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The Euclidean remainders

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Abstract. The Euclidean algorithm applied to arbitrary real numbers $r_{-1} > r_0 > 0$ is closely related to the continued fraction expansion of r_{-1}/r_0, but an explicit formula relating the remainders to the digits of the continued fraction is not found in the English language literature. (A German language reference for this is: Oskar Perron: Die Lehre von den Kettenbruechen Band I B.G. Teubner, Stuttgart (1971)). In this note, we give a short and self-contained derivation of an explicit formula for the remainders r_n in terms of continuant polynomials, from which the well-know fact that r_n goes to zero at least as fast as ϕ^{-n} (where ϕ is the golden ratio) follows immediately.

1. Introduction

Given real numbers $r_{-1} > r_0 > 0$, the Euclidean algorithm

$$a_0 = \left\lfloor \frac{r_{-1}}{r_0} \right\rfloor, \qquad r_{i+1} = r_{i-1} - r_i a_i, \qquad a_{i+1} = \left\lfloor \frac{r_i}{r_{i+1}} \right\rfloor, \qquad i \geqslant 0$$

produces a strictly decreasing sequence $0 \leqslant r_{i+1} < r_i$ as long as $r_i > 0$, and it does not require r_{-1} and r_0 to be integers. It is well-known (see for example [**3**]) that r_i decreases slowest if $r_{-1}/r_0 = \phi = (\sqrt{5}+1)/2$ (or the ratio of two consecutive Fibonacci numbers in the integer case). This corresponds to the case that all digits of the continued fraction $[a_0; a_1, \ldots]$ for r_{-1}/r_0 are equal to 1, where $[x_0; x_1, \ldots, x_n]$ is defined recursively by

$$[x_0] = x_0, \qquad [x_0; x_1, \ldots, x_n] = x_0 + \frac{1}{[x_1; x_2, \ldots, x_n]},$$

and (see [**2**, p.40]),

$$\frac{r_{-1}}{r_0} = [a_0; a_1, \ldots, a_{n-1}, t_n], \quad t_n = \frac{r_{n-1}}{r_n}, \quad n \geqslant 1.$$

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2. Continuant polynomials

The *continuant polynomials* $K_n(x_1, x_2, \ldots, x_n)$ are defined by

$$K_0 = 1, \quad K_1(x_1) = x_1,$$

$$K_n(x_1, \ldots, x_n) = x_n K_{n-1}(x_1, \ldots, x_{n-1}) + K_{n-2}(x_1, \ldots, x_{n-2}), \; n \geqslant 2, \tag{2.1}$$

and satisfy the identities ([**1**, p. 303-304])

$$[x_0; x_1, \ldots, x_n] = \frac{K_{n+1}(x_0, x_1, \ldots, x_n)}{K_n(x_1, x_2, \ldots, x_n)}, \tag{2.2}$$

$$\begin{aligned} &K_n(x_0, \ldots, x_{n-1}) K_{n-2}(x_1, \ldots, x_{n-2}) \\ &-K_{n-1}(x_0, \ldots, x_{n-2}) K_{n-1}(x_1, \ldots, x_{n-1}) = (-1)^n. \end{aligned} \tag{2.3}$$

Using the recursive definition (2.1), we can express the last entry of a continued fraction $y = [x_0; x_1, \ldots, x_n]$ as a rational function of y and the other entries, in terms of continuants:

$$x_n = \frac{y K_{n-2}(x_1, \ldots, x_{n-2}) - K_{n-1}(x_0, \ldots, x_{n-2})}{K_n(x_0, \ldots, x_{n-1}) - y K_{n-1}(x_1, \ldots, x_{n-1})}. \tag{2.4}$$

The following algebraic property of continuant polynomials is a simple consequence of (2.1), (2.2), (2.3) and (2.4).

LEMMA 2.1. *Let y_0, x_i, $i \geqslant 0$, be indeterminates, and define y_i, $i \geqslant 1$, by*

$$y_i = x_i + \frac{1}{y_{i+1}}, \quad i \geqslant 0.$$

Then

$$y_1 y_2 \cdots y_n = K_n(x_1, \ldots, x_{n-1}, y_n) \tag{2.5}$$

holds for all $n \geqslant 1$.

PROOF. From the definition of y_i, we have

$$y_0 = [x_0; x_1, \ldots, x_{i-1}, y_i] \quad \text{for all } i \geqslant 1,$$

and from (2.4) we find

$$y_i = \frac{y_0 K_{i-2}(x_1, \ldots, x_{i-2}) - K_{i-1}(x_0, \ldots, x_{i-2})}{K_i(x_0, \ldots, x_{i-1}) - y_0 K_{i-1}(x_1, \ldots, x_{i-1})} = \frac{-c_{i-1}}{c_i} \quad \text{for all } i \geqslant 2,$$

where

$$c_i = K_i(x_0, \ldots, x_{i-1}) - y_0 K_{i-1}(x_1, \ldots, x_{i-1}).$$

Also,

$$y_1 = \frac{1}{y_0 - x_0} = \frac{-1}{c_1}$$

and so

$$\begin{aligned} y_1 y_2 \cdots y_n &= \frac{(-1)(-c_1)(-c_2)(-c_3)\cdots(-c_{n-1})}{c_1 c_2 c_3 \cdots c_n} = \frac{(-1)^n}{c_n} \\ &= \frac{(-1)^n}{K_n(x_0, \ldots, x_{n-1}) - y_0 K_{n-1}(x_1, \ldots, x_{n-1})} \qquad (2.6) \end{aligned}$$

Substituting

$$y_0 = [x_0; x_1, \dots, x_{n-1}, y_n] = \frac{K_{n+1}(x_0, x_1, \dots, x_{n-1}, y_n)}{K_n(x_1, \dots, x_{n-1}, y_n)}$$

in (2.6), and using (2.1), (2.2) and (2.3), the result follows. □

3. Convergence of the Euclidean remainders

Theorem 3.1. Let a_n, t_n, r_n be the sequences arising when dividing r_{-1} by r_0 as defined before. Then, if $r_n > 0$,

$$r_n = \frac{r_0}{K_n(a_1, \dots, a_{n-1}, t_n)} \leqslant \frac{r_0}{F_{n+1}},$$

where F_n are the Fibonacci numbers.

Proof. Note that

$$t_n = \frac{r_{n-1}}{r_n} = a_n + \left\{\frac{r_{n-1}}{r_n}\right\} = a_n + \frac{r_{n+1}}{r_n} = a_n + \frac{1}{t_{n+1}},$$

where $\{x\}$ denotes the fractional part of x. So the formula for r_n follows from Lemma 2.1 by substituting $y_n = t_n$ and $x_n = a_n$, and using $t_1 \cdots t_n = r_0/r_n$. The inequality is a consequence of the fact that the continuant polynomials are increasing in each entry, $a_i \geqslant 1$, $t_n > a_n \geqslant 1$, and $K_n(1, \cdots, 1) = F_{n+1}$. □

The remainders r_n will be eventually zero if and only if r_{-1}/r_0 is rational. The theorem, together with the formula $F_n = \lfloor \phi^n/\sqrt{5} + 1/2 \rfloor$ ([**1**, 6.124]), shows that if r_{-1}/r_0 is irrational, the generating function

$$f(z) = \sum_{n=0}^{\infty} r_n z^n.$$

has radius of convergence $\geqslant 1/\phi$, with equality holding if $a_n = 1$ for all large enough n. We leave to the interested reader to prove that in the latter case, r_{-1}/r_0 will be of form $(a + b\phi)/(c + d\phi)$ for some non-negative integers a, b, c, d, and also to prove that the radius of convergence will be finite if there is some M such that $a_n \leqslant M$ for all n.

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