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# An elementary evaluation of a quartic integral

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Abstract. We provide an elementary evaluation for the integral

$$N_{0,4}(a;m) \quad := \quad \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}$$

where $m \in \mathbb{N}$ and $a \in (-1,\infty)$:

$$N_{0,4}(a;m) \quad = \quad \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a)$$

for $P_m(a)$ a polynomial in $a$.

## 1. Introduction

Honors Integral Calculus is a course taught at Tulane University to the best incoming students. Most of them are proficient at the mechanical aspects of single variable Calculus. In the discussion on definite integrals we encouraged the students to use both tables of integrals (such as Gradshteyn and Ryzhik [**10**]) and the symbolic integration package Mathematica 5.1 as sources of interesting problems and also as checks for the material presented in class.

After covering techniques of integration we looked at Wallis formula (2.1) and observed that the numbers generated are rational multiples of $\pi$ with denominators a pure power of 2. This prompted a discussion of the divisibility properties of the binomial coefficients. We proved that the central binomial coefficients $C_m$ are always even and that $\frac{1}{2}C_m$ is odd if and only if $m$ is a power of 2; the proof we gave is outlined in Section 3. A direct Mathematica calculation of Wallis integral gives (4.7), which leads naturally to a discussion of Legendre's relation (4.8).

The trigonometric form of Wallis's integral can also be used to discuss the evaluation of some finite sums. In Section 2 we discuss the sum

$$\sum_{i=0}^{[m/2]} 2^{-2i} \binom{m}{2i} \binom{2i}{i} \quad = \quad 2^{-m} \binom{2m}{m} \tag{1.1}$$

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that appears in a new proof of Wallis' formula. This result provides an excellent opportunity to introduce students to the wonderful world of *proving identities by machine*, which is described in the beautiful book [**13**] where the sum (1.1) is the first example (p. 113). Details are given in Section 2.

At this point we were asked if it was possible to evaluate the more general integral

$$J_{n,m} \quad := \quad \int_0^\infty \frac{dx}{(x^n+1)^{m+1}}.$$

This requires the introduction of the beta function, and in Section 4 we review the properties of $\Gamma$ and $B$ that are needed to demonstrate the evaluation

$$J_{4,m} \quad := \quad \int_0^\infty \frac{dx}{(x^4+1)^{m+1}} = \frac{\pi}{m!\, 2^{2m+3/2}} \prod_{k=1}^{m} (4k-1). \tag{1.2}$$

We then asked our students to consider the integral

$$N_{0,4}(a;m) \quad := \quad \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}, \tag{1.3}$$

which is a natural generalization of the integral in (1.2). We have presented a proof of the formula

$$N_{0,4}(a;m) \quad = \quad \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a) \tag{1.4}$$

in [**3**] using hypergeometric functions, and in [**5**] we proved it as a consequence of the Taylor series expansion

$$\sqrt{a+\sqrt{1+c}} \quad = \quad \sqrt{a+1} + \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} N_{0,4}(a;k-1)\, c^k. \tag{1.5}$$

The goal of this paper is to present a completely elementary proof of (1.4).

Many more examples of interesting Mathematics encountered while trying to evaluate definite integrals can be found in our book *Irresistible Integrals: Symbolics, Analysis and Experiments in the Evaluation of Integrals*. The motivation of the work presented here and in [**6**] is to provide all proofs of the evaluations appearing in the table [**10**]. This table contains a large number of formulas but the Mathematics behind their proofs is not directly available. In each new edition, the editors include additional evaluations and the number of integrals evaluated there is very large. In our task to prove them we have chosen first those that are connected to our work. For example, in the last edition (in the year 2000) one finds a beautiful formula: let

$$u(x) \quad = \quad 1 + \frac{4x^2}{3(1+x^2)^2}, \tag{1.6}$$

then [**10**, no. 3.248.5] states that

$$\int_0^\infty \frac{dx}{(1+x^2)^{3/2}\sqrt{u(x)+\sqrt{u(x)}}} \quad = \quad \frac{\pi}{2\sqrt{6}}. \tag{1.7}$$

This is appealing to us due to our work on the double square root function that produced (1.5). *Unfortunately (1.7) is incorrect.* This error now yields two new problems.
*The direct problem*: find a variation of the right hand side in (1.7) that is the correct answer for the definite integral.
*The inverse problem*: find a variation of the integrand that integrates to $\pi/2\sqrt{6}$.

The inverse problem deals with the issue of typos: perhaps the number 4 appearing in the integrand in (1.7) should have been a 14.

We cannot solve either one of them.

## 2. Wallis formula

We now show that

$$J_{2,m} = \int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \frac{\pi}{2^{2m+1}}\binom{2m}{m}, \tag{2.1}$$

where $m$ is a nonnegative integer. The change of variables $x = \tan\theta$ converts $J_{2,m}$ to its trigonometric form

$$J_{2,m} = \int_0^{\pi/2} \cos^{2m}\theta\, d\theta = \frac{\pi}{2^{2m+1}}\binom{2m}{m}, \tag{2.2}$$

which is known as Wallis formula. The proof of (2.2) is elementary and sometimes found in calculus books (see e. g. [**11**, page 492]). It consists of first writing $\cos^2\theta = 1-\sin^2\theta$ and using integration by parts to obtain the recursion

$$J_{2,m} = \frac{2m-1}{2m} J_{2,m-1}, \tag{2.3}$$

and then verifying that the right side of (2.2) satisfies the same recursion and that both sides yield $\pi/2$ for $m = 0$.

In order to motivate the calculations described in Section 6, we present a new proof of Wallis formula. We have

$$J_{2,m} = \int_0^{\pi/2} \cos^{2m}\theta\, d\theta = \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^m d\theta.$$

Now introduce $\psi = 2\theta$ and expand and simplify the result by observing that the odd powers of cosine integrate to zero. The inductive proof of (2.2) requires

$$J_{2,m} = 2^{-m}\sum_{i=0}^{[m/2]} \binom{m}{2i} J_{2,i}. \tag{2.4}$$

Note that $J_{2,m}$ is uniquely determined by (2.4) along with the initial value $J_{2,0} = \pi/2$. Thus (2.2) now follows from the identity

$$f(m) := \sum_{i=0}^{[m/2]} 2^{-2i}\binom{m}{2i}\binom{2i}{i} = 2^{-m}\binom{2m}{m} \tag{2.5}$$

since (2.5) can be written as

$$A_m \;=\; 2^{-m} \sum_{i=0}^{[m/2]} \binom{m}{2i} A_i,$$

where

$$A_i \;=\; \frac{\pi}{2^{2i+1}} \binom{2i}{i}.$$

It remains to verify the identity (2.5). This can be done *mechanically* using the theory developed by Wilf and Zeilberger, which is explained in [**12, 13**]; the sum in (2.5) is the example used in [**13**] (page 113) to illustrate their method. The command

$$ct(binomial(m, 2i)\, binomial(2i, i) 2^{-2i}, 1, i, m, N)$$

produces

$$f(m+1) \;=\; \frac{2m+1}{m+1}\, f(m), \tag{2.6}$$

and one checks that $2^{-m}\binom{2m}{m}$ satisfies the same recursion. Note that (2.3) and (2.6) are equivalent since

$$J_{2,m} \;=\; \frac{\pi}{2^{m+1}} f(m).$$

## 3. The quadratic denominators

The expression (2.1) shows that the integral $J_{2,m}$ is a rational multiple of $\pi$ with denominator a pure power of 2. We now show that the central binomial coefficient $C_m$ appearing in (2.1) is also even, so the power of 2 in the denominator is at most $2m$, and that this bound is optimal because it is possible for $\frac{1}{2}C_m$ to be odd. We introduce the notation $\nu_2(n)$ for the exponent of 2 that exactly divides $n$.

PROPOSITION 3.1. *The central binomial coefficient $C_m$ is even, and $\frac{1}{2}C_m$ is odd precisely when $m$ is a power of 2.*

PROOF. The proof is based on the expression for the power of 2 that divides $m!$. This is given by

$$\nu_2(m!) \;=\; \sum_{k=1}^{\infty} \left[ \frac{m}{2^k} \right], \tag{3.1}$$

which is easy to see. In the product defining $m!$ one can divide out every even number by 2, and there are $[m/2]$ such numbers. In the remaining number there are $[m/4]$ even ones (these were multiples of 4 to begin with), and so on. Note that the sum is finite.

Now

$$\nu_2((2m)!) = \sum_{k=1}^{\infty} \left[ \frac{m}{2^{k-1}} \right] = \sum_{k=0}^{\infty} \left[ \frac{m}{2^k} \right] = m + \nu_2(m!),$$

so that

$$\nu_2(C_m) \;=\; m - \nu_2(m!). \tag{3.2}$$

Thus $\frac{1}{2}C_m$ is odd if and only if

$$\sum_{k=1}^{\infty}\left[\frac{m}{2^k}\right] = m-1. \tag{3.3}$$

We claim that $m$ in (3.3) can be replaced by its odd part. Indeed, writing $m = 2^a \cdot b$ with $a$ and $b$ positive integers and $b$ odd, (3.3) is equivalent to

$$\begin{aligned} 2^a \cdot b - 1 &= \sum_{k=1}^{\infty}\left[\frac{b}{2^{k-a}}\right] = \sum_{k=1-a}^{\infty}\left[\frac{b}{2^k}\right] = \sum_{k=1-a}^{0}\left[\frac{b}{2^k}\right] + \sum_{k=1}^{\infty}\left[\frac{b}{2^k}\right] \\ &= b(2^a-1) + \sum_{k=1}^{\infty}\left[\frac{b}{2^k}\right], \end{aligned}$$

i. e. to

$$\sum_{k=1}^{\infty}\left[\frac{b}{2^k}\right] = b-1. \tag{3.4}$$

It remains to show that $b=1$. Clearly (3.4) holds for $b=1$. If $b>1$, then there exists a positive integer $N$ such that $2^N < b < 2^{N+1}$, and (3.4) becomes

$$\sum_{k=1}^{N}\left[\frac{b}{2^k}\right] = b-1, \tag{3.5}$$

but for $b \geqslant 3$ (and odd),

$$\sum_{k=1}^{N}\left[\frac{b}{2^k}\right] = \sum_{k=1}^{N}\left[\frac{b-1}{2^k}\right] \leqslant (b-1)(1-2^{-N}) \leqslant b-2.$$

We conclude that $b=1$ and thus that $m$ is a power of 2. □

An alternate proof of the proposition follows from the following facts:
a) $m \geqslant \nu_2(m!)$ (this follows directly from (3.2)).
b) $\nu_2((2^n)!) = 2^{n-1} + 2^{n-2} + \cdots + 1 = 2^n - 1$ (this follows directly from (3.1).
c) If $a$ is the largest integer such that $m = 2^a + b$, then $\nu_2(m!) = \nu_2((2^a)!) + \nu_2(b!) = 2^a - 1 + \nu_2(b!)$ (we leave this as an exercise).

The proof of the proposition is now immediate, since for $m = 2^a$,

$$\nu_2(C_m) = 2^a - \nu_2(2^a!) = 2^a - (2^a-1) = 1,$$

and for $m = 2^a + b\,(0 < b < 2^a)$,

$$\begin{aligned} \nu_2(C_m) &= 2^a + b - \nu_2((2^a+b)!) \\ &= 2^a + b - (2^a - 1 + \nu_2(b!)) \\ &> b + 1 - b = 1. \end{aligned}$$

A generalization of c) is the following formula for $\nu_2(m!)$ due to Legendre:

$$\nu_2(m!) = m - s_2(m), \tag{3.6}$$

where $s_2(m)$ is the sum of the binary digits of $m$. It follows that $\nu_2(C_m) = s_2(m)$ and we have a third proof of Proposition 3.1.

## 4. Gamma and Beta functions

In order to generalize the notion of factorials, Euler [**9**] introduced two functions: the *gamma function*, defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt, \tag{4.1}$$

and the *beta function*, given by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt. \tag{4.2}$$

The history of these functions is presented in [**8**].

Integration by parts produces the functional equation

$$\Gamma(x+1) = x\,\Gamma(x), \tag{4.3}$$

and since $\Gamma(1) = 1$, we see that

$$\Gamma(m) = (m-1)!$$

for $m$ a positive integer.

Among the many representations for these functions, we mention

$$B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\, dt, \tag{4.4}$$

which is obtained from (4.2) via the change of variable $t \mapsto t/(t+1)$. We employ two other properties of $\Gamma$ and $B$:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{4.5}$$

and

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{4.6}$$

Elementary proofs of (4.5) and (4.6) can be found in [**7**] and [**1**, page 71], respectively.

A *Mathematica* calculation of $J_{2,m}$ yields

$$J_{2,m} = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(m+1/2)}{\Gamma(m+1)}, \tag{4.7}$$

from which we see the connection between $J_{2,m}$ and the functions $\Gamma$ and $B$, and combining (4.7) with (2.1) gives Legendre's relation,

$$\Gamma(m+1/2) = \frac{\Gamma(2m)\Gamma(1/2)}{\Gamma(m)2^{2m-1}}. \tag{4.8}$$

It can easily be shown that (4.7) and (4.8) are also true for noninteger values of $m$ ($m > -1/2$ and $m > 0$ respectively).

## 5. A Beta function calculation

We next evaluate an extension of (2.1),

$$J_{4,m} = \int_0^\infty \frac{dx}{(x^4+1)^{m+1}}. \tag{5.1}$$

The change of variables $t = x^4$ and and (4.4) yield

$$J_{4,m} = \frac{1}{4} B\left(\tfrac{1}{4}, m+\tfrac{3}{4}\right).$$

Using (4.5) and (4.3) we obtain

$$J_{4,m} = \frac{1}{4m!}\Gamma(1/4)\Gamma(m+3/4) = \frac{\Gamma(3/4)\Gamma(1/4)}{m!\,2^{2m+2}} \times \prod_{k=1}^{m}(4k-1).$$

The symmetry formula (4.6) gives $\Gamma(1/4)\Gamma(3/4) = \pi\sqrt{2}$, so we conclude that

$$\int_0^\infty \frac{dx}{(x^4+1)^{m+1}} = \frac{\pi}{m! 2^{2m+3/2}} \prod_{k=1}^{m}(4k-1). \tag{5.2}$$

## 6. Reduction to a polynomial

In this section we prove that, apart from a simple algebraic factor, the integral $N_{0,4}(a;m)$ is a polynomial in $a$.

Theorem 6.1. *Let*

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}$$

*and define*

$$P_m(a) = \frac{1}{\pi} 2^{m+3/2}(a+1)^{m+1/2} N_{0,4}(a;m).$$

*Then $P_m(a)$ is a polynomial in $a$ of degree $m$ with rational coefficients, and is given by*

$$P_m(a) = \sum_{j=0}^{m} \binom{2m+1}{2j} (a+1)^j \sum_{k=0}^{m-j} \binom{m-j}{k} \binom{2(m-k)}{m-k} 2^{-3(m-k)} (a-1)^{m-k-j}. \tag{6.1}$$

We start with the substitutions $x = \tan\theta$ and $u = 2\theta$, which yield

$$\begin{aligned} N_{0,4}(a;m) &= \int_0^{\pi/2} \left( \frac{\cos^4\theta}{\sin^4\theta + 2a\sin^2\theta\cos^2\theta + \cos^4\theta} \right)^{m+1} \times \frac{d\theta}{\cos^2\theta} \\ &= 2^{-(m+1)} \int_0^{\pi} \left( \frac{(1+\cos u)^2}{(1+a)+(1-a)\cos^2 u} \right)^{m+1} \times \frac{du}{1+\cos u} \\ &= 2^{-(m+1)} \sum_{j=0}^{m} \binom{2m+1}{2j} \\ &\times \int_0^{\pi} \left[ (1+a)+(1-a)\cos^2 u \right]^{-(m+1)} \cos^{2j} u \, du, \end{aligned} \tag{6.2}$$

where in the last step we have used the fact that

$$\int_0^{\pi} \left[ (1+a)+(1-a)\cos^2 u \ \right]^{-(m+1)} \cos^{j} u \, du \quad = \quad 0$$

for odd $j$.

We now compute the integral appearing in (6.2). Let

$$I_m^j(a) \quad = \quad \int_0^{\pi} \left[ (1+a)+(1-a)\cos^2 u \right]^{-(m+1)} \cos^{2j} u \, du.$$

The substitution $v = 2u$ then gives

$$\begin{aligned} I_m^j(a) &= 2^{m-j} \int_0^{2\pi} \left[ (3+a)+(1-a)\cos v \right]^{-(m+1)} (1+\cos v)^j \, dv \\ &= 2^{m-j+1} \int_0^{\pi} \left[ (3+a)+(1-a)\cos v \right]^{-(m+1)} (1+\cos v)^j \, dv, \end{aligned}$$

where we have used the symmetry of cosine about $v = \pi$ in the last step.

For each fixed value of the index $j$, the integrand is a rational function of $\cos v$, so the substitution $z = \tan(v/2)$ is a natural one. It yields

$$\begin{aligned} I_m^j(a) &= 2 \times \int_0^{\infty} \left[ 2 + (1+a)z^2 \right]^{-(m+1)} \times (1+z^2)^{m-j} dz \\ &= 2(1+a)^{-(m+1)} \int_0^{\infty} \sum_{k=0}^{m-j} \binom{m-j}{k} \left[ z^2 + \frac{2}{1+a} \right]^{-m-1+k} \left[ \frac{a-1}{a+1} \right]^{m-j-k} dz. \\ &= \pi \times 2^{-1/2-3m} (1+a)^{-m-1/2} \times \\ &\times \sum_{k=0}^{m-j} \binom{2(m-k)}{m-k} \binom{m-j}{k} 2^{3k} (a+1)^j (a-1)^{m-j-k} \end{aligned} \tag{6.3}$$

where we have used $z = \sqrt{2/(1+a)}\tan\varphi$ and Wallis's formula (2.2). Thus

$$
\begin{aligned}
N_{0,4}(a;m) &= \pi \times \sum_{j=0}^{m} \binom{2m+1}{2j} (a+1)^{-m-1/2+j} \times \\
&\times \sum_{k=0}^{m-j} \binom{m-j}{k} \binom{2(m-k)}{m-k} 2^{3k-4m-3/2} (a-1)^{m-j-k}.
\end{aligned}
\tag{6.4}
$$

The expression (6.4) given for $N_{0,4}(a;m)$ allows the explicit evaluation of this integral for a given value of the parameter $a$.

**Some examples**. Using Theorem 6.1 we obtain

$$
\begin{aligned}
P_0(a) &= 1 & P_1(a) &= \tfrac{1}{2}(2a+3) \\
P_2(a) &= \tfrac{3}{8}(4a^2+10a+7) & P_3(a) &= \tfrac{1}{16}(40a^3+140a^2+172a+77)
\end{aligned}
$$

so that

$$
\begin{aligned}
N_{0,4}(a,0) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)} &= \frac{\pi}{2^{3/2}(a+1)^{1/2}} \cdot 1 \\
N_{0,4}(a;1) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^2} &= \frac{\pi}{2^{7/2}(a+1)^{3/2}} \cdot (2a+3) \\
N_{0,4}(a;2) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^3} &= \frac{3\pi}{2^{13/2}(a+1)^{5/2}} \cdot (4a^2+10a+7) \\
N_{0,4}(a;3) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^4} &= \frac{\pi}{2^{17/2}(a+1)^{7/2}} \cdot (40a^3+140a^2+172a+77).
\end{aligned}
$$

## 7. A triple sum for the coefficients

We now write

$$
P_m(a) = \sum_{l=0}^{m} d_l(m) a^l
$$

and derive an expression for the coefficients $d_l(m)$ in terms of $m$ and $l$.

We start by reversing the order of summation in (6.1) to produce

$$
\begin{aligned}
P_m(a) &= \sum_{l=0}^{m}\sum_{k=0}^{m}\sum_{\nu=0}^{m}\sum_{j=0}^{m} 2^{-3k}\binom{2k}{k}\binom{2m+1}{2\nu}\binom{m-\nu}{m-k}\binom{\nu}{j}\binom{k-\nu}{l-j} \\
&\times (-1)^{k-\nu-l+j} a^l.
\end{aligned}
\tag{7.1}
$$

where we have extended all the sums to $m$ (this is valid since the added terms vanish).

The last step is to restrict the ranges of the sums in (7.1) to nonzero terms. Consideration of the binomial coeffcients involved leads to a) $\nu+l-j \leqslant k \leqslant m$ (from $\binom{k-\nu}{l-j}$), b) $j \leqslant \nu \leqslant k-l+j \leqslant m-l+j$ (from $\binom{\nu}{j}$ and $\binom{k-\nu}{l-j}$), c) $0 \leqslant j \leqslant l$ (from

$\binom{k-\nu}{l-j}$), and d) $0 \leqslant l \leqslant m$, so that (7.1) becomes

$$\begin{aligned} P_m(a) &= \sum_{l=0}^{m}\sum_{j=0}^{l}\sum_{\nu=j}^{m-l+j}\sum_{k=\nu+l-j}^{m} 2^{-3k}\binom{2k}{k}\binom{2m+1}{2\nu}\binom{m-\nu}{m-k}\binom{\nu}{j}\binom{k-\nu}{l-j} \\ &\times (-1)^{k-\nu-l+j}a^l, \end{aligned} \tag{7.2}$$

and thus,

$$\begin{aligned} d_l(m) &= \sum_{j=0}^{l}\sum_{\nu=j}^{m-l+j}\sum_{k=\nu+l-j}^{m} 2^{-3k}\binom{2k}{k}\binom{2m+1}{2\nu}\binom{m-\nu}{m-k}\binom{\nu}{j}\binom{k-\nu}{l-j} \\ &\times (-1)^{k-\nu-l+j}. \end{aligned} \tag{7.3}$$

For each pair of values $(m, l)$, the evaluation of (7.3) requires the calculation of $(l+1)(m-l+1)(m-l+2)/2$ terms, each involving a product of 5 binomial coefficients. The calculations of $P_m(a)$ and $N_{0,4}(a;m)$ therefore amount to a quadruple sum with $(m+1)(m^3+9m^2+26m+24)/24 \sim m^4/24$ terms.

## 8. The quartic denominators

The expression (7.3) for the coefficients of $P_m(a)$ makes it clear that the denominator of $d_l(m)$ is a power of 2. Define $dmax$ to be the maximum exponent of 2 that appears in the denominator of $d_l(m)$. From (7.3) we immediately get the crude bound

$$dmax \leqslant 3m.$$

The largest contribution to $dmax$ comes from the term $k = m$, but this coefficient is multiplied by the central binomial coefficient $C_m$. In Section 3 we showed that $\nu_2(C_m) = m - \nu_2(m!)$, so the crude bound can be improved to

$$dmax \leqslant D := 2m + \sum_{i=1}^{\infty}\left[\frac{m}{2^i}\right].$$

Optimal bounds for $\nu_2(m!)$ have been provided by Berndt-Bhargava [**2**, page 653], yielding

$$3m - \frac{\log(m+1)}{\log 2} \leqslant D \leqslant 3m-1.$$

We conclude that the contribution of $C_m$ to the denominators of the coefficients of $P_m(a)$ is asymptotically negligible as $m \to \infty$. The sharp bound $dmax \leqslant 2m-1$ is discussed in the next section. We have been unable to use elementary methods to prove this result.

## 9. A single sum formula for $P_m(a)$

The representations

$$P_m(a) = 2^{-2m}\sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}(a+1)^k \tag{9.1}$$
$$= 2^{-2m}\sum_{l=0}^{m}\sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{l}a^l$$

are significant improvements over (6.1) and (7.2), and are presented in [**3**] and [**5**]. Note that the coefficients of $P_m(a)$ are positive and given by

$$d_l(m) = 2^{-2m}\sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{l}, \tag{9.2}$$

the denominators of which are bounded by $2^{2m}$. Every term in the sum (9.2) is even with the possible exception of the first term when $l = 0$. This term is the central binomial coefficient $C_m$. From Proposition 3.1 we know that $C_m$ is even and that $\frac{1}{2}C_m$ can be odd. We conclude that the bound is $2^{2m-1}$ and that this is optimal:

Proposition 9.1. *The denominators of $P_m(a)$ are powers of 2 bounded by $2^{2m-1}$.*

As a result of (9.1), the integral $N_{0,4}(a;m)$ can now be rewritten more simply:

Theorem 9.1. *The integral $N_{0,4}(a;m)$ is given explicitly by*

$$\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} = \frac{\pi}{2^{3m+3/2}(a+1)^{m+1/2}} \times \sum_{l=0}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}(a+1)^l$$
$$= \frac{\pi}{2^{3m+3/2}(a+1)^{m+1/2}} \times \sum_{l=0}^{m}\sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{l}a^l.$$

We expect that this formula will be included in the next edition of [**10**].

We give a simple number-theoretic consequence of Theorem 9.1. The special case $a = 0$ in (5.2) yields

$$\sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m} = \frac{2^m}{m!}\prod_{k=1}^{m}(4k-1), \tag{9.3}$$

and since the left-hand side is an integer, we conclude:

Proposition 9.2. *The odd part of $m!$ divides the product $\prod_{k=1}^{m}(4k-1)$.*

## 10. A finite sum

We have produced two expressions for the polynomial $P_m(a)$, (6.1) and (9.1). The corresponding formulae for the value $P_m(1)$ lead to

$$\sum_{k=0}^{m} 2^{-2k} \binom{2k}{k} \binom{2m-k}{m} = \sum_{k=0}^{m} 2^{-2k} \binom{2k}{k} \binom{2m+1}{2k}. \tag{10.1}$$

The identity (10.1) can be verified using D. Zeilberger's package EKHAD [**13**]. Indeed, EKHAD tells us that both sides of (10.1) satisfy the recursion

$$(2m+3)(2m+2)f(m+1) = (4m+5)(4m+3)f(m),$$

and they obviously agree at $m = 1$. An elementary proof of (10.1) would be desirable.

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