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A warning about symbolic integration

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Abstract. An example of an integral whose symbolic evaluation is given incorrectly in `Mathematica 9.0` is provided. An analytic derivation of the correct value is presented.

1. Introduction

Before the advent of the computer, assembling lists of formulae was undoubtedly tedious. Occasional errors in integral tables, such as formula 3.747.10 in [**1**, p. 434],

$$\int_0^\infty \frac{\tan ax}{x}\, dx = \frac{\pi}{2}, \tag{1.1}$$

are forgivable in light of the astounding scope of that work. The same expression appears as formula 604 in [**5**, p. 331]. In modern times, automated symbolic computation gives hope for error-free integration. The integral (1.1) is returned unevaluated by `Mathematica` version 9.0 and Wolfram Alpha produces the answer "(integral does not converge)". The editors of the latest edition [**2**] are trying to provide a list of entries that is error-free.

Unfortunately, computer methods are also susceptible to error. For example, `Mathematica` version 9.0 evaluates

$$\int_0^\infty \frac{\sinh 2x\, dx}{e^{3x}-1} = \frac{1}{36}(9+2\sqrt{3}\pi + \ln 729) \approx 0.7354. \tag{1.2}$$

The relevant entry 3.545.2 in [**1**, p. 383] is

$$J(a,b) := \int_0^\infty \frac{\sinh ax}{e^{bx}-1}\, dx = \frac{1}{2a} - \frac{\pi}{2b}\cot\frac{\pi a}{b}. \tag{1.3}$$

The same expression appears as formula 702 in [**5**, p. 336]. For $a=2$ and $b=3$, (1.3) gives

$$J(2,3) = \frac{1}{36}(9+2\sqrt{3}\pi) \approx 0.5523. \tag{1.4}$$

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Clearly, either (1.2) or (1.3) is incorrect. In fact, as shown below, (1.2) is wrong. The intention of this note is to warn the reader that, in spite of the tools made available by symbolic languages, errors in evaluating integrals persist.

2. An elementary evaluation of the special case

The change of variables $t = e^{-x}$ gives

$$J(2,3) = \frac{1}{2}\int_0^1 \frac{1+t+t^2+t^3}{1+t+t^2}\,dt = \frac{1}{4} + \frac{1}{2}\int_0^1 \frac{dt}{1+t+t^2}. \tag{2.1}$$

The second integral can be computed by completing the square and a trigonometric change of variables to obtain

$$\int_0^1 \frac{dt}{t^2+t+1} = \int_0^1 \frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}} = \frac{4}{3}\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{ds}{s^2+1} \tag{2.2}$$

and this yields the value in (1.4).

3. A generalization

The proof of the formula (1.3) begins with the change of variables $t = px$ to reduce the number of parameters by one. This is not needed, but it is useful in the computation of more complicated integrals. One finds that

$$J(a,b) = \int_0^\infty \frac{\sinh ax}{e^{px}-1}\,dx = \frac{1}{p}\int_0^\infty \frac{\sinh bt}{e^t-1}\,dt \tag{3.1}$$

with $b = a/p$. The second change of variables $u = e^{-t}$ gives

$$J(a,b) = \frac{1}{2b}\int_0^1 \frac{u^{-b}-u^b}{1-u}\,du. \tag{3.2}$$

This form of $J(a,b)$ is now evaluated using elementary properties of the *digamma function*

$$\psi(a) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{3.3}$$

where $\Gamma(x)$ is the classical *gamma function*, defined by the eulerian integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt. \tag{3.4}$$

This function usually comes with its companion *beta function* defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \tag{3.5}$$

and linked to Γ by the identity

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}. \tag{3.6}$$

The reader will find information about these functions in [**4**].

An integral representation for the difference of two values of the digamma function is given next. This proof has also been given in [**3**].

Proposition 3.1. Let p, $q \in \mathbb{R}$. Then

$$\int_0^1 \frac{x^{p-1} - x^{q-1}}{1-x}\, dx = \psi(q) - \psi(p). \tag{3.7}$$

Proof. Consider first

$$I(\epsilon) = \int_0^1 x^{p-1}(1-x)^{\epsilon-1}\, dx - \int_0^1 x^{q-1}(1-x)^{\epsilon-1}\, dx, \tag{3.8}$$

that avoids the apparent singularity at $x = 1$. The integral $I(\epsilon)$ can be expressed in terms of the beta function as $I(\epsilon) = B(p,\epsilon) - B(q,\epsilon)$. The relation (3.6) gives

$$I(\epsilon) = \Gamma(\epsilon)\left(\frac{\Gamma(p)}{\Gamma(p+\epsilon)} - \frac{\Gamma(q)}{\Gamma(q+\epsilon)}\right). \tag{3.9}$$

Now use $\Gamma(1+\epsilon) = \epsilon\Gamma(\epsilon)$ to write

$$I(\epsilon) = \Gamma(1+\epsilon)\left(\frac{\Gamma(p)-\Gamma(p+\epsilon)}{\epsilon}\,\frac{1}{\Gamma(p+\epsilon)} - \frac{\Gamma(q)-\Gamma(q+\epsilon)}{\epsilon}\,\frac{1}{\Gamma(q+\epsilon)}\right), \tag{3.10}$$

and obtain (3.7) by letting $\epsilon \to 0$. □

The conclusion is that the generalized integral is given by

$$J(a,b) = \frac{1}{2p}\left(\psi(b+1) - \psi(1-b)\right), \text{ with } b = a/p. \tag{3.11}$$

To simplify this formula, use the identity $\psi(b+1) = \psi(b) + \frac{1}{b}$ that comes from $\Gamma(b+1) = b\Gamma(b)$ via logarithmic differentiation and $\psi(b) - \psi(1-b) = -\pi\cot\pi b$ obtained from $\Gamma(b)\Gamma(1-b) = \dfrac{\pi}{\sin\pi b}$. The value in (1.3) follows from here.

It is curious that `Mathematica` evaluates the generalization $J(a,b)$ correctly.

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