2)}. \tag{10.12}$$

Proof. Consider the generating function

$$\begin{aligned}
\sum_{n\geqslant 0}\left(2^{n+1}-1\right)\frac{B_{n+1}}{n+1}z^{n+1} &= \sum_{n\geqslant 0}\frac{B_{n+1}}{n+1}(2z)^{n+1} - \sum_{n\geqslant 0}\frac{B_{n+1}}{n+1}z^{n+1} \\
&= \left(\frac{2z}{e^{2z}-1} - 1\right) - \left(\frac{z}{e^z-1} - 1\right) \\
&= \frac{z}{e^z-1}\left(\frac{2}{e^z+1} - 1\right) = \frac{z}{e^z-1}\frac{1-e^z}{1+e^z} \\
&= -\frac{z}{e^z+1}
\end{aligned}$$

so that

$$\sum_{n\geqslant 0} -2\left(2^{n+1}-1\right)\frac{B_{n+1}}{n+1}z^n = \frac{2}{e^z+1},$$

which coincides with the generating function of $E_n(0)$, or equivalently with that of $-E_n(1)$ since

$$\sum_{n\geqslant 0}\frac{E_n(1)}{n!}z^n = \frac{2e^z}{1+e^z} = \frac{2}{1+e^{-z}} = \sum_{n\geqslant 0}\frac{E_n(0)}{n!}(-z)^n,$$

since the even terms in this series vanish.

The first integral is computed using (10.6) as

$$\int_0^1 E_n(t)\,dt = \frac{E_{n+1}(1)-E_{n+1}(0)}{n+1}$$

and applying identities (10.9) and (10.10). The second integral is equally simple. □

Lemma 10.2. The complementary case of integral (10.12) is

$$\int_0^{\frac{1}{2}} E_{2n-1}(t)\,dt = 2^{-2n}\frac{E_{2n}}{2n},\ n\geqslant 1.$$

Proof. Using (10.6),

$$\int_0^{\frac{1}{2}} E_{2n-1}(t)\,dt = \frac{1}{2n}\left[E_{2n}(1/2)-E_{2n}(0)\right]$$

and the result now follows from identity (10.2). □

Remark 10.3. The equivalent of the difference equation (1.11) for Bernoulli polynomials is, in the Euler case,

$$\frac{1}{2}\left[E_n(x+1)+E_n(x)\right] = x^n, \tag{10.13}$$

as can be shown by inspecting the generating functions of both sides. Note that the discrete difference operator $\Delta f(x) = f(x+1)-f(x)$ in (1.11) is now replaced by a discrete average operator $\delta f(x) = \frac{1}{2}\left[f(x)+f(x+1)\right]$.

Remark 10.4. Faulhaber identity for Bernoulli polynomials (1.10) extends to the Euler case as follows:

$$\sum_{k=0}^{p-1}(-1)^{p-1-k}k^n = \frac{E_n(p)+(-1)^p E_n(0)}{2}.$$

10.1. Fourier expansion. The Fourier expansion of the Euler polynomials can be computed in the same way as that of Bernoulli polynomials: it requires the computation, using integration by parts, of the following integrals:

$$\int_0^1 E_n(x)\cos(\pi m x)\,dx = \begin{cases}1 & m=n=0\\ 0 & n+m \text{ odd}\end{cases}$$

and

$$\int_0^1 E_{2n+1}(x)\cos(\pi(2m+1)x)\,dx = \frac{2(-1)^{n+1}(2n+1)!}{((2m+1)\pi)^{2n+2}},\ m,n \geqslant 0$$

and

$$\int_0^1 E_{2n}(x)\sin(\pi(2m+1)x)\,dx = \frac{2(-1)^{n}(2n)!}{((2m+1)\pi)^{2n+1}},\ m,n \geqslant 0.$$

Using these integrals provides the Fourier expansion of Euler polynomials as follows.

Theorem 10.5. For $x \in [0,1]$ and $n \geqslant 1$, the following identities hold

$$E_{2n}(x) = (-1)^n \frac{4(2n)!}{\pi^{2n+1}} \sum_{k\geqslant 0} \frac{\sin((2k+1)\pi x)}{(2k+1)^{2n+1}}$$

and

$$E_{2n-1}(x) = (-1)^n \frac{4(2n-1)!}{\pi^{2n}} \sum_{k\geqslant 0} \frac{\cos((2k+1)\pi x)}{(2k+1)^{2n}}.$$

11. An integral representation

Similarly to Bernoulli polynomials, Euler polynomials have an integral representation of the form

$$E_n(x) = \int_{-\infty}^{+\infty} \left(x - \frac{1}{2} + it\right)^n \operatorname{sech}(\pi t)\,dt. \tag{11.1}$$

The proof is identical to that of (7.1) for the Bernoulli polynomials, starting now from entry 3.981.3 in [**15**]

$$\int_0^\infty \operatorname{sech}(ax)\cos(tx)\,dx = \frac{\pi}{2a}\operatorname{sech}\left(\frac{\pi t}{2a}\right).$$

Identities such as (10.5), (10.7) and (10.8) are easily deduced from (11.1). The reader will find details and extensions in [**16**].

12. Additional Integrals from Prudnikov

12.1. Moments of Euler polynomials. Prudnikov 2.4.3.1 is

$$\int_0^1 x^m E_n(x)\,dx = 4(-1)^n\left(2^{m+n+2}-1\right)\frac{m!n!}{(m+n+2)!}B_{m+n+2}$$
$$+ 2(-1)^n m!n! \sum_{k=0}^{m-1} \frac{2^{n+k+2}-1}{(m-k)!(n+k+2)!}B_{n+k+2}.$$

The easier version

$$\int_0^1 x^m E_n(x)\,dx = \sum_{k=0}^{n} \binom{n}{k} \frac{E_{n-k}(0)}{m+k+1}$$

is obtained by using the Taylor expansion (10.3)

$$\int_0^1 x^m E_n(x)\,dx = \sum_{k=0}^{n} \binom{n}{k} E_{n-k}(0) \int_0^1 x^{m+k} dx = \sum_{k=0}^{n} \binom{n}{k} \frac{E_{n-k}(0)}{m+k+1}.$$

12.2. Laplace transform over $[0,1]$**.** Prudnikov 2.4.3.2 is the equivalent for Euler polynomials of identity (9.4) for Bernoulli polynomials:

$$\int_0^1 e^{ax} E_n(x)\,dx = (-1)^n \frac{e^a+1}{a^{n+2}} \left(a \tanh \frac{a}{2} - a - 2 \sum_{k=1}^{n+1} \frac{2^k-1}{k!} a^k B_k \right),$$

It can be obtained using the same technique as in the case of Bernoulli polynomials.

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