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Quadratic and Quartic Integrals Using the Method of Brackets

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Abstract. The method of brackets is used to evaluate quadratic and quartic type integrals appearing in the classical table of integrals by Gradshteyn and Ryzhik in terms of hypergeometric functions. Some generalizations are also presented.

1. Introduction

The method of brackets, introduced in [**15**], is a method for the evaluation of definite integrals (mostly over the half-line). A variety of entries in the classical table of integral by Gradshteyn and Ryzhik [**16**] have been evaluated in [**10**] by this method. Many other types of integrals have appeared recently in the literature. For instance, Coffey [**5**] considers integrals of Russel type generalizing [**18**] and [**1**], Bravo et al. [**4**] studied integrals of Frullani type and Gonzalez et al [**9**] evaluates problems coming from the moments of the hydrogen atom. Many other examples appear in [**11**] and [**13**].

Many of the integrals considered by previous authors have their origin in the evaluation of Feynman diagrams [**6, 14, 15, 8, 12, 17, 19, 21**]. The results presented here provide new evaluations of some entries in the integrals appearing in Gradshteyn and Ryzhik [**16**]. The integrands are of quadratic or quartic type, which have also been considered elsewhere. For instance, the basic integral that we consider has been evaluated only for $n \in \mathbb{N}$:

$$\int_0^\infty \frac{dx}{(ax^2+2bx+c)^n} = \frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial c^{n-1}}\Big[\frac{1}{\sqrt{ac-b^2}}\cot^{-1}\frac{b}{\sqrt{ac-b^2}}\Big], \quad (a>0, ac>b^2) \tag{1.1}$$

Here we present an evaluation for arbitrary n in terms of hypergeometric functions.

The method of brackets has been partly inspired by the negative dimensional integration method that arose in elementary particle physics applications due to Halliday and Ricotta [**17**] and was further used by Suzuki [**20, 21, 22**].

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The paper contains some elementary examples, given to illustrate the method. These include a generalization of the Gaussian integral and an example of a path integral from Feynman and Hibbs [**7**]. The next section evaluates some quadratic type integrals and their generalizations. In the final section we evaluate a quartic integral and then conclude the section with the evaluation of a generalized quartic integral.

2. Ramanujan's Master Theorem and the Method of Brackets

At this point it is important to recall the Ramanujan's master theorem which forms the basis for the method of brackets. This result states that if a complex-valued function has an expansion of the form

$$f(x) = \sum_{k=0}^{\infty} \frac{\phi(k)}{k!}(-x)^k,$$

then the Mellin transform of f is given by

$$\int_0^{\infty} x^{s-1} f(x)dx = \Gamma(s)\phi(-s), \tag{2.1}$$

with $\Gamma(s)$ the classical gamma function.

The next step is to recall the basic rules of the method of brackets. The paper then presents a variety of integral evaluations using this procedure.

Definition 2.1. The method makes the use of brackets which is defined as

$$\langle a \rangle = \int_0^{\infty} x^{a-1} dx. \tag{2.2}$$

The bracket itself is a divergent integral, but when it appears inside a summation, then it acts like a delta function. The formal rules for operating with these brackets are described as follows.

Rules 2.2. For any function $f(x)$, its power series is written in the form

$$f(x) = \sum_{n=0}^{\infty} \phi_n a_n x^{\alpha n+\beta-1}. \tag{2.3}$$

The symbol

$$\phi_n := \frac{(-1)^n}{\Gamma(n+1)}$$

will be called the `indicator` of n. The symbol ϕ_n is used to keep track of the indices appearing in the bracket sums.

Rules 2.3. For any $\alpha \in \mathbb{C}$, the expression

$$G = (a_1 + a_2 + \cdots + a_r)^{\alpha}, \tag{2.4}$$

is given by a bracket series

$$G = \sum_{m_1,\cdots,m_r} \phi_{1,2,\cdots,r}\, a_1^{m_1} \cdots a_r^{m_r} \frac{\langle -\alpha + m_1 + \cdots + m_r \rangle}{\Gamma(-\alpha)}. \tag{2.5}$$

The notation above is

$$\phi_{1,2,\cdots,r} \equiv \phi_{m_1}\phi_{m_2}\cdots\phi_{m_r} \quad \text{and} \quad \sum_{m_1,\cdots,m_r} \equiv \sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}.$$

PROOF. Start with the expression

$$\frac{\Gamma(n)}{k^n} = \int_0^\infty e^{-kx}\,x^{n-1}\,dx$$

and substitute $k = (a_1 + a_2 + \cdots + a_r)$ to obtain

$$\begin{aligned}\frac{\Gamma(n)}{(a_1+a_2+\cdots+a_r)^n} &= \int_0^\infty e^{-(a_1+a_2+\cdots+a_r)x}\,x^{n-1}\,dx,\\ &= \int_0^\infty e^{-a_1x}e^{-a_2x}\cdots e^{-a_rx}\,x^{n-1}\,dx.\end{aligned}$$

Expand each exponential term in power series to produce

$$\frac{\Gamma(n)}{(a_1+a_2+\cdots+a_r)^n} = \sum_{n_1,n_2\cdots n_r}\phi_{1,2\cdots r}(a_1)^{n_1}(a_2)^{n_2}\cdots(a_r)^{n_r}\int_0^\infty x^{a_1+a_2+\cdots+a_r+n-1}\,dx,$$

and integrating over x yields the result. □

The next rule assigns the value of a one-dimensional bracket series with one indicator.

Rules 2.4. The bracket series

$$\sum_n \phi_n f(n)\langle an+b\rangle, \tag{2.6}$$

is assigned the value

$$\frac{1}{a}f(n^*)\Gamma(-n^*). \tag{2.7}$$

where n^* solves the equation $an+b=0$. This definition extends to higher dimensional bracket series. For example, a two-dimensional bracket series

$$\sum_{n_1,n_2}\phi_{n_1,n_2}f(n_1,n_2)\langle a_{11}n_1+a_{12}n_2+c_1\rangle\langle a_{21}n_1+a_{22}n_2+c_2\rangle, \tag{2.8}$$

is assigned the value

$$\frac{1}{|a_{11}a_{22}-a_{12}a_{21}|}f(n_1^*,n_2^*)\Gamma(-n_1^*)\Gamma(-n_2^*). \tag{2.9}$$

where n_1^*, n_2^* is the unique solution to the linear system

$$\begin{aligned} a_{11}n_1+a_{12}n_2+c_1 &= 0,\\ a_{21}n_1+a_{22}n_2+c_2 &= 0, \end{aligned} \tag{2.10}$$

obtained by the vanishing of the expressions in the brackets. A similar rule generalized to higher dimensional series is,

$$\sum_{n_1 \cdots n_r} \phi_{1,\cdots,r} f(n_1,\cdots,n_r)\langle a_{11}n_1+\cdots a_{1r}n_r+c_1\rangle \cdots \langle a_{r1}n_1+\cdots a_{rr}n_r+c_r\rangle,$$

is assigned the value

$$\frac{1}{|\det(A)|} f(n_1^*,\cdots,n_r^*)\Gamma(-n_1^*)\cdots f(-n_r^*), \tag{2.11}$$

where A is the matrix of coefficients (a_{ij}) and $\{n_i^*\}$ is the solution of the linear system obtained by the vanishing of the brackets. The value is not defined if the matrix A is not invertible. It could be shown that the above rule is really the Ramanujan's master theorem at work.

Proof: Consider the following bracket series

$$G=\sum_n \phi_n f(n)\langle an+b\rangle = \sum_n \phi_n \int_0^\infty f(n)\, x^{an+b-1}\, dx \tag{2.12}$$

and the change of variables $x^a = t$ yields

$$G=\sum_n \phi_n \int_0^\infty f(n)\,(t)^{n+\frac{b}{a}-\frac{1}{a}} \frac{dt}{a\, t^{1-\frac{1}{a}}}$$

which simplifies to

$$G=\sum_n \phi_n \int_0^\infty f(n) t^{n+\frac{b}{a}-1}\frac{dt}{a}.$$

The Ramanujan's master theorem now gives

$$G=\frac{1}{a} f\left(-\frac{b}{a}\right)\Gamma\left(\frac{b}{a}\right). \tag{2.13}$$

which is rule 2.4 for the one dimensional case. We can similarly proof the higher dimensional cases. Above proof also shows the bracket indeed acts like a delta function when used inside a summation.

Note. In the case where the assignment leaves free parameters, any divergent series in these parameters are discarded. In case several choices of free parameters are available, the series that converge in a common region are added to contribute to the integral.

Notation. The value of the integral is denoted by $I_{1,\cdots,n}$ where $i_1, i_2, \cdots, i_n$ are the free variables contributing to the solution.

3. Examples

The first few examples already appear in the literature. This has a pedagogical value, it illustrates how the method of brackets is applied.

Example 3.1. `Generalised Gaussian Integral`. This is the integral

$$I = \int_0^\infty e^{-x^p}\, dx. \tag{3.1}$$

To evaluate this entry by the method of brackets, expand the function appearing inside the integral, to produce

$$e^{-x^p} = \sum_{n=0}^{\infty} \phi_n (x^p)^n,$$

so that (3.1) becomes

$$\begin{aligned} I &= \sum_{n=0}^{\infty} \phi_n \int_0^\infty x^{pn+1-1} dx \\ &= \sum_{n=0}^{\infty} \phi_n \langle pn+1 \rangle. \end{aligned} \tag{3.2}$$

The value of the bracket series is given by computing parameter n from the vanishing of the bracket:

$$pn + 1 = 0. \tag{3.3}$$

This gives

$$n = -\frac{1}{p}. \tag{3.4}$$

The value of the integral is then given by using Rule 3:

$$I = \frac{1}{p}\Gamma\left(\frac{1}{p}\right).$$

For special case $p = 2$, one obtains the familiar Gaussian integral.

Example 3.2. `An integral from Feynman and Hibbs`. The integral considered here is taken from the appendix given in the book of Feynman and Hibbs [**7**]. The integral is complex and difficult to evaluate using the conventional methods. Here we present an evaluation using the method of brackets.

The integral is

$$I = \int_0^\infty e^{ia/x^2 + ibx^2} dx. \tag{3.5}$$

The evaluation begins with the expansion of the functions appearing in the integral

$$\begin{aligned} e^{ia/x^2} &= \sum_{n1} \phi_{n_1} (-ia)^{n_1} (x)^{-2n_1}, \\ e^{ibx^2} &= \sum_{n2} \phi_{n_2} (-ib)^{n_2} (x)^{2n_2}. \end{aligned}$$

Replacing this in (3.5) gives

$$\begin{aligned} I &= \sum_{n_1,n_2} \phi_{1,2} \int_0^\infty (-ia)^{n_1} (-ib)^{n_2} x^{2n_2-2n_1+1-1}\, dx, \\ &= \sum_{n_1,n_2} \phi_{1,2} (-ia)^{n_1} (-ib)^{n_2} \langle 2n_2 - 2n_1 + 1 \rangle. \end{aligned} \tag{3.6}$$

The evaluation of this bracket series, by solving a system of equations coming from the vanishing of brackets, yields a 2×2 system of rank 1. The solutions are given by choosing n_1 or n_2 as the free variable.

1) n_1 **as the free variable**. Then

$$n_2^* = \frac{2n_1 - 1}{2}.$$

and rule 2.4 produces

$$I_1 = \sum_{n_1} \phi_1 (-ia)^{n_1} (-ib)^{n_2^*} \frac{\Gamma(-n_2^*)}{2}.$$

Substituting the value of n_2^* and summing up the resulting series in n_1 produces the solution

$$I_1 = \frac{(-1)^{1/4} \sqrt{\pi} \cos(2\sqrt{ab})}{2\sqrt{b}}. \tag{3.7}$$

2) n_2 **as the free variable** now yields

$$n_1^* = \frac{2n_2 + 1}{2},$$

and rule 2.4 now gives

$$I_1 = \sum_{n_2} \phi_2 (-ia)^{n_1^*} (-ib)^{n_2} \frac{\Gamma(-n_1^*)}{2}.$$

Substituting the values and summing up the series over n_2 now produces

$$I_2 = \frac{i(-1)^{1/4} \sqrt{\pi} \sin(2\sqrt{ab})}{2\sqrt{b}}. \tag{3.8}$$

Since both the solutions have the same region of convergence, they full result is $I_1 + I_2$, which reduces to

$$I = I_1 + I_2 = \frac{(-1)^{1/4} e^{2i\sqrt{ab}} \sqrt{\pi}}{2\sqrt{b}}. \tag{3.9}$$

This is the final result.

4. Quadratic Integrals

This section contains the evaluation of some examples of `quadratic type`. These are integrals taken from [**16**] where the integrand contains the factor $ax^2 + 2bx + c$.

4.1. Entry 3.252.1**.** The first integral considered here is

$$I = \int_0^\infty \frac{dx}{(ax^2 + 2bx + c)^n}. \tag{4.1}$$

The value of the integral, as given in Gradshteyn and Ryzhik [**16**], is

$$\int_0^\infty \frac{dx}{(ax^2 + 2bx + c)^n} = \frac{(-1)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial c^{n-1}} \left[\frac{1}{\sqrt{ac - b^2}} \cot^{-1} \frac{b}{\sqrt{ac - b^2}} \right], \tag{4.2}$$

with $a > 0, ac > b^2$ and $n \in \mathbb{N}$. A proof of this formula, for $n \in \mathbb{N}$, is given in [**2**]. The solution presented here is valid for $n \notin \mathbb{N}$, restricted only by convergence conditions.

To use the method of brackets, start by expanding the denominator as a bracket series as

$$(ax^2 + bx + c)^{-n} = \sum_{n_1, n_2, n_3} \phi_{1,2,3} \frac{(ax^2)^{n_1} (2bx)^{n_2} (c)^{n_3} \langle n + n_1 + n_2 + n_3 \rangle}{\Gamma(n)}. \tag{4.3}$$

Replace this expansion in the (4.1) and integrating over x produces the following bracket series, with three indices and two brackets:

$$I = \sum_{n_1, n_2, n_3} \phi_{1,2,3} \frac{(a)^{n_1} (2b)^{n_2} (c)^{n_3} \langle n + n_1 + n_2 + n_3 \rangle \langle 2n_1 + n_2 + 1 \rangle}{\Gamma(n)}.$$

The method now yields the following system of equations

$$\begin{aligned} n + n_1 + n_2 + n_3 &= 0, \\ 2n_1 + n_2 + 1 &= 0. \end{aligned}$$

There are three variables and two equation so there are 3 different solutions, taking one free variable each time.

1) n_2 **as the free variable**. The solution obtained using rule 2.4 is

$$I_2 = \frac{\sqrt{\pi} c^{\frac{1}{2} - n} \Gamma\left(n - \frac{1}{2}\right) {}_1F_0\left(n - \frac{1}{2};; \frac{b^2}{ac}\right)}{2\sqrt{a}\,\Gamma(n)} - \frac{bc^{-n} {}_2F_1\left(1, n; \frac{3}{2}; \frac{b^2}{ac}\right)}{a}, \tag{4.4}$$

with the conditions

$$a \neq 0, \; c \neq 0, \; \left| \frac{b^2}{ac} \right| < 1. \tag{4.5}$$

This evaluation is valid for all the values of $n \in \mathbb{R}^+$, restricted only to the region of convergence of the integral.

2) n_1 **and** n_3 **as the free variables**. The solution obtained using n_1 and n_3 have the same region of convergence hence they both are to be added to get the full answer.

This gives

$$I_{1,3} = \frac{a^{n-1}b^{1-2n}\Gamma(1-n)\Gamma\left(n-\frac{1}{2}\right)\, {}_1F_0\left(n-\frac{1}{2};;\frac{ac}{b^2}\right)}{2\sqrt{\pi}} + \frac{c^{1-n}\Gamma(n-1)\, {}_2F_1\left(\frac{1}{2},1;2-n;\frac{ac}{b^2}\right)}{2b\Gamma(n)},$$

for $a \neq 0, c \neq 0, b \neq 0, |ac/b^2| < 1$. Observe that the solution above is also valid when $n \notin \mathbb{N}$.

NOTE 4.1. Now (4.2) and (4.4) produces the identity

$$\frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial c^{n-1}}\Big[\frac{1}{\sqrt{ac-b^2}}\cot^{-1}\frac{b}{\sqrt{ac-b^2}}\Big] = \frac{\sqrt{\pi}c^{\frac{1}{2}-n}\Gamma\left(n-\frac{1}{2}\right)\, {}_1F_0\left(n-\frac{1}{2};;\frac{b^2}{ac}\right)}{2\sqrt{a}\Gamma(n)} - \frac{bc^{-n}\, {}_2F_1\left(1,n;\frac{3}{2};\frac{b^2}{ac}\right)}{a}. \tag{4.6}$$

4.2. Entry 3.252.3. We now consider the integral

$$I = \int_0^\infty \frac{dx}{(ax^2+2bx+c)^{n+\frac{3}{2}}}, \tag{4.7}$$

which appears as entry 3.252.3 in Gradshteyn and Ryzhik [**16**]:

$$\int_0^\infty \frac{dx}{(ax^2+2bx+c)^{n+\frac{3}{2}}} = \frac{(-2)^n}{(2n+1)!!}\frac{\partial^n}{\partial c^n}\Big(\frac{1}{\sqrt{c}(\sqrt{ac}+b)}\Big), \tag{4.8}$$

for $a \geqslant 0, \quad c > 0, \quad b > -\sqrt{ac}$.

In order to evaluate this example by the method of brackets, expand the denominator as a bracket series:

$$(ax^2+2bx+c)^{-n-\frac{3}{2}} = \sum_{n_1,n_2,n_3}^{\infty} \phi_{1,2,3} a^{n_1}(2b)^{n_2}c^{n_3}x^{2n_1+n_2}\frac{\langle n+\frac{3}{2}+n_1+n_2+n_3\rangle}{\Gamma(n+\frac{3}{2})}. \tag{4.9}$$

Replacing this expansion in (4.7), and integrating over x produces the bracket series:

$$I = \sum_{n_1,n_2,n_3}^{\infty} \phi_{1,2,3} a^{n_1}(2b)^{n_2}c^{n_3}\frac{\langle 2n_1+n_2+1\rangle\langle n+\frac{3}{2}+n_1+n_2+n_3\rangle}{\Gamma(n+\frac{3}{2})}. \tag{4.10}$$

The usual procedure now gives the linear system

$$2n_1+n_2+1 = 0,$$
$$n+\frac{3}{2}+n_1+n_2+n_3 = 0.$$

This is a system of rank 2, yielding three expressions for the integral.

1) n_2 **as the free variable**. This yields

(4.11)

$$I_2 = \frac{c^{-n}}{2a\left(ac-b^2\right)}\left(\frac{\sqrt{\pi}a^{3/2}\Gamma(n+1)\,{}_1F_0\left(n;;\frac{b^2}{ac}\right)}{\Gamma\left(n+\frac{3}{2}\right)}+\frac{2\left(b^3-abc\right)\,{}_2F_1\left(1,n+\frac{3}{2};\frac{3}{2};\frac{b^2}{ac}\right)}{c^{3/2}}\right),$$

for $c>0,\ b^2<|ac|$. This solution is valid for all $n\in\mathbb{R}^+$.

2) n_1 **and** n_3 **as the free variable.** The solution obtained using n_1 and n_3 have the same region of convergence hence they both are to be added to get the full answer. The final solution is:

(4.12)

$$I_{1,3} = \frac{1}{2}\Gamma\left(-n-\frac{1}{2}\right)\left(\frac{a^{n+\frac{1}{2}}b^{-2n}\Gamma(n+1){}_1F_0\left(n;;\frac{ac}{b^2}\right)}{\sqrt{\pi}\left(b^2-ac\right)}-\frac{c^{-n-\frac{1}{2}}\,{}_2\tilde{F}_1\left(\frac{1}{2},1;\frac{1}{2}-n;\frac{ac}{b^2}\right)}{b}\right),$$

for $b>0,\ |ac|<b^2$. Here ${}_2\tilde{F}_1$ is the regularized Hypergeometric ${}_2F_1$, defined by ${}_2\tilde{F}_1(a,b;c;z) = {}_2F_1(a,b;c;z)/\Gamma(c)$. This solution is valid for all $n\in\mathbb{R}^+$ except half-integers $\frac{1}{2},\frac{3}{2}$, ...etc.

NOTE 4.2. Using (4.8) and (4.11) gives the relation

$$\frac{(-2)^n}{(2n+1)!!}\frac{\partial^n}{\partial c^n}\Big(\frac{1}{\sqrt{c}(\sqrt{ac}+b)}\Big) = \frac{c^{-n}}{2a\left(ac-b^2\right)}\left(\frac{\sqrt{\pi}a^{3/2}\Gamma(n+1)\,{}_1F_0\left(n;;\frac{b^2}{ac}\right)}{\Gamma\left(n+\frac{3}{2}\right)}\right.$$
$$\left.+\frac{2\left(b^3-abc\right)\,{}_2F_1\left(1,n+\frac{3}{2};\frac{3}{2};\frac{b^2}{ac}\right)}{c^{3/2}}\right). \tag{4.13}$$

4.3. Entry 3.252.4. The integral considered now is entry 3.252.4 in [**16**], defined by

$$I = \int_0^\infty \frac{x\,dx}{(ax^2+2bx+c)^n} \tag{4.14}$$

The value of this integral is given as

(4.15)

$$\int_0^\infty \frac{x\,dx}{(ax^2+2bx+c)^n} = \begin{cases} \frac{(-1)^n}{(n-1)!!}\frac{\partial^{n-2}}{\partial c^{n-2}}\Big(\frac{1}{2(ac-b^2)}-\frac{b}{2(ac-b^2)^{\frac{3}{2}}}\cot^{-1}\big(\frac{b}{\sqrt{ac-b^2}}\big)\Big),\ ac>b^2,\\ \frac{(-1)^n}{(n-1)!!}\frac{\partial^{n-2}}{\partial c^{n-2}}\Big(\frac{1}{2(ac-b^2)}+\frac{b}{4(b^2-ac)^{\frac{3}{2}}}\ln\Big(\frac{b+\sqrt{b^2-ac}}{b-\sqrt{b^2-ac}}\Big)\Big),\ b^2>ac>0,\\ \frac{a^{n-2}}{2(n-1)(2n-1)b^{2n-2}},\ ac=b^2. \end{cases}$$

The procedure is now standard: expand the denominator as a bracket-series:

$$(ax^2+2bx+c)^{-n} = \sum_{n_1,n_2,n_3}^{\infty}\phi_{1,2,3}a^{n_1}(2b)^{n_2}c^{n_3}x^{2n_1+n_2}\frac{\langle n+n_1+n_2+n_3\rangle}{\Gamma(n)}. \tag{4.16}$$

and substitute the expansion in (4.14), integrate over x to produce the bracket series

$$I = \sum_{n_1,n_2,n_3}^{\infty} \phi_{1,2,3} a^{n_1}(2b)^{n_2} c^{n_3} \frac{\langle 2n_1+n_2+2\rangle\langle n+n_1+n_2+n_3\rangle}{\Gamma(n)}. \tag{4.17}$$

This is evaluated from the linear system

$$2n_1+n_2+2=0,$$
$$n+n_1+n_2+n_3=0,$$

which produces

1) n_2 as the free variable

$$I_2 = \frac{c^{1-n}\Gamma(n-1)\,{}_2F_1\left(1,n-1;\frac{1}{2};\frac{b^2}{ac}\right)}{2a\Gamma(n)} - \frac{\sqrt{\pi}bc^{\frac{1}{2}-n}\Gamma\left(n-\frac{1}{2}\right)\,{}_1F_0\left(n-\frac{1}{2};;\frac{b^2}{ac}\right)}{2a^{3/2}\Gamma(n)}, \tag{4.18}$$
$$\left(c>0, b>0, \left|\frac{b^2}{ac}\right|<1\right).$$

with a solution valid for all $n \in \mathbb{R}^+$.

2) n_1 and n_3 as the free variable

$$I_{1,3} = -\frac{\Gamma(1-n)\left(2a^{n-2}b^{4-2n}\Gamma\left(n-\frac{1}{2}\right)\,{}_1F_0\left(n-\frac{1}{2};;\frac{ac}{b^2}\right) - \sqrt{\pi}c^{2-n}\,{}_2\tilde{F}_1\left(1,\frac{3}{2};3-n;\frac{ac}{b^2}\right)\right)}{4\sqrt{\pi}b^2}, \tag{4.19}$$
$$\left(a>0, b>0, \left|\frac{ac}{b^2}\right|<1\right),$$

giving an expression valid for all $n \notin \mathbb{N}$.

The case $ac = b^2$ is treated in the same manner.

Note 4.3. As in the previous examples, this evaluation and (4.15) and (4.19) we can write the following identity

$$\frac{(-1)^n}{(n-1)!!}\frac{\partial^{n-2}}{\partial c^{n-2}}\left(\frac{1}{2(ac-b^2)} - \frac{b}{2(ac-b^2)^{\frac{3}{2}}}\cot^{-1}\left(\frac{b}{\sqrt{ac-b^2}}\right)\right) =$$
$$\frac{c^{1-n}\Gamma(n-1)\,{}_2F_1\left(1,n-1;\frac{1}{2};\frac{b^2}{ac}\right)}{2a\Gamma(n)} - \frac{\sqrt{\pi}bc^{\frac{1}{2}-n}\Gamma\left(n-\frac{1}{2}\right)\,{}_1F_0\left(n-\frac{1}{2};;\frac{b^2}{ac}\right)}{2a^{3/2}\Gamma(n)}.$$

4.4. Generalization. We now present a generalization of the quadratic type integrals in the form

$$I = \int_0^\infty \frac{x^n\,dx}{(ax^2+2bx+c)^m}. \tag{4.20}$$

The evaluation by the method of brackets begins with an expansion of the denominator as a bracket series.

$$(ax^2+2bx+c)^{-m} = \sum_{n_1,n_2,n_3} \phi_{1,2,3} \frac{(ax^2)^{n_1}(2bx)^{n_2}(c)^{n_3}\langle m+n_1+n_2+n_3\rangle}{\Gamma(m)}.$$

Substituting in the (4.20) and integrating over x produces the bracket series:

$$I = \sum_{n_1,n_2,n_3} \phi_{1,2,3} \frac{(a)^{n_1}(2b)^{n_2}(c)^{n_3}\langle m+n_1+n_2+n_3\rangle\langle 2n_1+n_2+n+1\rangle}{\Gamma(m)}.$$

This is evaluated by solving

$$\begin{aligned} m+n_1+n_2+n_3 &= 0,\\ 2n_1+n_2+n+1 &= 0. \end{aligned}$$

This is a system of rank 2. The solutions are expressed in terms of a single free variable:

1) n_2 as the free variable

$$\begin{aligned} I_2 = {} & \frac{a^{-\frac{n}{2}-\frac{1}{2}}\Gamma\left(\frac{n+1}{2}\right)c^{\frac{1}{2}(n-2m)+\frac{1}{2}}\Gamma\left(m-\frac{n}{2}-\frac{1}{2}\right)\,{}_2F_1\left(m-\frac{n}{2}-\frac{1}{2},\frac{n+1}{2};\frac{1}{2};\frac{b^2}{ac}\right)}{2\Gamma(m)} \\ & -\frac{ba^{-\frac{n}{2}-1}\Gamma\left(\frac{n}{2}+1\right)c^{\frac{1}{2}(n-2m)}\Gamma\left(m-\frac{n}{2}\right)\,{}_2F_1\left(m-\frac{n}{2},\frac{n}{2}+1;\frac{3}{2};\frac{b^2}{ac}\right)}{\Gamma(m)}, \end{aligned} \tag{4.21}$$

for $b^2 < |ac|$.

2) n_1 and n_3 as the free variables. The solution corresponding to n_1 and n_3 as the free variables have the same region of convergence and are added together to get the full answer. The final answer after simplification is

$$\begin{aligned} I_{1,3} = {} & \frac{2^{-2m+n+1}a^{m-n-1}\,b^{-2m+n+1}\Gamma(2m-n-1)\Gamma(-m+n+1)\,{}_2F_1\left(m-\frac{n}{2}-\frac{1}{2},m-\frac{n}{2};m-n;\frac{ac}{b^2}\right)}{\Gamma(m)} \\ & +\frac{2^{-n-1}b^{-n-1}\Gamma(n+1)c^{-m+n+1}\Gamma(m-n-1)\,{}_2F_1\left(\frac{n+1}{2},\frac{n+2}{2};-m+n+2;\frac{ac}{b^2}\right)}{\Gamma(m)}, \end{aligned} \tag{4.22}$$

with $|ac| < b^2$. The above evaluation for some general powers n and m is not given in Gradshteyn and Ryzhik. Special values for m and n will reproduce the previous examples.

4.5. Special case $n = 0$. This is given in terms of the free indices.

1) n_2 **as the free variables**

Simplifying we get

$$I_2 = \frac{\sqrt{\pi}c^{\frac{1}{2}-m}\Gamma\left(m-\frac{1}{2}\right)\ {}_1F_0\left(m-\frac{1}{2};;\frac{b^2}{ac}\right)}{2\sqrt{a}\Gamma(m)} - \frac{bc^{-m}\ {}_2F_1\left(1,m;\frac{3}{2};\frac{b^2}{ac}\right)}{a}. \tag{4.23}$$

which is same as (4.4)

2) n_1 **and** n_3 **as the free variables**. Simplifying the answer yields

(4.24)

$$I_{1,3} = \frac{2^{1-2m}a^{m-1}b^{1-2m}\Gamma(1-m)\Gamma(2m-1)_1F_0\left(m-\frac{1}{2};;\frac{ac}{b^2}\right) + \frac{c^{1-m}\Gamma(m-1)\ {}_2F_1\left(\frac{1}{2},1;2-m;\frac{ac}{b^2}\right)}{2b}}{\Gamma(m)}.$$

The Legendre duplication formula on $\Gamma(2m-1)$ gives (4.6).

5. Quartic Integrals

The quartic integral

$$I = \int_0^\infty \frac{dx}{(ax^4+2bx^2+c)^m}, \tag{5.1}$$

can be scaled to the special case $a = c = 1$. This has been investigated by Amdeberhan et al. [**3**]. The above integral has been evaluated by the method of brackets for the special case of $a = 1$ and $c = 1$ in [**11**]. Here we evaluate it for some general a and c.

First expand the denominator as a bracket series

$$(ax^4+2bx^2+c)^{-m} = \sum_{n_1,n_2,n_3} \phi_{1,2,3}\frac{(ax^4)^{n_1}(2bx^2)^{n_2}(c)^{n_3}\langle m+n_1+n_2+n_3\rangle}{\Gamma(m)}.$$

Substituting the above expansion in (5.1) and integrating over x yields the bracket series

$$I = \sum_{n_1,n_2,n_3} \phi_{1,2,3}\frac{(a)^{n_1}(2b)^{n_2}(c)^{n_3}\langle m+n_1+n_2+n_3\rangle\langle 4n_1+2n_2+1\rangle}{\Gamma(m)}.$$

This is solved in terms of the linear system

$$m+n_1+n_2+n_3 = 0,$$
$$4n_1+2n_2+1 = 0.$$

There are three variables and two equation so there are 3 different solution by taking one free variable each time. The following are the solutions obtained:

1) n_2 as the free variable

$$I_2 = \frac{\Gamma\left(\frac{1}{4}\right) c^{\frac{1}{4}-m} \Gamma\left(\frac{1}{4}(4m-1)\right) {}_2F_1\left(\frac{1}{4}, m-\frac{1}{4}; \frac{1}{2}; \frac{b^2}{ac}\right)}{4\sqrt[4]{a}\Gamma(m)} - \frac{b\Gamma\left(\frac{3}{4}\right) c^{-m-\frac{1}{4}} \Gamma\left(\frac{1}{4}(4m+1)\right) {}_2F_1\left(\frac{3}{4}, m+\frac{1}{4}; \frac{3}{2}; \frac{b^2}{ac}\right)}{2a^{3/4}\Gamma(m)},$$

for $b^2 < |ac|$.

2) n_1 and n_3 as the free variables. The solution corresponding to n_1 and n_3 as the free variables have the same region of convergence and are added together to get the full answer. The final answer after simplification is

$$I_{1,3} = \frac{2^{-2m-\frac{1}{2}} a^{m-\frac{1}{2}} b^{\frac{1}{2}-2m} \Gamma\left(\frac{1}{2}(1-2m)\right) \Gamma\left(\frac{1}{2}(4m-1)\right) {}_2F_1\left(m-\frac{1}{4}, m+\frac{1}{4}; m+\frac{1}{2}; \frac{ac}{b^2}\right)}{\Gamma(m)} + \frac{\sqrt{\frac{\pi}{2}} c^{\frac{1}{2}-m} \Gamma\left(\frac{1}{2}(2m-1)\right) {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; \frac{3}{2}-m; \frac{ac}{b^2}\right)}{2\sqrt{b}\Gamma(m)},$$

for $|ac| < b^2$.

5.1. Generalization. Next we evaluate a more general quartic integral given by

$$I = \int_0^\infty \frac{dx}{x^n(ax^4+2bx^2+c)^m}. \tag{5.2}$$

The above integral occurs as a recursion relation in entry 2.161.6 of the table [**16**].

The evaluation method is now standard. First expand the denominator as a bracket series

$$(ax^4+2bx^2+c)^{-m} = \sum_{n_1,n_2,n_3} \phi_{1,2,3} \frac{(ax^4)^{n_1}(2bx^2)^{n_2}(c)^{n_3}\langle m+n_1+n_2+n_3\rangle}{\Gamma(m)}.$$

Substituting the above expansion in (5.2) and integrating over x produces the bracket series

$$I = \sum_{n_1,n_2,n_3} \phi_{1,2,3} \frac{(a)^{n_1}(2b)^{n_2}(c)^{n_3}\langle m+n_1+n_2+n_3\rangle\langle 4n_1+2n_2-n+1\rangle}{\Gamma(m)}.$$

This is evaluated by solving the two linear equations

$$m+n_1+n_2+n_3 = 0,$$
$$4n_1+2n_2-n+1 = 0.$$

There are three variables and two equation so there are 3 different solution by taking one free variable each time.

1) n_2 as the free variable

$$I_2 = a^{\frac{n-3}{4}} c^{\frac{1}{4}(-4m-n-1)} \left(\frac{\sqrt{a}\sqrt{c}\,\Gamma\left(\frac{1}{4}-\frac{n}{4}\right)\Gamma\left(m+\frac{n}{4}-\frac{1}{4}\right)\,{}_2F_1\left(\frac{1-n}{4},\frac{1}{4}(4m+n-1);\frac{1}{2};\frac{b^2}{ac}\right)}{4\Gamma(m)} - \frac{b\,\Gamma\left(\frac{3}{4}-\frac{n}{4}\right)\Gamma\left(m+\frac{n}{4}+\frac{1}{4}\right)\,{}_2F_1\left(\frac{3}{4}-\frac{n}{4},m+\frac{n}{4}+\frac{1}{4};\frac{3}{2};\frac{b^2}{ac}\right)}{2\Gamma(m)} \right),$$

for $b^2 < |ac|$.

2) n_1 and n_3 as the free variables. The solution corresponding to n_1 and n_3 as the free variables have the same region of convergence and are added together to get the full answer. The final solution is

$$I_{1,3} = 2^{-\frac{n}{2}-\frac{3}{2}} b^{-\frac{n}{2}-\frac{1}{2}} \left(\frac{2^{1-2m} b^{1-2m} a^{\frac{1}{2}(2m+n-1)}\Gamma\left(\frac{1}{2}(-2m-n+1)\right)\Gamma\left(\frac{1}{2}(4m+n-1)\right)}{\Gamma(m)} \times {}_2F_1\left(m+\frac{n}{4}-\frac{1}{4}, m+\frac{n}{4}+\frac{1}{4}; m+\frac{n}{2}+\frac{1}{2}; \frac{ac}{b^2}\right) + \frac{2^n b^n \Gamma\left(\frac{1-n}{2}\right) c^{\frac{1}{2}(-2m-n+1)}\Gamma\left(\frac{1}{2}(2m+n-1)\right)\,{}_2F_1\left(\frac{1}{4}-\frac{n}{4},\frac{3}{4}-\frac{n}{4};-m-\frac{n}{2}+\frac{3}{2};\frac{ac}{b^2}\right)}{\Gamma(m)} \right),$$

for $|ac| < b^2$.

The special case $n = 0$ gives the previous results.

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