SCIENTIA
Series A: *Mathematical Sciences*, Vol. 29 (2019), 61–63
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

A small trove of functional equations

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Abstract. A new proof is presented for an old algebraic identity which is then used to produce the general functional relation

$$\sum_{k=0}^{n-1} \frac{(m)_k}{k!} g(m,k) + \sum_{k=0}^{m-1} \frac{(n)_k}{k!} g(k,n) = g(0,0),$$

where g is an Euler transform, and a related integral identity. Several examples are given.

1. Introduction

We begin with the expression

$$f_0(m,n,x) := \frac{1}{n}\, {}_2F_1(1, m+n; n+1, x) \tag{1.1}$$

By the hypergeometric linear transformation 7.3.1.(5) of [1]

(1.2)

$$\begin{aligned} f_0(n,m,1-x) &= -\frac{1}{n}\, {}_2F_1(1, m+n; n+1; x) + \frac{\Gamma(m)\Gamma(n)}{x^n\Gamma(m+n)}\, {}_2F_1(m, 1-n; 1-n; x) \\ &= -f_0(m,n,x) + \frac{B(m,n)}{x^n(1-x)^m}. \end{aligned}$$

Next, by equation 7.3.1(123) of [1] we find

$$f_0(m,n,x) = \frac{B(m,n)}{x^n}\left[\frac{1}{(1-x)^m} - \sum_{k=0}^{n-1} \frac{(m)_k}{k!} x^k\right]. \tag{1.3}$$

Consequently,

$$x^n(1-x)^m[f_0(m,n,x) + f_0(n,m,1-x)] = B(m,n) \tag{1.4}$$

2000 *Mathematics Subject Classification.* Primary C3320 Secondary C3305.
Key words and phrases. Euler transform, Hypergeometric function, functional identity.

is independent of x. By (1.3) this becomes

$$S(m,n,x) := (1-x)^m \sum_{k=0}^{n-1} \frac{(m)_k}{k!} x^k + x^n \sum_{k=0}^{m-1} \frac{(n)_k}{k!}(1-x)^k = 1. \tag{1.5}$$

The identity (1.5), which is the subject of an engaging recent historical essay by T. H. Koornwinder and M. J. Schlosser [2] can be traced back to an exchange between Samuel Pepys and Isaac Newton in 1693, and may have even earlier roots.

2. Applications

If (1.5) is multiplied by any function $f(x)$ and integrated over an interval [a,b] one has the formal identity

$$\sum_{k=0}^{n-1} \frac{(m)_k}{k!} F(m,k) + \sum_{k=0}^{m-1} \frac{(n)_k}{k!} F(k,n) = F(0,0) \tag{2.1}$$

with

$$F(m,n) := \int_a^b x^n (1-x)^m f(x) dx. \tag{2.2}$$

There are many interesting cases, especially for hypergeometric functions, as the following two examples indicate.

Example 2.1. From

$$\int_0^1 x^n (1-x)^m \ln x \, dx = B(m+1, n+1)[\psi(n+1) - \psi(m+n+2)] \tag{2.3}$$

equation (2.1) yields

$$\sum_{k=0}^{n-1} \frac{m}{(m+k)(m+k+1)} [\psi(k+1) - \psi(m+k+2)] - \sum_{k=0}^{m-1} \frac{n}{(n+k)(n+k+1)} \psi(n+k+2) = -\frac{m}{m+n}\psi(n+1) - 1. \tag{2.4}$$

Example 2.2. From

$$\int_0^1 x^k (1-x)^n \, {}_2F_1(a,b;c;zx) dx = \frac{k!n!}{(k+n+1)!} \, {}_3F_2(a,b,k+1;c,k+n+2;z) \tag{2.5}$$

we get the extended contiguity relation

$$\sum_{k=0}^{m-1} \frac{n}{(n+k)(n+k+1)} \, {}_3F_2(a,b,n+1;c,n+k+2;z) + \sum_{k=0}^{n-1} \frac{m}{(m+k)(m+k+1)} \, {}_3F_2(a,b,k+1;c,m+k+2;z) = {}_3F_2(a,b,1;c,2;z). \tag{2.6}$$

If (1.4) is multiplied by any function $g(x)$ which we shall assume possesses the reflection property $g(1-x) = g(x)$, then by integrating over [0,1] we find

$$\frac{1}{n}G(m,n) + \frac{1}{m}G(n,m) = B(m,n)G(0,0), \tag{2.7}$$

where

$$G(m,n) = \int_0^1 x^n(1-x)^m \ {}_2F_1(1, m+n; n+1; x)g(x)dx. \tag{2.8}$$

Furthermore, by Carlson's theorem [3, Section 5.81] , m and n need not be positive integers. In particular, for $m = n = \nu$ one has

$$\int_0^1 x^\nu(1-x)^\nu \ {}_2F_1(1, 2\nu; \nu+1; x)g(x)dx = \frac{\nu\Gamma^2(\nu)}{2\Gamma(2\nu)} \int_0^1 g(x)dx. \tag{2.9}$$

A simple consequence of (2.9) (just set $\nu = 1$) is that for any function f

$$\int_0^1 f[x(1-x)]dx = 2\int_0^1 xf[x(1-x)]dx. \tag{2.10}$$

Example 2.3. For $\operatorname{Re}\nu \geqslant 0$

$$\int_0^1 x^\nu(1-x)^\nu \ {}_2F_1(1, 2\nu; 1+\nu; x)\sin\pi x \, dx = \frac{\Gamma(\nu)\Gamma(1+\nu)}{\pi\Gamma(2\nu)} \tag{2.11}$$

A related result is the curious algebraic identity: For $m,\ n = 1, 2, 3, \cdots$ and arbitrary $z > -1/2$, $(0!! = 1)$

$$\sum_{k=0}^{n-1} \binom{m-1+k}{k} \frac{(2m)!!}{\prod_{l=0}^{m}[2(k+z)+2l+1]} + \sum_{k=0}^{m-1} \binom{n-1+k}{k} \frac{(2k)!!}{\prod_{l=0}^{k}[2(n+z)+2l+1]} = \frac{1}{2z+1}. \tag{2.12}$$

References

[1] A.P. Prudnikov, Yu. Brychkov and A. Marichev, *Integrals, Products and Series*, Vol.3 [Gordon and Breach Publishers, NY. 1987]

[2] T. H. Koornwinder and M.J. Schlosser, *On an identity by Chaundy and Bullard. II. More history*, arXiv: 1205.6362v2 [Math.CA] 26 June 2012.

[3] E.C. Titchmarsh, *The Theory of Functions* [Oxford University Press, 1939] Section 5.81.

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Received 18 08 2019 revised 27 08 2019