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The valuation tree for $n^2 + 7$

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Abstract. The 2-adic valuation of an integer x is the highest power of 2 which divides x. It is denoted by $\nu_2(x)$. The goal of the present work is to describe the sequence $\{\nu_2(n^2+a)\}$ for $1 \leqslant a \leqslant 7$. The first six cases are elementary. The last case considered here, namely $a = 7$, presents distinct challenges. It is shown here how to represent this family of valuations in the form of an infinite binary tree, with two symmetric infinite branches.

1. Introduction

For a prime p and an integer x, the p-adic valuation $\nu_p(x)$ is the highest power of p which divides x. The problems considered here deal with p-adic valuations of sequences generated by a polynomial. In detail, we consider the sequence

$$V_p(f) = \{\nu_p(f(n)) : \ n \in \mathbb{N}\} \tag{1.1}$$

for a polynomial f with integer coefficients. An important ingredient in the analysis of $V_p(f)$ is the ring of p-adic integers $\mathbb{Z}_p$. One description of this is ring is via series:

$$x \in \mathbb{Z}_p \text{ if and only if } x = \sum_{k=k_0}^{\infty} c_k p^k \tag{1.2}$$

where $k_0 \geqslant 0$ and $0 \leqslant c_k \leqslant p-1$ are integers. The series is convergent in the p-adic norm

$$|x|_p = p^{-k_0} \tag{1.3}$$

and $x \in \mathbb{Z}_p$ is invertible in $\mathbb{Z}_p$ if and only if $|x|_p = 1$; that is, $k_0 = 0$.

The next result appears in [**?**].

Theorem 1.1. Let p be a prime and $f \in \mathbb{Z}[x]$ be a polynomial, irreducible over $\mathbb{Z}$. Then $V_p(f)$ is either periodic or unbounded. Moreover, $V_p(f)$ is periodic if and only if f has no zeros in $\mathbb{Z}_p$. In the periodic case, the minimal periodic length is a power of p.

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2. Some elementary examples

This section contains the description of $\{\nu_2(n^2+a)\}$ for some small values of a, where the analysis is elementary.

Proposition 2.1. The 2-adic valuation of n^2+1 is given by

$$\nu_2(n^2+1) = \begin{cases} 0 & \text{if } n \equiv 0 \bmod 2 \\ 1 & \text{if } n \equiv 1 \bmod 2. \end{cases} \tag{2.1}$$

PROOF. If n is even, then n^2+1 is odd and so $\nu_2(n^2+1)=0$. On the other hand, if n is odd, say $n=2m+1$, then $n^2+1 = 2(2[m^2+m]+1)$. Therefore n^2+1 is twice an odd number and so $\nu_2(n^2+1)=1$. ☐

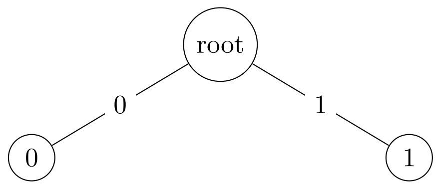

FIGURE 1. The complete tree for $\nu_2(n^2+1)$

Figure 1 contains a `tree representation` of the set of valuations $\{\nu_2(n^2+1)\}$. The process creates a collections of levels, formed by vertices with a set of indices associated to them, denoted by $\mathcal{I}(v)$.

A vertex v is split to a next level below if the set of values $\{\nu_2(n^2+1)\}$, for indices $n \in \mathcal{I}(v)$, does not reduce to a singleton; that is, there are indices $n_1,\ n_2 \in \mathcal{I}(v)$, such that $\nu_2(n_1^2+1) \neq \nu_2(n_2^2+1)$. In the case a vertex splits, then the two new descendants are at one level higher and if $\mathcal{I}(v)$ has the form $\{n \equiv j \bmod 2^a\}$, then the index sets of the descendants are $\{n \equiv j \bmod 2^{a+1}\}$ (for the left-one) and $\{n \equiv j+2^a \bmod 2^{a+1}\}$ (for the right-one). A vertex which does not split is called `terminal`. Starting with the root of the tree, all the vertices that split from the k^{th} level form the $(k+1)^{st}$ level.

The process begins with the root v_0 with $\mathcal{I}(v_0)=\mathbb{N}$. The data $\nu_2(1^2+1)=1$ and $\nu_2(2^2+1)=0$, shows that v_0 splits. The vertices at the second level are $\{n \equiv 0 \bmod 2\}$ (for the left-one) and $\{n \equiv 1 \bmod 2\}$ (for the right-one). Now consider the vertex v_1 with $\mathcal{I}(v_1) = \{n \equiv 0 \bmod 2\}$ and v_2 with $\mathcal{I}(v_2) = \{n \equiv 1 \bmod 2\}$. For each $n \in \mathcal{I}(v_1)$, $n^2+1 = (2m)^2+1 \equiv 1 \bmod 2$ and $\{\nu_2(n^2+1)\}$ reduces to $\{0\}$, showing that v_1 is a terminal vertex, with assigned value 0. Similarly, for the vertex v_2, with $\mathcal{I}(v_2) = \{n \equiv 1 \bmod 2\}$, one sees directly that $\{\nu_2(n^2+1)\} \equiv \{1\}$, showing that v_2 is also terminal, with assigned value 1. Therefore, the set of valuations $\{\nu_2(n^2+1)\}$ is represented by a finite tree, containing two levels.

Proposition 2.2. The 2-adic valuation of n^2+2 is given by

$$\nu_2(n^2+2) = \begin{cases} 1 & \text{if } n \equiv 0 \bmod 2 \\ 0 & \text{if } n \equiv 1 \bmod 2. \end{cases} \tag{2.2}$$

PROOF. If n is odd, then n^2+2 is also odd and then $\nu_2(n^2+2)=0$. On the other hand, if n is even, say $n=2m$, then $n^2+2=2(2m^2+1)$, twice an odd number and it follows that $\nu_2(n^2+2)=1$. $\square$

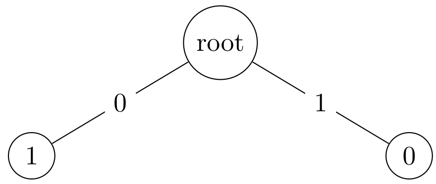

FIGURE 2. The complete tree for $\nu_2(n^2+2)$

Proposition 2.3. The 2-adic valuation of n^2+3 is given by

$$\nu_2(n^2+3)=\begin{cases}0 & \text{if } n\equiv 0 \bmod 2\\ 2 & \text{if } n\equiv 1 \bmod 2.\end{cases} \tag{2.3}$$

PROOF. For n even, n^2+3 is odd, so $\nu_2(n^2+3)=0$. On the other hand, if $n=2m+1$, then $n^2+3=4\,([m(m+1)]+1)$ and this is 4 times an odd number. $\square$

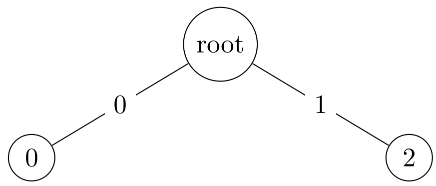

FIGURE 3. The complete tree for $\nu_2(n^2+3)$

The next result gives the valuations of n^2+4. The proof is similar to the one given in the previous examples, so it is omitted.

Proposition 2.4. The 2-adic valuation of n^2+4 is given by

$$\nu_2(n^2+4)=\begin{cases}0 & \text{if } n\equiv 1 \bmod 2\\ 2 & \text{if } n\equiv 0 \bmod 4\\ 3 & \text{if } n\equiv 2 \bmod 4.\end{cases} \tag{2.4}$$

In the tree, the presence of a vertex v inside a square and without a numerical label, as in the second level of Figure 5, indicates that the valuation of n^2+4 for indices n in $\mathcal{I}(v)$ is not constant, and so the vertex must be split to the next level.

The reader is invited to draw trees for the sequences $\{\nu_2(n^2+5)\}$ and $\{\nu_2(n^2+6)\}$.

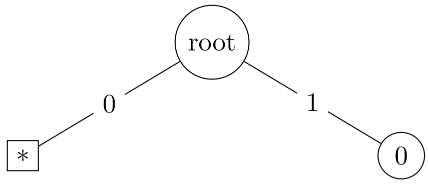

FIGURE 4. The first level for the tree of $\nu_2(n^2+4)$

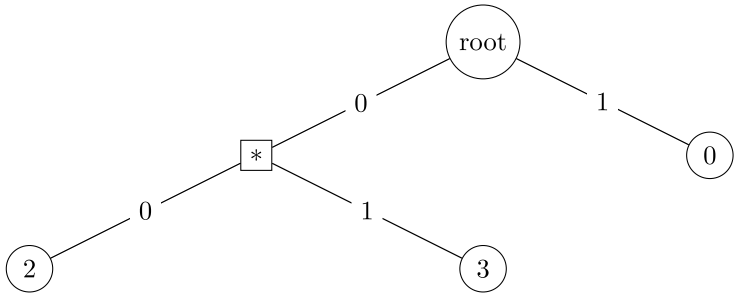

FIGURE 5. The complete tree for $\nu_2(n^2+4)$

3. The labeling of the classes

Given a polynomial $f(x)$ with integer coefficients, the sequence $\{\nu_2(f(n)) : n \in \mathbb{N}\}$ has been described via a tree. This is called the *valuation tree attached to* f. The vertices correspond to some selected classes

$$C_{m,j} = \{2^m i + j : i \in \mathbb{N}\}, \tag{3.1}$$

starting with the root vertex v_0 for $C_{0,0} = \mathbb{N}$. The procedure to select the classes is explained below in the example $f(x) = x^2 + 16$. Some notation for the vertices of the tree is introduced next.

Definition 3.1. A residue class $C_{m,j}$ is called *terminal* for the tree attached to f, if the valuation $\nu_2(f(2^m i + j))$ is independent of the index $i \in \mathbb{N}$. Otherwise it is called *non-terminal*. The same terminology is given to vertices. In the tree, terminal vertices are marked by their constant valuation and non-terminal vertices are marked with a star.

Example 3.2. The construction of the tree for $\nu_2(n^2+16)$ starts with the fact that $\nu_2(1^2+16) = 0$ and $\nu_2(2^2+16) = 2 \neq 0$, showing that the root node v_0 is non-terminal. This node is split into two vertices that form the first level. These correspond to $C_{1,0} = \{2i : i \in \mathbb{N}\}$ and $C_{1,1} = \{2i+1 : i \in \mathbb{N}\}$. For the class $C_{1,0}$, the valuation

$$\nu_2((2i)^2+16) = 2 + \nu_2(i^2+4)$$

depends on i, so $C_{1,0}$ is non-terminal. For the class $C_{1,1}$,

$$\nu_2((2i+1)^2+16) = \nu_2(4i^2+4i+17) = 0,$$

showing that $C_{1,1}$ is a terminal class with valuation 0. Figure 6 shows the root and the first level of the tree associated to $\nu_2(n^2+16)$.

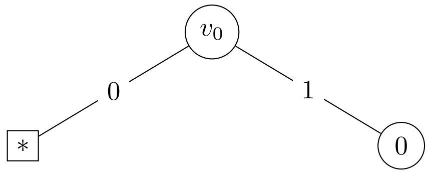

FIGURE 6. The root and the first level of the tree for $\nu_2(n^2+16)$

The class $C_{1,0}$ is now split into $C_{2,0}=\{4i : i\in\mathbb{N}\}$ and $C_{2,2}=\{4i+2 : i\in\mathbb{N}\}$. These two classes form the second level. For $C_{2,0}$, the valuation

$$\nu_2((4i)^2+16)=4+\nu_2(i^2+1)$$

shows that this class is non-terminal. In the class $C_{2,2}$,

$$\nu_2((4i+2)^2+16)=\nu_2(16i^2+16i+20)=2+\nu_2(4i^2+4i+5)=2.$$

Therefore $C_{2,2}$ is a terminal class with valuation 2.

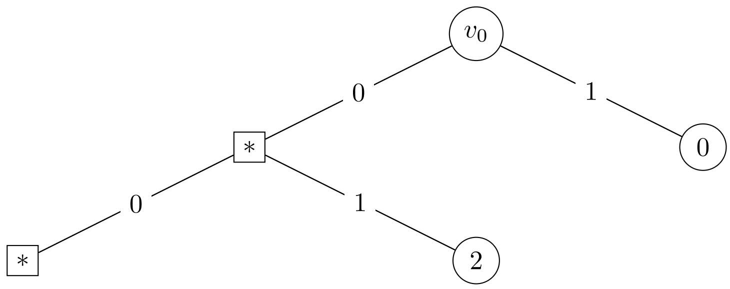

FIGURE 7. Two levels of the tree for $\nu_2(n^2+16)$

The third level contains the two classes $C_{3,0}$ and $C_{3,4}$ descending from $C_{2,0}$. The first class is terminal with valuation 4, since

$$\nu_2((8i)^2+16)=4+\nu_2(4i^2+1)=4.$$

The second class is also terminal, with valuation 5, since

$$\nu_2((8i+4)^2+16)=\nu_2(64i^2+64i+32)=5+\nu_2(2i^2+2i+1)=5.$$

Therefore, every class in the third level is terminal and the tree is complete.

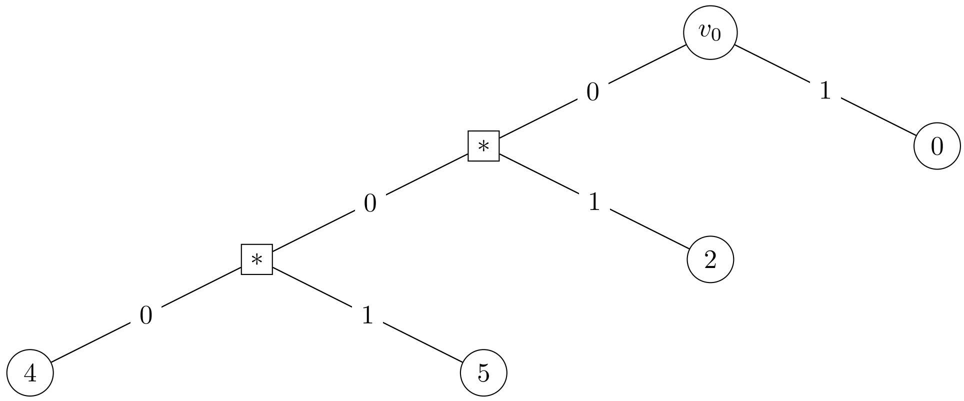

FIGURE 8. The complete tree for $\nu_2(n^2+16)$

4. The 2-adic tree valuation of n^2+7

It is apriori surprising that the valuations of $\nu_2(n^2+7)$ present a more erratic behavior than the cases considered before. The goal of this section is to describe the set $\{\nu_2(n^2+7)\}$ via its valuation tree.

Construction of the tree. The values $\nu_2(1^2+7) = 3 \neq \nu_2(2^2+7) = 0$, show that the root vertex v_0 has to be split into two classes to form the first level:

$$C_{1,0} = \{2n : n \in \mathbb{N}\} \text{ and } C_{1,1} = \{2n+1 : n \in \mathbb{N}\}. \tag{4.1}$$

It is easy to check that the class $C_{1,0}$ is terminal since $\nu_2((2n)^2+7) = 0$. Figure 9 shows the root vertex and the first level of the tree for $\nu_2(n^2+7)$.

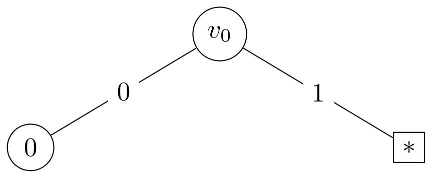

FIGURE 9. The root and the first level of the tree for $\nu_2(n^2+7)$

Lemma 4.1. Assume $n \equiv 1 \bmod 2$, so that $n \in C_{1,1}$, then $\nu_2(n^2+7) \geqslant 3$.

PROOF. Write $n = 2n_1+1$, then $n^2+7 = 4(n_1^2+n_1+2) = 8\left(\frac{n_1(n_1+1)}{2}+1\right)$. This gives the result since $n_1(n_1+1)$ is even. $\square$

The class $C_{1,1}$ is non-terminal, since $1, 3 \in C_{1,1}$ and $\nu_2(1^2+7) = 3 \neq \nu_2(3^2+7) = 4$. Therefore the vertex corresponding to $C_{1,1}$ is split into the classes $C_{2,1}$ and $C_{2,3}$, to form the third level. Recall that $C_{2,1} = \{4n+1\}$ and $C_{2,3} = \{4n+3\}$. Neither of these vertices is terminal. For instance, every number in $C_{2,1}$ has valuation at least 3 and

$$(4n + 1)^2 + 7 = 16n^2 + 8n + 8 \equiv 0 \bmod 2^4 \tag{4.2}$$

provided n is odd. A similar argument shows that $C_{2,3}$ is not terminal. The rest of this section is devoted to the proof of the next result. A similar discussion has been presented in [**?**].

Theorem 4.2. Let v be a non-terminating node at the k-th level for the valuation tree of $\nu_2(n^2 + 7)$. Then v splits into two vertices at the $(k + 1)$-level. Exactly one of them terminates, say with constant valuation ν_k. The second one has valuation at least $\nu_k + 1$.

`Proof`. Start with the class $C_{1,1} = \{2n + 1\}$. As described above, this is non-terminal vertex, splitting into the classes $C_{2,1} = \{4n + 1\}$ and $C_{2,3} = \{4n + 3\}$. The class $C_{2,1}$ is non-terminal since $1, 5 \in C_{2,1}$ and $\nu_2(1) = 3$ and $\nu_2(5) = 5$. Similarly $3, 7 \in C_{2,3}$ and $\nu_2(3) = 4$ and $\nu_2(7) = 3$. Figure 10 shows the tree up to the first four levels.

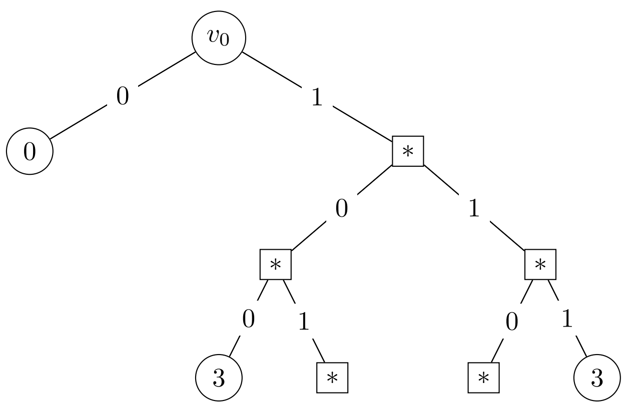

FIGURE 10. First four levels of the tree for $\nu_2(n^2 + 7)$

The vertex $C_{2,1}$ now splits into $C_{3,1} = \{8n + 1\}$ and $C_{3,5} = \{8n + 5\}$. For the class $C_{3,1}$ observe that

$$\nu_2((8n + 1)^2 + 7) = \nu_2(8[(8n^2 + 2n) + 1]) = \nu_2(8) + \nu_2((8n^2 + 2n) + 1) = 3. \tag{4.3}$$

This is shown in the left-most branch in the figure. Therefore $C_{3,1}$ is a terminal class. For the other one, $C_{2,3}$, it is easily seen that it splits into $C_{3,3}$ and $C_{3,7}$. The class $C_{3,7}$ is terminal, with value 3, as shown on the right-most branch and the class $C_{3,3}$ is not, so it continues.

At this point, the part of the tree that has not been described yet, splits into two branches:

$$\mathcal{B}_1 = \{8n + 3\} \quad \text{and} \quad \mathcal{B}_2 = \{8n + 5\}. \tag{4.4}$$

In view of an apparent symmetry, only the case $\mathcal{B}_1$ is discussed in detail. Note 4.4 presents information on this symmetry.

4.1. The 2-valuation of $(8n+3)^2+7$. The class $C_{3,3} = \{8n+3\}$ has valuation at least 3. Now observe first that this class is not terminal because $\nu_2((8\times 1+3)^2+7) = 7$ and $\nu_2((8\times 2+3)^2+7) = 4$. At the next level, the classes have the form $\{8(2n+j)+3\}$, where $j = 0$ or 1. Now

$$[8(2n+j)+3]^2 + 7 = 2^4\left[2\{2(2n+j)^2 + 3n\} + 3j + 1\right], \tag{4.5}$$

shows that $\nu_2\left([8(2n+j)+3]^2+7\right) \geqslant 4$ for any $n \in \mathbb{N}$ and any $j \in \{0, 1\}$. Moreover,

$$[8(2n+j)+3]^2 + 7 \equiv 2^4(3j+1) \bmod 2^5. \tag{4.6}$$

In particular, when $j = 0$, it follows that $[8(2n+j)+3]^2 + 7 \equiv 2^4 \not\equiv 0 \bmod 2^5$. This proves that, for $j = 0$, one has $\nu_2\left([8(2n+j)+3]^2+7\right) = 4$, independent of n. On the other hand, for $j = 1$, the congruence (4.6) shows that $\nu_2\left([8(2n+j)+3]^2+7\right) \geqslant 5$.

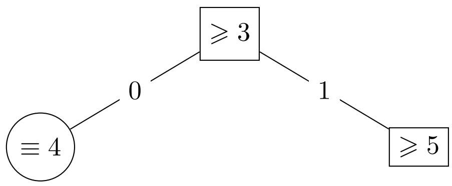

FIGURE 11. The root and the fourth level of the tree for $\nu_2(n^2+7)$

The main step of the proof is given next.

Proposition 4.3. Assume $C_{\alpha,\beta} = \{2^\alpha n + \beta\}$ is a non-terminal class at level α, where the valuation is $\geqslant \alpha+1$. Then, at the next level, this class splits into two classes

$$C_{\alpha+1,\beta} = \{2^\alpha(2n+0)+\beta\} \quad \text{and} \quad C_{\alpha+1,2^\alpha+\beta} = \{2^\alpha(2n+1)+\beta\} \tag{4.7}$$

one of which is terminal with valuation $\equiv \alpha+1$ and the second one is non-terminating with valuation $\geqslant \alpha+2$.

PROOF. The starting point is the class $C_{4,11} = \{2^4 n+11\}$. It has been established that

$$\nu_2((16n+11)^2+7) \geqslant 5. \tag{4.8}$$

Indeed, $(16n+11)^2+7 = 2^5(8n^2+11n+4)$, so (4.8) holds. Moreover, for n odd, the valuation is $\equiv 5$ and for n even, this valuation is $\geqslant 6$. This shows the class $C_{4,11}$ splits into two classes with the stated properties. The conclusion of the proposition is valid at the initial step.

Now take a non-terminal class $C_{\alpha,\beta}$ with valuation $\geqslant \alpha+1$; that is,

$$\nu_2\left((2^\alpha n+\beta)^2+7\right) \geqslant \alpha+1. \tag{4.9}$$

Observe that this implies $(2^\alpha n+\beta)^2+7\equiv 0 \bmod 2^{\alpha+1}$. In particular, this implies $\beta^2+7\equiv 0 \bmod 2^{\alpha+1}$ and so β is odd. Write $\beta^2+7=2^{\alpha+1}\gamma$.

This class splits into the two classes

$$C_{\alpha+1,2^\alpha j+\beta}=\{2^\alpha(2n+j)+\beta:\ n\in\mathbb{N}\}\,. \tag{4.10}$$

Now

$$\begin{aligned}[2^\alpha(2n+j)+\beta]^2+7 &\equiv 2^{\alpha+1}\beta j+\beta^2+7 \bmod 2^{\alpha+2}\\ &\equiv 2^{\alpha+1}(\beta j+\gamma) \bmod 2^{\alpha+2}.\end{aligned}$$

Since β is odd, the congruence $\beta j+\gamma\equiv 1 \bmod 2$ has a unique solution (either 0 or 1) in the variable j. For that value of j,

$$[2^\alpha(2n+j)+\beta]^2+7\not\equiv 0 \bmod 2^{\alpha+2}$$

and this proves $\nu_2\,[2^\alpha(2n+j)+\beta]^2+7]=\alpha+1$, independent of n. For the other value of j,

$$[2^\alpha(2n+j)+\beta]^2+7\equiv 0 \bmod 2^{\alpha+2}$$

and this proves $\nu_2\,[2^\alpha(2n+j)+\beta]^2+7]\geqslant\alpha+2$, independent of n. This concludes the proof. ☐

The proof of Theorem 4.2 is complete.

Note 4.4. It remains to verify the symmetry of the branches

$$\mathcal{B}_1=\{8n+3\}\quad\text{and}\quad\mathcal{B}_2=\{8n+5\}. \tag{4.11}$$

An informal description is presented here.

Every index in the branch $\mathcal{B}_1$ has the form $8n+3$. Then

$$(8n+3)^2+7=2^4(4n^2+3n+1) \tag{4.12}$$

shows that $\nu_2((8n+3)^2+7)\geqslant 4$. To move to the next level in the tree, write $n=2m+j$, with $m\in\mathbb{N}$ and $j\in\{0,1\}$. Then

$$(8n+3)^2+7=[8(2m+j)+3]^2+7\equiv 2^4(3j+1) \bmod 2^5. \tag{4.13}$$

Therefore, for $j=0$, then $(8n+3)^2+7\not\equiv 0 \bmod 2^5$ and thus its valuation is always 4, independently of n. This is the terminal vertex. On the other hand, for $j=1$, it follows that $(8n+3)^2+7\equiv 0 \bmod 2^5$ and its valuation is at least 5. The identity

$$\left([8(2m+j)+3]^2+7\right)-\left([8(2m+1-j)+5]^2+7\right)=2^5(8j-5)(2m+1) \tag{4.14}$$

proves that

$$[8(2m+j)+3]^2+7\equiv[8(2m+1-j)+5]^2+7 \bmod 2^5. \tag{4.15}$$

Therefore the roles of 0, 1 for the index j in the branch $\mathcal{B}_1$ are interchanged in the branch $\mathcal{B}_2$. This phenomena persists at all levels: if there is a movement to the left, in the branch $\mathcal{B}_1$ to advance to the next level; then there is a movement to the right on $\mathcal{B}_2$. This produces the symmetry of the two branches mentioned above. Some details are given below.

The equation

$$x^2+7\equiv 0 \bmod 2^k$$

has exactly 4 non-congruent solutions in the set $\{1, 2, \ldots, 2^k - 1\}$ for $k \geqslant 3$. If r_{k_1} and r_{k_2} are the solutions yielding odd multiples of 2^k, then the four non-congruent solutions to $x^2 + 7 \equiv 0 \bmod 2^{k+1}$ are $r_{k_1} \pm 2^{k-1}$ and $r_{k_2} \pm 2^{k-1}$. Exactly two yield odd multiples of 2^{k+1} and two yield even multiples. We compute the valuation for one of the odd multiples (call it r_k) :

$$\nu_2((2^k t \pm r_k)^2 + 7) = \begin{cases} k & \text{if } c \text{ odd} \\ \geqslant k+1 & \text{if } c \text{ even} \end{cases}$$

and

$$\nu_2((2^k t \pm (r_k + 2^{k-1}))^2 + 7) = \begin{cases} k & \text{if } c \text{ even} \\ \geqslant k+1 & \text{if } c \text{ odd} \end{cases}$$

since r_k must be odd.

Recalling the proposed form of the four branches

$$\begin{aligned} &2^k t + r_k \bmod 2^k, \\ &2^k t + r_k + 2^{k-1} \bmod 2^k, \\ &2^k t + 2^k - r_k \bmod 2^k, \\ &2^k t + 2^k - (r_k + 2^{k-1}) \bmod 2^k, \end{aligned}$$

it follows that either:

(1) The branches $2^k t \pm r_k$ terminate with valuation equal to k and the branches $2^k t + r_k + 2^{k-1}$ and $2^k t + 2^k - (r_k + 2^{k-1})$ continue with valuation greater than or equal to $k+1$, or

(2) The branches $2^k t \pm r_k$ continue with valuation greater than or equal to $k+1$ and the branches $2^k t + r_k + 2^{k-1}$ and $2^k t + 2^k - (r_k + 2^{k-1})$ terminate with valuation equal to k.

This fact explains the symmetry in the 2-adic valuation tree of $x^2 + 7$, since the $2^k t \pm r_k$ branches lie on the *outside* (far left and far right) of each level and the $2^k t \pm (r_k + 2^{k-1})$ branches lie on the *inside.*

5. The range of the valuation $\nu_2(n^2 + 7)$

The proof of Theorem 4.2 shows that, given $k \geqslant 4$, there is a class such that $\nu_2(n^2 + 7) = k$ for all indices n in the class. As a matter of fact, these classes appear one by level. This gives the main part of the next statement.

Theorem 5.1. The range of $\nu_2(n^2 + 7)$ is $\mathbb{N} \setminus \{1, 2\}$.

Small values of the range of $\nu_2(n^2 + 7)$ admit easy characterization. An example is discussed next.

Lemma 5.2. The equation $\nu_2(n^2 + 7) = 0$ is equivalent to $n \equiv 0 \bmod 2$.

Proof. The valuation is 0 if and only if $n^2 + 7 \equiv 1 \bmod 2$. In turn, this is equivalent to $n \equiv 0 \bmod 2$. $\square$

There are two elements missing from this range.

Lemma 5.3. The equations $\nu_2(n^2+7)=1$ or 2 have no solutions.

Proof. Any solution satisfies $n^2+7=2t$. Therefore $n^2+7=2t$ and this implies n is odd, say $n=2m+1$. Then

$$n^2+7=4m^2+4m+8=4(m^2+m+2)=8\left(\frac{m(m+1)}{2}+1\right). \tag{5.1}$$

This implies $\nu_2(n^2+7)\geqslant 3$ and establishes the result. □

The next result continues describing the appearance of small values in the range of $\nu_2(n^2+7)$.

Lemma 5.4. The equation $\nu_2(n^2+7)=3$ is equivalent to $n\equiv\pm1 \bmod 2^3$.

Proof. If $n=8t+1$, then $n^2+7=(8t+1)^2+7=8(8t^2+2t+1)$. Thus, $\nu_2(n^2+7)=3$. The case of $n=8t+7$ is similar. Conversely, if $\nu_2(n^2+7)=3$, then $n^2+7=2^3s$, with s odd. Therefore $n^2\equiv 1 \bmod 8$ and this implies $n\equiv\pm1 \bmod 8$, as claimed. □

The position of indices with valuation 4, 5 are established in a similar manner.

Lemma 5.5. The equation $\nu_2(n^2+7)=4$ is equivalent to $n\equiv\pm3 \bmod 2^4$. Similarly, $\nu_2(n^2+7)=5$ is equivalent to $n\equiv\pm5 \bmod 2^5$.

The previous results suggest a clear pattern.

Problem 5.6. Given $k\in\mathbb{N}$ at least 3, prove there is $\alpha_k\in\mathbb{N}$ such that

$$\nu_2(n^2+7)=k \text{ if and only if } n\equiv\pm\alpha_k \bmod 2^k.$$

A characterization of the sequence $\{\alpha_k\}$ would be desirable.

Finally, the next table shows the smallest index n for which $\nu_2(n^2+7)=i$. This index is called λ_i.

i	3	4	5	6	7	8	9	10	11	12	13	14	15	16
λ_i	1	3	5	21	11	53	75	331	843	1867	3915	8011	181	16565

Figure 12. The minimum index n for which $\nu_2(n^2+7)=i$

Conclusions

The sequence of valuations $\{\nu_2(n^2+a)\}$, for $1\leqslant a\leqslant 6$ have been represented in terms of a finite tree. This corresponds to a closed-form expression for this valuation. The case $a=7$ produces an infinite tree, with two branches. The experimental behavior of the tree makes it unlikely that such a closed-form expression exists for $\nu_2(n^2+7)$.

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