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A note on Solvable Equations including De Moivre's quintic

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Abstract. In this note we examine a simple algorithm for producing solvable polynomial equations and study the solutions of the quintic equation with real coefficients

$$x^5 + 5ax^3 + 5a^2x + b = 0$$

solved in radicals by De Moivre. the result is used to sum a hypergeometric series for several arguments.

Introduction

It is common knowledge that there are no elementary formulas for the solution of polynomial equations of degree higher than 4. However, specific equations of any degree exist which have algebraic solutions. In the n-th degree case of a trinomial equation it is known that the roots are expressible in terms of hypergeometric functions and one of the aims of this note is to show by example that solvability can lead to the exact evaluation of such functions in elementary terms..

In the last twenty five years attention has been devoted to determining all sextic equations which are solvable in terms of radicals in the sense of Galois; we mention [1], [2], [3] in particular. These studies were group-theoretic and little attention was paid to explicit solutions. We begin by presenting an elementary procedure for generating multiple-parameter solvable equations of arbitrary degree, which is illustrated here for degrees 5 and 6.

Let

$$F(x) = \sum_{k=0}^{n} A_k x^k = 0 \tag{1.1}$$

be an algebraic equation with $A_n = 1$, but otherwise having arbitrary complex coefficients. Then, for $a \neq 0$,

$$G(u) = u^n F(u - a/u) = 0 \tag{1.2}$$

is a 2n-th degree equation. Next, select the coefficients A_k so that (1.2) reduces to a solveable equation with respect to some power u^m (where $m|2n$). For example, in the

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case $n = 6$, let us choose $A_5 = 0,\ A_4 = 6a,\ A_2 = 9a^2$ and $A_1 = 6aA_3$ resulting in

$$u^{12} + A_3u^9 + (A_0 - 2a^3)u^6 - A_3a^3u^3 + a^6 = 0 \tag{1.3}$$

a quartic equation in u^3, which therefore has twelve algebraic roots, half of which are roots of the three parameter sextic

$$x^6 + 3ax^4 + A_3x^3 + 9a^2x^2 + 3A_3ax + A_0 = 0. \tag{1.4}$$

As an example, $A_3 = a = 1,\ A_0 = 2$ gives

$$u^{12} + u^9 - 2u^3 + 1 = 0 \tag{1.5}$$

having root

$$u^3 = -\frac{1}{4} + \frac{i\sqrt{3}}{4} - \frac{1}{2}\sqrt{\frac{3}{2} - \frac{3i\sqrt{3}}{2}}. \tag{1.6}$$

Therefore, the irreducible sextic

$$x^6 + 3x^4 + x^3 + 9x^2 + 3x + 2 = 0 \tag{1.7}$$

has the root

$$x = \sqrt[3]{-\frac{1}{4} + \frac{i\sqrt{3}}{4} - \frac{1}{2}\sqrt{\frac{3}{2} - \frac{i\sqrt{3}}{2}}} + \frac{1}{\sqrt[3]{-\frac{1}{4} + \frac{5i\sqrt{3}}{4} - \frac{1}{2}\sqrt{\frac{3}{2} - \frac{5i\sqrt{3}}{2}}}} \tag{1.8}$$

De Moivre's Solvable Quintic

We start with the general quintic equation (over the real numbers)

$$\sum_{j=0}^{5} a_jx^j = 0 \tag{2.1}$$

with $a_5 = 1$. Using Vietá's substitution, $x = u - a/u$ with $a \neq 0$ real, expanding and multiplying by u^5, the resulting 10-th degree equation is solvable (in radicals) only for $a_1 = aa_3 = 5a^2$, $a_2 = a_4 = 0$, in which case

$$u^{10} + a_0u^5 - a^5 = 0. \tag{2.2}$$

By solving this quadratic equation (in u^5) and extracting the fifth roots, after eliminating duplicates, we see that the roots of the De Moivre equation[(writing b in place of a_0)

$$x^5 + 5ax^3 + 5a^2x + b = 0 \tag{2.3}$$

are

$$\begin{aligned}
x_1 &= 2^{-1/5}[e^{-3\pi i/5}(\sqrt{\gamma} - b)^{1/5} + e^{-2\pi i/5}\sqrt{\gamma} + b)^{1/5}] \\
x_2 &= -2^{-1/5}[e^{-\pi i/5}(\sqrt{\gamma} - b)^{1/5} + e^{-4\pi i/5}(\sqrt{\gamma} + b)^{1/5}] \\
x_3 &= 2^{-1/5}[e^{-2\pi i/5}(\sqrt{\gamma} - b)^{1/5} + e^{-3\pi i/5}(\sqrt{\gamma} + b)^{1/5}]. \\
x_4 &= -2^{-1/5}[e^{\pi i/5}(\sqrt{\gamma} - b)^{1/5} + e^{4\pi i/5}(\sqrt{\gamma} + b)^{1/5}] \\
x_5 &= 2^{-1/5}[(\sqrt{\gamma} - b)^{1/5} + (\sqrt{\gamma} + b)^{1/5}] \\
\gamma &= 4a^5 + b^2.
\end{aligned} \tag{2.4}$$

(The asterisk denotes the complex conjugate.)

By means of the Tschirnhausen transformation [5] $y = x^2 + \alpha x + \beta$ any quintic can be reduced to "principal" form $y^5 + Ay^2 + By + C = 0$. This can be carried out efficiently using Mathematica by means of the command

$Resultant[x^5 + 5ax^3 + 5a^2x + b, y - (x^2 + \alpha x + \beta), x]$ which returns $y^5 + (10a - 5\beta)y^4 + (35a^2 + 5a\alpha^2 - 40a\beta + 10\beta^2)y^3 + \ldots$ from which we easily determine $\alpha = \sqrt{a}$ and $\beta = 2a$. Thus, the principal form for the De Moivre quintic is

$$y^5 + (5\sqrt{a}b - 10a^3)y^2 + 15a^4y - (9a^{5/2}b + 22a^5 + b^2) = 0. \tag{2.5}$$

Jerrard[4] found that by means of a quartic Tschirnhausen transformation a principal quintic can be further reduced to the trinomial Bring form

$$z^5 - z + t = 0. \tag{2.6}$$

The details can be mechanised to some extent[5,6], but are still quite cumbersome. To present the result it is convenient to introduce the further abbreviations: $\alpha = 8a^{5/2} - b$, $\beta = 2a^{5/2} - b$, $\delta = 176a^5 + 36a^{5/2}b - b^2$. Also $d_0 = 675(\beta/\alpha)(2a^{5/2} + b)a^2b$, $d_1 = \Delta/\alpha$, $d_2 = 25(\beta/\alpha)(8a^5 - 3a^{5/2}b - b^2)$, $d_3 = 75(\beta/\alpha)a^{3/2}(16a^{5/2} + 3b)$ and $d_4 = 75\beta a^{1/2}$, where

$$\begin{aligned}\Delta = 25[&\frac{\delta\gamma^{1/2}\alpha}{3^{2/3}a^{1/2}}(\beta\sqrt{3a^{3/2}(11a^{5/2} - b)} - 9a^2\gamma^{1/2})^{1/3} + \\ &\frac{1}{3^{1/3}\alpha\gamma^{1/2}}\frac{16a^{15/2} + 4a^5b + 4a^{5/2}b^2 + b^3}{(\beta\sqrt{3a^{3/2}(11a^{5/2} - b)} - 9a^2\gamma^{1/2})^{1/3}} - \\ &(800a^{17/2} - 318a^6b + 227a^{7/2}b^2 - 12ab^3)].\end{aligned} \tag{2.7}$$

The quantity Δ has six possible values, due to the choice of branches in the square and cube roots, so it takes a bit of experimentation to select the appropriate one.

Next, we require the coefficients

$$c_0 = \frac{5^6\delta^3\gamma^2}{\alpha^5}[5625\beta^2 f_1(a,b) + \frac{25}{a}\beta\Delta f_2(a,b) + 3a^{1/2}\Delta^2(188a^5 + 86a^{5/2}b + 9b^2)] \tag{2.8}$$

$$c_1 = \frac{5^5\delta^2\gamma a^{1/2}}{\alpha^4}[5625\beta^2 g_1(a,b) + \frac{25}{a}\beta\Delta g_2(a,b) + 9a^3\Delta^2(4a^{5/2} + b)],$$

where

$$\begin{aligned}f_1(a,b) &= 2622464a^{35/2} - 1339776a^{15}b + 103828a^{25/2}b^2 + 218210a^{10}b^3 - \\ &\quad 17365a^{15/2}b^4 - 2858a^5b^5 + 297a^{5/2}b^6 - b^7 \\ f_2(a,b) &= 209024a^{25/2} - 30616a^{10}b - 34508a^{15/2}b^2 - \\ &\quad 98a^5b^3 + 530a^{5/2}b^4 - b^5, \\ g_1(a,b) &= 133120a^{35/2} - 83520a^{15}b + 26804a^{25/2}b^2 + 1789a^{10}b^2 - \\ &\quad 1082a^{15/2}b^4 + 386a^5b^5 - 48a^{5/2}b^6 + b^7, \\ g_2(a,b) &= 11200a^{25/2} - 3656a^{10}b - 754a^{15/2}b^2 + 62a^5b^3 - 38a^{5/2}b^4 + b^5.\end{aligned} \tag{2.9}$$

The result is that the roots of the Bring-Jerrard quintic

$$z^5 - z + t = 0 \tag{2.10}$$

with

$$t = -e^{-i\pi/4} c_0 c_1^{-5/4} \tag{2.11}$$

are

$$z_j = e^{-i\pi/4} c_1^{-1/4} \sum_{k=0}^{4} d_k (x_j^2 + \sqrt{a} x_j + 2a)^k. \qquad (2.12).$$

Now it has been shown by James Cockle[7] (the first Chief Magistrate of Queensland, Australia) and others[8] that one of the roots of (2.10) is

$$z_0 = t\ {}_4F_3(1/5, 2/5, 3/5, 4/5; 1/2, 3/4, 5/4; \frac{3125}{256} t^4). \tag{2.13}$$

Therefore, it has been shown that

$$ {}_4F_3(1/5, 2/5, 3/5, 4/5; 1/2, 3/4, 5/4; -\frac{3125}{256} \frac{c_0^4}{c_1^5}) = $$

$$ -\frac{c_1}{c_0} \sum_{k=0}^{4} d_k (x_0^2 + \sqrt{a} x_0 + 2a)^k \tag{2.14}$$

where x_0 is one of the five values in (2.4). (Which one it is can be determined numerically).

As a very simple example, let us take the case $b = 2a^{5/2}$. Then the principal quintic (2.5) is already in Bring form, which is easily scaled by $y = e^{i\pi/4} 15^{1/4} a z$ into (2.6) with $t = 44 e^{-i\pi/4} 15^{-5/4}$. Therefore, from (2.13) we find

$$ {}_4F_3(1/5, 2/5, 3/5, 4/5; 1/2, 3/4, 5/4; -\frac{14641}{243}) = $$

$$ \frac{15}{44} [(\sqrt{2}+1)^{2/5} + (\sqrt{2}-1)^{2/5} - (\sqrt{2}+1)^{1/5} + (\sqrt{2}-1)^{1/5}]. \tag{2.15}$$

At the other extreme, the algorithm described here produces formulas such as

$$ {}_4F_3(1/5, 2/5, 3/5, 4/5; 1/2, 3/4, 5/4; z) = Z \tag{2.16}$$

where

$$z = ((13+\sqrt{182})^{2/3}((78913^{2/3} + 24713^{1/6}\sqrt{14})(13+\sqrt{182})^{1/3} + 2275(13+\sqrt{182})^{2/3} - 13^{1/3}(6799 + 542\sqrt{182}))^4)/(1521((-713^{2/3} + 313^{1/6}\sqrt{14})(13+\sqrt{182})^{1/3} - 161(13+\sqrt{182})^{2/3} + 13^{1/3}(133 + 10\sqrt{182}))^5) \qquad *2.17)$$

and

$$Z = (5((-713^{2/3} + 313^{1/6}\sqrt{14})(13+\sqrt{182})^{1/3} - 161(13+\sqrt{182})^{2/3} + 13^{1/3}(133+10\sqrt{182}))(-3(-5+\sqrt{26})^{4/5}(13+\sqrt{182})^{1/3}(1103 + 3465(5+\sqrt{26})^{1/5} + 5355(5+\sqrt{26})^{2/5} + 3780(5+\sqrt{26})^{3/5}) + 3(-5+\sqrt{26})^{3/5}(13+\sqrt{182})^{1/3}(8872 + 1485\sqrt{26} + 4412(5+\sqrt{26})^{1/5} + 6930(5+\sqrt{26})^{2/5} + 3780(5+\sqrt{26})^{4/5}) + (-5+\sqrt{26})^{1/5}(-14213^{2/3} +$$

$$14213^{1/3}(13+\sqrt{182})^{2/3}+(13+\sqrt{182})^{1/3}(34114+5355\sqrt{26}+$$
$$13236(5+\sqrt{26})^(3/5)+10395(5+\sqrt{26})^{4/5}-$$
$$9(5+\sqrt{26})^{2/5}(-2797+180\sqrt{26})))-(-5+\sqrt{26})^{2/5}(-14213^{2/3}+$$
$$14213^{1/3}(13+\sqrt{182})^{2/3}+(13+\sqrt{182})^{1/3}(33742+6480\sqrt{26}+$$
$$20790(5+\sqrt{26})^{3/5}+16065(5+\sqrt{26})^{4/5}+$$
$$9(5+\sqrt{26})^{1/5}(2797+180\sqrt{26})))+(5+\sqrt{26})^{1/5}(14213^{2/3}-3309(5+\sqrt{26})^{3/5}(13+\sqrt{182})^{1/3}+$$
$$3(5+\sqrt{26})^{2/5}(-8872+1485\sqrt{26})(13+\sqrt{182})^{1/3}+$$
$$(-34114+5355\sqrt{26})(13+\sqrt{182})^{1/3}-14213^{1/3}(13+\sqrt{182})^{2/3}+ \quad (2.18)$$
$$2(5+\sqrt{26})^{1/5}(7113^{2/3}+(-16871+3240\sqrt{26})(13+\sqrt{182})^{1/3}-7113^{1/3}(13+$$
$$\sqrt{182})^{2/3}))))/(568(13+\sqrt{182})^{1/3}((78913^{2/3}+$$
$$24713^{1/6}\sqrt{14})(13+\sqrt{182})^{1/3}+2275(13+\sqrt{182})^{2/3}-13^{1/3}(6799+542\sqrt{182})))$$

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References

[1] Thomas R. Hagedorn, J. Algebra **233**, 704 (2000)
[2] Gene Ward Smith, Mathematics of Computation **69**, 775 (1999).
[3] Caroll W. Boswell, Classification and Solutions of Sixth Degree Equations PhD Thesis,Department of Mathematics,. Clarkson University April (2000); C.W. Boswell and M.L. Glasser, ArXiv Math-Phys/0504001
[4] B.K. Spearman and K,S. Williams, Far East J. Math. Sci.**1**, 137-143 (1999)
[5] V.S. Adamchik and D.J. Jeffrey, ACM SIGSAM Bulletin**37**, 90 (2003).
[6] Victor Adamchik, Private communication.
[7] J. Cockle, Phil. Mag.**20**, 145 (1860).
[8] M.L. Glasser, J. Comp. Appl. Math., **118**, 169 (2000).

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